CHAPTER XI

SOME PHYSICAL APPLICATIONS

11.1. SCOPE OF THE CHAPTER

This chapter deals with the exploration of some physical applications of exterior differential forms. In the first four successive sections we discuss the analytical mechanics for which exterior forms prove to be a very powerful tool to reveal its various fundamental properties. We first investigate in Sec. 11.2 the behaviour of a dynamical system with m degrees of freedom, whose constraints are holonomic and are not changing with time. We further assume that forces acting on the system are derivable from a time-independent potential. Such a system is depicted by the Lagrangian function. Then, the Lagrange equations are given, and by defining the generalised momenta, the Hamiltonian function and the Hamilton equations are introduced. It is shown that the generalised coordinates and momenta are local coordinates of a $2m$-dimensional symplectic manifold S. A symplectic 2-form then provides an isomorphism between the module of 1-forms on the manifold S and its tangent bundle. This enables us to define Hamiltonian vector fields and to express equations of motions as an exterior equation on S. We then introduce the Poisson bracket of 1-forms on S in Sec. 11.3 and we examine properties of these brackets. We further show that 1-forms constitute a Lie algebra with respect to a product identified as a Poisson bracket. Making use of the relations involving such Poisson brackets, we obtain Poisson brackets of 0-forms, namely, differentiable functions and we show that these functions also constitute a Lie algebra with respect to Poisson brackets. Then the connection between Poisson brackets and equations of motion is established. We deal with canonical transformations in Sec. 11.4 that are characterised as mappings under which the symplectic form remains invariant. It turns out that these transformations leave also the Hamilton equations of motions invariant. Afterwards, we discuss non-conservative mechanics in Sec. 11.5. Dynamical system now occupies a $2m+1$-dimensional non-symplectic manifold. The Hamilton equations are then reduced again

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to an exterior form equation by means of a 1-form involving a time-dependent Hamiltonian function and a 2-form that is its exterior derivative. It is shown that the canonical transformations leave this 2-form invariant. It is then proven that the Hamilton equations remain invariant under canonical transformations. The structural properties of canonical transformations are investigated. We finalise the study of analytical mechanics by exploring the Hamilton-Jacobi theory that help reduce the Hamilton equations to their simplest possible form. Our next topic is the electromagnetic theory studied in Sec. 11.6. The Maxwell equations are expressed as vanishing divergences of two second order antisymmetric tensors on a 4-dimensional manifold and it is found that these equations are equivalent to an exterior system involving two 2-forms. The general solution of these equations is constructed by employing the homotopy operator. When constitutive relations are taken into account, it is shown that this solution leads to the classical solution that are expressed in terms of scalar and vectorial potentials satisfying wave equations. In the final Sec. 11.7, the classical thermodynamics is briefly treated in a rather elementary level. A thermodynamic system whose state is determined by external and internal variables, and the empirical temperature is considered. An isothermal work function is defined assuming that external agents are conservative. By employing the first law of thermodynamics which states that the work done by external effects plus the heat energy input is equal to the rate of change of the internal energy and the physical fact that thermodynamic functions are additive, admissible versions of work and heat energy forms are obtained. Furthermore, the thermodynamic (absolute) temperature is introduced by an appropriate transformation, the existence of the entropy is proven under the conditions of complete integrability of the heat form. Then, the relations between the internal energy, free energy and heat forms are illustrated.

11.2. CONSERVATIVE MECHANICS

Let us consider a dynamical system consisting of several particles and rigid bodies moving in the space $\mathbb{R}^3$. In this space, the position of a particle is determined by at most 3 numbers corresponding to its coordinates implying that a particle has *three degrees of freedom*. On the other hand the position of a rigid body is prescribed by at most 6 numbers (for instance, 3 coordinates of one of its points, frequently of its centroid, and 3 Euler angles prescribing its orientation in the space). Therefore, a rigid body has *six degrees of freedom*. We can thus represent the position of a dynamical system as a point in some space $\mathbb{R}^N$ where $1 \leq N \leq \infty$ and the time evolution of such a system can be depicted by a curve in this space.

However, the system may possess ***constraints*** that restrict its motion so that it has a lower degrees of freedom. For instance, if we restrict the motion of a particle to a plane, then it has only two degrees of freedom. For another example, let us consider a rigid body. Although it has infinitely many particles, due to the fact that the distance between any two particles does not change during the motion, its degrees of freedom become just six and its motion is completely determined by specifying only six functions depending on time. Constraints that can be expressed by functional relations are called the ***holonomic constraints*** while they are known as the ***anholonomic constraints*** if they are prescribed by non-integrable differential forms. Moreover, if their structure is rigid, i.e., it does not change with time they are called the ***scleronomic constraints*** whereas if it varies with time they are named as the ***rheonomic constraints***. We first consider a system with scleronomic holonomic constraints. Let us assume that the system has now m degrees of freedom with constraints. The position of the system, thereby of every member of the system are completely determined by m variables $\mathbf{q} = \{q^1, q^2, \dots, q^m\}$ called the ***generalised coordinates*** through the relations $x^i = x^i(\mathbf{q}), i = 1, 2, \dots, m$. If we denote the time by t, the functions $\mathbf{q}(t) = \{q^i(t),\ i = 1, 2, \dots, m\}$ now describe fully the evolution of the dynamical system. This coordinate transformation produces a differentiable m-dimensional submanifold M of the simple manifold $\mathbb{R}^N$. We call this manifold, which might acquire quite a complicated structure due to this transformation, as the ***configuration manifold***. Hence, the motion of the system is represented by a curve on this manifold. If we can find this curve on M, then we can carry it over the physical space by using appropriate coordinate transformations. This task is, of course, conceptually quite simple, but it may prove to be rather difficult to realise it operationally.

It is evident that the generalised coordinates need not to be determined uniquely. A new set of generalised coordinates $\mathbf{Q} = \{Q^1, Q^2, \dots, Q^m\}$ for the configuration manifold may be defined by the help of functions

$$Q^i = \mathcal{Q}^i(q^1, q^2, \dots, q^m), \ \ i = 1, 2, \dots, m$$

However, in order that the degrees of freedom of the system are preserved, the new coordinates should be functionally independent. Therefore, we have to be sure that the condition

$$\det \boldsymbol{\mathfrak{Q}} = \det\Big[\frac{\partial \mathcal{Q}^i}{\partial q^j}\Big] \neq 0$$

must be satisfied. That we are somewhat free in choosing the generalised coordinates suggests the possibility of searching for a particular choice of them to simplify the investigation of the system to a great extent.

Example 11.2.1. Let us consider three particles moving in a plane with masses m_1, m_2, m_3. The masses m_1 and m_2 are connected by a rod of length l_1 whereas m_2 and m_3 are connected by a rod of length l_2. Both rods are assumed to be rigid and massless. Connections are provided by freely rotating joints. This system is, of course, taking place in the manifold $\mathbb{R}^6$ with the coordinate cover $(x_1, y_1, x_2, y_2, x_3, y_3)$. But, if we denote the angles between rods and the horizontal line by θ_1 and θ_2, we can write

$$\begin{aligned} x_2 &= x_1 + l_1 \cos\theta_1, \qquad y_2 = y_1 + l_1 \sin\theta_1, \\ x_3 &= x_1 + l_1 \cos\theta_1 + l_2 \cos\theta_2, y_3 = y_1 + l_1 \sin\theta_1 + l_2 \sin\theta_2. \end{aligned}$$

Since the motion of the system is now determined by generalised coordinates $(x_1, y_1, \theta_1, \theta_2)$, it has four degrees of freedom. So the system will evolve with time on a 4-dimensional configuration manifold $M^4 \subset \mathbb{R}^6$ with a coordinate cover $(x_1, y_1, \theta_1, \theta_2)$. This manifold may be defined by the following algebraic equations

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_1^2, \ (x_3 - x_2)^2 + (y_3 - y_2)^2 = l_2^2. \qquad \blacksquare$$

When the constraints are both scleronomic and holonomic, the kinetic energy of the system can be expressed as follows

$$T = \frac{1}{2} g_{ij}(q^1, q^2, \dots, q^m)\, \dot{q}^i \dot{q}^j = \frac{1}{2} g_{ij}(\mathbf{q})\, \dot{q}^i \dot{q}^j \geq 0. \tag{11.2.1}$$

An overdot denotes as usual the time derivative. $\mathbf{G}(\mathbf{q}) = [g_{ij}(\mathbf{q})]$ must be a symmetric and positive definite $m \times m$ matrix. The functions $\dot{q}^i(t)$ are called the ***generalised velocities***. We know that the generalised velocities at a point $\mathbf{q} \in M$ take place in the tangent space of the manifold M at that point. The ***velocity phase space*** is the tangent bundle $T(M)$ of the configuration manifold M. It is a $2m$-dimensional differentiable manifold whose coordinate cover is $(q^i, \dot{q}^i)$. If the system is ***conservative***, then there exists a scalar-valued ***potential function***

$$V = V(q^1, q^2, \dots, q^m) = V(\mathbf{q})$$

and the gradient $\{\partial V / \partial q^i\}$ of this function with respect to the generalised coordinates determines, somewhat indirectly, the actual forces acting on the physical system. The differentiable function $L : T(M) \to \mathbb{R}$ defined by the relation

$$L(\mathbf{q}, \dot{\mathbf{q}}) = T - V \tag{11.2.2}$$

is called the ***Lagrangian function*** of the system and the dynamical evolution of the system is governed by the following ***Lagrange equations***

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}^{\,i}}\Big) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, 2, \ldots, m. \tag{11.2.3}$$

These are a set of second order ordinary differential equations satisfied by functions $q^i(t)$. One must easily recognise that these equations are none other than Euler-Lagrange equations for functions $q^i(t)$ extremising the *action functional*

$$A(\mathbf{q}) = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}\,)\, dt$$

[*see* (10.3.5)].

Lagrange had obtained the equations (11.2.3) and similar equations corresponding to more general systems in 1760. However, the importance of these equations and, particularly of the approach leading to these equations has been fully understood only after he has published in 1788 *Mécanique Analytique*, which is a groundbreaking and probably one of the most influential books in the history of science. In this work, Lagrange has succeeded to convert the rational mechanics to a branch of mathematical analysis. In contrast to the geometrical approach prevalent at that time, his priding himself on not including even a single figure in his book[1] is a striking statement reflecting his new philosophy to which he had subscribed in treating the rational mechanics.

For scleronomic systems the kinetic energy given by (11.2.1) enables us to equip the configuration manifold M with a metric so that M becomes a complete Riemannian manifold. We define the metric tensor by using the coefficient functions $g_{ij}(\mathbf{q})$ in the expression for the kinetic energy just like in (5.9.1) as follows

$$\boldsymbol{\mathcal{G}} = g_{ij}(\mathbf{q})\, dq^i \otimes dq^j \in \mathfrak{T}(M)^0_2.$$

Therefore, we can introduce an inner product on $T(M)$ by the relation

$$(U, V) = \boldsymbol{\mathcal{G}}(U, V) = g_{ij} u^i v^j, \quad U, V \in T(M).$$

The arc element on the manifold M in the direction of the generalised velocity vector is then given by

$$ds^2 = g_{ij}\, dq^i dq^j = g_{ij}\, \dot{q}^{\,i}\, \dot{q}^{\,j}\, dt^2 = 2T\, dt^2$$

or $ds = \sqrt{2T}\, dt$. Therefore, in such kind of systems when we insert the

[1]"On ne trouvera point de Figures dans cet Ouvrage."

Lagrangian function

$$L = \frac{1}{2}\, g_{ij}(\mathbf{q})\, \dot{q}^{\,i}\, \dot{q}^{\,j} - V(\mathbf{q})$$

into the equations (11.2.3), we arrive at the following equations of motion on the manifold M

$$\big[g_{ij}(\mathbf{q})\, \dot{q}^{\,j}\big]^{\cdot} - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\, \dot{q}^{\,j}\, \dot{q}^{\,k} + \frac{\partial V}{\partial q^i} = 0.$$

On evaluating the time derivatives above and arranging the resulting terms, we obtain

$$g_{ij}\, \ddot{q}^j + \Big(\frac{\partial g_{ij}}{\partial q^k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\Big)\dot{q}^{\,j}\, \dot{q}^{\,k} + \frac{\partial V}{\partial q^i} = 0.$$

Nevertheless, if we notice that only the symmetric part with respect to indices j and k of the expression within parentheses in the above equations would survive, then it is straightforward to see that these set of equations can be cast into the form

$$g_{ij}\, \ddot{q}^j + \frac{1}{2}\Big(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\Big)\dot{q}^{\,j}\, \dot{q}^{\,k} + \frac{\partial V}{\partial q^i} = 0.$$

If we utilise the relation $g^{jk} g_{ik} = \delta^j_i$ and recall the definition (7.4.5) of the Christoffel symbols of the second kind, we end up with the following set of second order, generally non-linear ordinary differential equations by inverting the coefficient matrix $[g_{ij}]$

$$\ddot{q}^i + \Gamma^i_{jk}(\mathbf{q})\, \dot{q}^{\,j}\, \dot{q}^{\,k} = \, - g^{ij}\frac{\partial V}{\partial q^j}.$$

When we suppose that $V = 0$, these equations reveal the fact that *points representing dynamical systems that are free of forces must move on some geodesics in the configuration manifold* [*see* (7.2.16)].

The set of second order differential equations (11.2.3) can be transformed into an equivalent but larger set of first order ordinary differential equations by introducing certain auxiliary variables. To this end, we shall select the new variables p_i with $i = 1, 2, \ldots, m$ that will be called the ***generalised momenta*** as follows

$$p_i = \frac{\partial L}{\partial \dot{q}^{\,i}}. \tag{11.2.4}$$

When the condition

$$\det\Big[\frac{\partial p_i}{\partial \dot q^{\,j}}\Big] = \det\Big[\frac{\partial^2 L}{\partial \dot q^{\,i}\partial \dot q^{\,j}}\Big] \neq 0$$

is met, then by resorting to the inverse mapping, (11.2.4) yields in principle

$$\dot q^{\,i} = \mathfrak{q}^i(p_1, \dots, p_m, q^1, \dots, q^m). \tag{11.2.5}$$

So long as the quantities $\dot q^{\,i}$ are given by (11.2.5), the ***Hamiltonian function*** $H = H(\mathbf{p}, \mathbf{q})$ can now be defined by the *Legendre transformation*

$$H(\mathbf{p}, \mathbf{q}) = p_i \dot q^{\,i} - L(\mathbf{q}, \dot{\mathbf{q}}). \tag{11.2.6}$$

When we evaluate the differential of the function (11.2.6) and employ the equations (11.2.4) and (11.2.3), we conclude that

$$\begin{aligned} dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i &= \dot q^{\,i} dp_i + p_i \, d\dot q^{\,i} - \frac{\partial L}{\partial q^i}\, dq^i - \frac{\partial L}{\partial \dot q^{\,i}}\, d\dot q^{\,i} \\ &= \dot q^{\,i} dp_i - \dot p_i \, dq^i \end{aligned}$$

from which we derive the first order ***Hamilton equations***

$$\dot q^{\,i} = \frac{\partial H}{\partial p_i}, \quad \dot p_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, 2, \dots, m. \tag{11.2.7}$$

In order to fully understand the exact nature of generalised momenta p_i, we wish to examine their behaviour under a coordinate transformation $Q^i = \mathcal{Q}^i(q^j)$ in the configuration manifold. To this end, let us define an $m \times m$ matrix $\boldsymbol{\mathcal{Q}}$ by

$$\boldsymbol{\mathcal{Q}} = \big[\mathfrak{Q}^i_j(\mathbf{q})\big] = \Big[\frac{\partial Q^i}{\partial q^j}\Big].$$

so that the time derivative of the coordinate transformation is expressible as

$$\dot Q^i = \frac{\partial Q^i}{\partial q^j}\, \dot q^{\,j} = \mathfrak{Q}^i_j \, \dot q^{\,j} \ \text{ or } \ \dot{\mathbf{Q}} = \boldsymbol{\mathfrak{Q}}\dot{\mathbf{q}} \ \text{ or } \ \dot{\mathbf{q}} = \boldsymbol{\mathfrak{Q}}^{-1}\dot{\mathbf{Q}}.$$

Making use of these relations we obtain

$$P_i = \frac{\partial L}{\partial \dot Q^i} = \frac{\partial L}{\partial \dot q^{\,j}} \frac{\partial \dot q^{\,j}}{\partial \dot Q^i} = (\boldsymbol{\mathfrak{Q}}^{-1})^j_i \, p_j = \frac{\partial q^j}{\partial Q^i} p_j$$

which means that the elements $\{p_1, p_2, \dots, p_m\}$ behaves like components of a covariant vector that is a member of the cotangent bundle, in other words, they are the components of a 1-form. Therefore, the coordinate cover of the

$2m$-dimensional differentiable manifold $S = T^*(M)$ is given by $\{q^i, p_i\}$. We name this manifold as the ***momentum phase space***, or in short, merely the ***phase space***. Thus, the vector fields $\{\dot{q}^i, \dot{p}_i\}$ satisfying the Hamilton equations (11.2.6) inhabit the tangent bundle $T(T^*(M))$. On the other hand, the 1-form defined by

$$\theta = p_i\, dq^i = p_1\, dq^1 + \cdots + p_m\, dq^m \in \Lambda^1(S) \tag{11.2.8}$$

and usually known as the ***Liouville form*** is a member of the cotangent bundle $T^*(M)$. From the exterior derivative of the form (11.2.8) we can generate a closed 2-form

$$\begin{aligned} \omega &= -\,d\theta = -\,dp_i \wedge dq^i = dq^i \wedge dp_i \\ &= dq^1 \wedge dp_1 + \cdots + dq^m \wedge dp_m \in \Lambda^2(S). \end{aligned} \tag{11.2.9}$$

Let us denote the coordinate cover of the manifold S by $\{x^a, a = 1, 2, \ldots, 2m\}$. These coordinates will represent the coordinates q^i when we take $a = i$ with $1 \le i \le m$, and the coordinates p_i when we take $a = m + i$ if we do not mind a slight abuse of notation due to the unfamiliar positions of superscripts and subscripts. Hence, the form ω can now be written as follows

$$\omega = \frac{1}{2}\,\omega_{ab}(\mathbf{x})dx^a \wedge dx^b, \quad 1 \le a, b \le 2m.$$

In this case, the coefficients ω_{ab} of the form ω can now be expressed by the $2m \times 2m$ antisymmetric matrix

$$\mathbf{J} = [\,\omega_{ab}] = \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \\ -\mathbf{I}_m & \mathbf{0} \end{bmatrix}. \tag{11.2.10}$$

where $m \times m$ identity matrix is denoted by $\mathbf{I}_m$. Since $\det \mathbf{J} = 1$, then the rank of 2-form ω is maximal, namely, it is $2m$. We shall see a little later that ω is also non-degenerate. Hence, ω is a symplectic form [*see p.* 46]. We shall call this form whose structure has been manifested by (11.2.9) as the ***canonical symplectic form***. The generalised coordinates $\{q^i, p_i\}$ that enable us to write the symplectic form locally in this way are also called ***canonical coordinates***. We refer a manifold S endowed with a symplectic form as a ***symplectic manifold***. Inasmuch as the rank of ω is $2m$, the Darboux class of the form θ is m. Consequently, Theorem 6.6.2 states that we can always find canonical coordinates that make it possible to write the symplectic form locally in the canonical form (11.2.9).

The matrix $\boldsymbol{J}$ is called a ***symplectic matrix*** [*see* Exercise **3.4**]. We can immediately see that this matrix enjoys the following properties

$$\boldsymbol{J}^{\mathrm{T}} = \begin{bmatrix} \mathbf{0} & -\mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0} \end{bmatrix} = -\boldsymbol{J}$$

$$\boldsymbol{J}^2 = \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \\ -\mathbf{I}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I}_m \\ -\mathbf{I}_m & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -\mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_m \end{bmatrix} = -\mathbf{I}_{2m}$$

whence we deduce that

$$\boldsymbol{J}^{-1} = -\boldsymbol{J} = \boldsymbol{J}^{\mathrm{T}}.$$

We would like now to introduce a mapping $S_\omega : T(S) \to T^*(S)$ between the tangent and cotangent bundles of a symplectic manifold S which may be equivalently interpreted as a mapping $S_\omega : \mathfrak{V}(S) \to \Lambda^1(S)$ between the tangent module $\mathfrak{V}(S)$ and the module of 1-forms $\Lambda^1(S)$. For each vector field $V \in T(S)$, we define this mapping by employing the symplectic form in the following fashion

$$S_\omega V = \mathbf{i}_V(\omega) \in \Lambda^1(S). \tag{11.2.11}$$

Because of the properties (5.4.7), we immediately see that S_ω *is a linear operator on the module* $\mathfrak{V}(S)$. The value of this 1-form on a vector field $U \in T(S)$ is naturally given by

$$S_\omega V(U) = \mathbf{i}_U\big(\mathbf{i}_V(\omega)\big) = \omega(V, U) \in \Lambda^0(S).$$

If we write

$$V = v^a \frac{\partial}{\partial x^a} = v^i \frac{\partial}{\partial q^i} + \mathfrak{v}_i \frac{\partial}{\partial p_i} \in T(S),$$

then (11.2.11) yields

$$S_\omega V = \mathbf{i}_V(\omega) = \omega_{ab}\, v^a\, dx^b = v^i\, dp_i - \mathfrak{v}_i\, dq^i \in \Lambda^1(S).$$

Let us consider a form $\alpha \in \Lambda^1(S)$ by

$$\alpha = \xi_i dq^i + \eta^i dp_i.$$

It is straightforward to observe immediately that this 1-form is the image $S_\omega V_\alpha$ of the vector

$$V_\alpha = \eta^i \frac{\partial}{\partial q^i} - \xi_i \frac{\partial}{\partial p_i}, \quad \alpha = \mathbf{i}_{V_\alpha}(\omega). \tag{11.2.12}$$

Hence, the operator S_ω will be surjective. On the other hand, if we write

$S_\omega V = 0$ we end up with the expression $\omega_{ab}\, v^a = 0$. Since the matrix $\mathbf{J} = [\omega_{ab}]$ is regular, we find only the trivial solution $v^a = 0$ or $V = 0$, that is, the operator S_ω is injective, and consequently it is bijective. Therefore, the operator S_ω is one of the isomorphisms between tangent and cotangent spaces. Thus, the inverse mapping $S_\omega^{-1} : \Lambda^1(S) \to \mathfrak{V}(S)$ assigns to each 1-form field

$$\alpha = \xi_i dq^i + \eta^i dp_i \in \Lambda^1(S)$$

a unique vector field

$$S_\omega^{-1}\alpha = V_\alpha = \eta^i \frac{\partial}{\partial q^i} - \xi_i \frac{\partial}{\partial p_i} \in T(S).$$

It is evident that one can write $\alpha = S_\omega V_\alpha$. Since the relation $\mathbf{i}_V(\omega) = \omega(V) = 0$ is satisfied if and only if $V = 0$, we gather that the form ω is *non-degenerate.*

Let us next consider the smooth function $H \in \Lambda^0(S)$. A vector field V_H complying with the condition

$$S_\omega V_H = \mathbf{i}_{V_H}(\omega) = dH \in \Lambda^1(S) \tag{11.2.13}$$

is called a ***Hamiltonian vector field***. Since S_ω is an isomorphism, when a function H is chosen, the Hamiltonian vector field corresponding to this function is *uniquely* determined through the relation $V_H = S_\omega^{-1}(dH)$. If we explicitly write dH as

$$dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i,$$

then the corresponding Hamiltonian vector field is given by the relation

$$V_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{11.2.14}$$

It is now obvious that trajectories of such a vector field will have to satisfy the Hamilton equations

$$\dot{q}^{\,i} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_{\,i} = -\frac{\partial H}{\partial q^i}$$

cited in (11.2.7). Thus, it would be then quite reasonable to state that the equation (11.2.13) is the *symplectic form of the Hamilton equations*.

Example 11.2.1. Suppose that $H = qp^2 - qp + ap$ where a is a constant. In that case, the Hamilton equations become

$$\frac{dq}{dt} = 2qp - q + a,$$
$$\frac{dp}{dt} = -p^2 + p.$$

The integration of these differential equations yields easily

$$p(t) = \frac{1}{1 - e^{-(t-c_1)}},$$
$$q(t) = \big(1 - e^{-(t-c_1)}\big)\big[c_2\, e^t\big(1 - e^{-(t-c_1)}\big) - a\big].$$

c_1 and c_2 are integration constants to be determined through the initial conditions. If we evaluate the given Hamiltonian function on these trajectories, we find that

$$H = q(t)p(t)^2 - q(t)p(t) + ap(t) = a + e^{c_1}c_2 = \mathit{constant}.$$

Clearly, this constant will generally be different on each trajectory in the phase space. ■

11.3. POISSON BRACKET OF 1-FORMS AND SMOOTH FUNCTIONS

(S, ω) is a symplectic manifold. We consider the forms $\alpha, \beta \in \Lambda^1(S)$. The ***Poisson bracket*** *of* 1-*forms* α *and* β *is also a* 1-*form* $\{\alpha, \beta\} \in \Lambda^1(S)$ defined by the following relation

$$\begin{aligned}\{\alpha, \beta\} &= -S_\omega([V_\alpha, V_\beta]) = -\mathbf{i}_{[V_\alpha, V_\beta]}(\omega) \\ &= -S_\omega([S_\omega^{-1}\alpha, S_\omega^{-1}\beta])\end{aligned} \tag{11.3.1}$$

where the vector fields $V_\alpha = S_\omega^{-1}\alpha$ and $V_\beta = S_\omega^{-1}\beta$ are generated from the forms α and β, respectively, through the isomorphism S_ω. Consequently, on the module $\mathfrak{V}(S)$ the expression

$$S_\omega^{-1}(\{\alpha, \beta\}) = -[S_\omega^{-1}\alpha, S_\omega^{-1}\beta] \tag{11.3.2}$$

would be valid. On the other hand, if we recall that $d\omega = 0$ we can write

$$\begin{aligned}\mathbf{i}_{[V_\alpha, V_\beta]}(\omega) &= \mathbf{i}_{£_{V_\alpha}V_\beta}(\omega) = [£_{V_\alpha}, \mathbf{i}_{V_\beta}](\omega) = £_{V_\alpha}\mathbf{i}_{V_\beta}(\omega) - \mathbf{i}_{V_\beta}£_{V_\alpha}(\omega) \\ &= £_{V_\alpha}\beta - \mathbf{i}_{V_\beta}\big(\mathbf{i}_{V_\alpha}(d\omega) + d\mathbf{i}_{V_\alpha}(\omega)\big) \\ &= £_{V_\alpha}\beta - \mathbf{i}_{V_\beta}\big(d\alpha\big)\end{aligned}$$

owing to the equality (5.11.7). However, if we take into account the Cartan magic formula $\mathbf{i}_{V_\beta}(d\alpha) = £_{V_\beta}\alpha - d\mathbf{i}_{V_\beta}(\alpha)$ and anticommutativity (5.4.4) of the interior product, the Poisson bracket becomes expressible as

$$\{\alpha,\beta\} = -\mathbf{i}_{[V_\alpha,V_\beta]}(\omega) = £_{V_\beta}\alpha - £_{V_\alpha}\beta + d\big(\mathbf{i}_{V_\alpha}\mathbf{i}_{V_\beta}(\omega)\big). \quad (11.3.3)$$

It then follows from (11.3.3) that *the Poisson bracket of two closed* 1*-forms is an exact* 1*-form*. In fact, when $d\alpha = 0$ and $d\beta = 0$, then we have

$$£_{V_\alpha}\beta = d\big(\mathbf{i}_{V_\alpha}\mathbf{i}_{V_\beta}(\omega)\big), \;\; £_{V_\beta}\alpha = -d\big(\mathbf{i}_{V_\alpha}\mathbf{i}_{V_\beta}(\omega)\big)$$

and (11.3.3) leads to

$$\begin{aligned}\{\alpha,\beta\} &= -d\big(\mathbf{i}_{V_\alpha}\mathbf{i}_{V_\beta}(\omega)\big) = -d\big(\omega(V_\beta,V_\alpha)\big) \\ &= d\big(\omega(V_\alpha,V_\beta)\big).\end{aligned} \quad (11.3.4)$$

From the definition of the Poisson bracket and the linearity of the operator S_ω, we see that the following properties are valid:

(i). $\{\alpha,\beta\} = -\{\beta,\alpha\}$ (*Antisymmetry*),
(ii). $\{\alpha, b\beta + c\gamma\} = b\{\alpha,\beta\} + c\{\alpha,\gamma\}, \quad b,c \in \mathbb{R}$ (*Linearity*).

Furthermore, the Poisson bracket satisfies the *Jacobi identity*

(iii). $\{\alpha,\{\beta,\gamma\}\} + \{\beta,\{\gamma,\alpha\}\} + \{\gamma,\{\alpha,\beta\}\} = 0.$

In order to see this, it only suffices to notice that one can write

$$\{\alpha,\{\beta,\gamma\}\} = -S_\omega([V_\alpha,[V_\beta,V_\gamma]])$$

and the operator S_ω is linear.
(iv). Let us consider a function $f \in \Lambda^0(S)$. The linearity of the operator S_ω and the relation (2.10.19) result in

$$\begin{aligned}\{\alpha, f\beta\} &= -S_\omega([V_\alpha, fV_\beta]) \\ &= -S_\omega\big(f[V_\alpha,V_\beta] + V_\alpha(f)V_\beta\big) \\ &= -fS_\omega([V_\alpha,V_\beta]) - V_\alpha(f)S_\omega V_\beta \\ &= f\{\alpha,\beta\} + V_\alpha(f)\beta.\end{aligned}$$

The properties (i), (ii) and (iii) demonstrate that the module $\Lambda^1(S)$ of 1-forms constitutes a Lie algebra with respect to the Poisson bracket if we rightly interpret the Poisson bracket as the Lie product of 1-forms. Since the Poisson bracket of two closed form is an exact, consequently, a closed form, we realise at once that closed 1-forms is a subalgebra of such a Lie algebra of 1-forms. It then obviously follows from the relation (11.3.1) that $\{\alpha,\beta\} = 0$ whenever $[V_\alpha, V_\beta] = 0$.

Let us next take into account 1-forms $df, dg \in \Lambda^1(S)$ that are exterior derivatives of functions $f, g \in \Lambda^0(S)$. We know that the isomorphism S_ω generates the vectors

$$\begin{aligned} V_f &= S_\omega^{-1} df = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}, \\ V_g &= S_\omega^{-1} dg = \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i}. \end{aligned}$$

In view of the relation (11.2.13), these vectors are Hamiltonian vector fields associated with functions f and g. Since df and dg are closed forms, then (11.3.4) yields

$$\{df, dg\} = -d\big(\mathbf{i}_{V_f}\mathbf{i}_{V_g}(\omega)\big) = d\{f, g\} \tag{11.3.5}$$

where the ***Poisson bracket*** *of the functions* f *and* g are defined by the following relation

$$\{f, g\} = -\mathbf{i}_{V_f}\mathbf{i}_{V_g}(\omega) = \mathbf{i}_{V_g}\mathbf{i}_{V_f}(\omega) \in \Lambda^0(S). \tag{11.3.6}$$

On the other hand, we can easily evaluate that

$$\begin{aligned} &\mathbf{i}_{V_g}(\omega) = \frac{\partial g}{\partial p_i} dp_i + \frac{\partial g}{\partial q^i} dq^i = dg, \\ &\mathbf{i}_{V_f}(dg) = V_f(g). \end{aligned}$$

Therefore, we conclude that the Poisson bracket of two functions f and g is determined by the expression

$$\{f, g\} = -V_f(g) = V_g(f) = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{11.3.7}$$

We can thereby deduce from the relations (11.3.2) and (11.3.5) that

$$S_\omega^{-1}(\{df, dg\}) = S_\omega^{-1} d\{f, g\} = -[S_\omega^{-1} df, S_\omega^{-1} dg].$$

This simply implies that Hamiltonian vector fields generated by functions f, g and $\{f, g\}$ are connected by the relation

$$V_{\{f,g\}} = -[V_f, V_g]. \tag{11.3.8}$$

The equation (11.3.8) amounts to say that if V_f and V_g are Hamiltonian vector fields, then their Lie product $[V_f, V_g]$ is also a Hamiltonian vector field. This, of course, means that Hamiltonian vector fields constitute a Lie subalgebra. We then observe from the expression (11.3.6) that the equality

$$\{f, g\} = -\{g, f\}$$

holds. On the other hand, for three functions $f, g, h \in \Lambda^0(S)$ (11.3.7) leads to the relations

$$\begin{aligned}&\{f,\{g,h\}\} = V_f V_g(h),\ \{g,\{h,f\}\} = V_g V_h(f) = -V_g V_f(h)\\ &\{h,\{f,g\}\} = -\{\{f,g\},h\} = V_{\{f,g\}}(h) = -[V_f, V_g](h)\end{aligned}$$

from which we deduce the identity

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = ([V_f, V_g] - [V_f, V_g])(h) = 0.$$

Hence, Poisson brackets on smooth functions verify the Jacobi identity as well. Accordingly, the module $\Lambda^0(S)$ equipped with the Poisson bracket is a Lie algebra. One readily sees that the Poisson bracket $\{f, g\}$ is a bilinear function on real numbers. Moreover, we find that

$$\begin{aligned}\{f, hg\} &= -V_f(hg) = -V_f(h)g - V_f(g)h\\ &= g\{f,h\} + h\{f,g\}.\end{aligned}$$

Let us now consider canonical local coordinates. Then (11.3.7) leads to the relations

$$\{q^k, q^l\} = 0,\ \{p_k, p_l\} = 0,\ \{q^k, p_l\} = -\{p_l, q^k\} = \delta^k_l$$

and for a function $f \in \Lambda^0(S)$ we obtain

$$\{f, q^i\} = -\{q^i, f\} = -\frac{\partial f}{\partial p_i},\ \{f, p_i\} = -\{p_i, f\} = \frac{\partial f}{\partial q^i}.$$

Hence, the Hamilton equations can now be written in the form

$$\dot{q}^i = \{q^i, H\}, \quad \dot{p}_i = \{p_i, H\}. \tag{11.3.9}$$

We shall next try to evaluate the change in a function $f \in \Lambda^0(S)$ on the flow $e^{tV_g}: S \to S$ generated by a Hamiltonian vector field V_g on the manifold S. We know that we can write $f(t) = e^{t£_{V_g}} f$ and the necessary and sufficient condition for the function f to remain invariant under this flow is $£_{V g}f = 0$. However, this condition means that

$$£_{V g}f = V_g(f) = \{f, g\} = 0. \tag{11.3.10}$$

Accordingly, if the Poisson bracket, or equivalently the Lie product, of functions f and g vanishes, then the function f has to remain 'constant' on the flow on S generated by the Hamiltonian vector field V_g, i.e., $f(\mathbf{q}, \mathbf{p}) = f(\mathbf{q}_0, \mathbf{p}_0)$ where $(\mathbf{q}, \mathbf{p}) = e^{tV_g}(\mathbf{q}_0, \mathbf{p}_0)$ although this constant may take different values on each trajectory. The relation (11.3.10) requires, of course, that the function g will, in turn, remain constant on the flow produced by the Hamiltonian vector field V_f . We already know that a trajectory of the Hamiltonian vector field V_H associated with the Hamiltonian function H determines the evolution of a dynamical system with particular initial conditions on the symplectic manifold S. The time rate of change of a function $f \in \Lambda^0(S)$ during the evolution of the dynamical system can now be calculated by

$$\frac{df}{dt} = \frac{\partial f}{\partial q^i}\dot{q}^{\,i} + \frac{\partial f}{\partial p_i}\dot{p}_i = \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i} = \{f, H\}.$$

Thus, in case a function $f \in \Lambda^0(S)$ verifies the condition

$$\{f, H\} = 0,$$

it remains constant in association with the evolution of the dynamical system. A relation between generalised coordinates and generalised momenta in the form

$$f(\mathbf{q}(t), \mathbf{p}(t)) = c = constant$$

corresponds to an ***integral of the motion*** and help us to reduce the number of the dependent variables $\mathbf{p}(t)$. Due to the property of the Poisson bracket, we clearly obtain $\{H, H\} = 0$. Hence, the Hamiltonian function H is an integral of the motion. The relation

$$H(\mathbf{p}, \mathbf{q}) = p_i\dot{q}^{\,i} - L(\mathbf{q}, \dot{\mathbf{q}}) = constant$$

is known as the ***conservation of energy***. As a matter of fact, when the kinetic energy is prescribed by (11.2.1) we find at once that

$$p_i = \frac{\partial L}{\partial \dot{q}^{\,i}} = g_{ij}\,\dot{q}^{\,j} \text{ and } p_i\dot{q}^{\,i} = g_{ij}\,\dot{q}^{\,i}\dot{q}^{\,j} = 2T.$$

Thus, the Hamiltonian function

$$H = 2T - T + V = T + V$$

represents now the total energy of the dynamical system that is conserved during the motion of the system.

Finally, we attempt to calculate the Lie derivative of the symplectic form ω with respect to a Hamiltonian vector field V_f. Since $d\omega = 0$, we easily obtain

$$\pounds_{V_f}\,\omega = d\,\mathbf{i}_{V_f}(\omega) = d(df) = d^2 f = 0. \tag{11.3.11}$$

Consequently, the symplectic form ω remains invariant under the flow produced by a Hamiltonian vector field. In other words, under the mapping $\phi_t = e^{tV_f} : S \to S$, we get

$$\omega^* = \phi_t^* \omega = e^{t\pounds_{V_f}} \omega = \omega.$$

The volume form of the symplectic manifold S is of course

$$\mu = dq^1 \wedge dq^2 \wedge \cdots \wedge dq^m \wedge dp_1 \wedge dp_2 \wedge \cdots \wedge dp_m \in \Lambda^{2m}(S).$$

It is quite easy now to prove the following theorem .

Theorem 11.3.1 (The Liouville Theorem). *Let (S, ω) be a $2m$-dimensional symplectic manifold and ϕ_t be the flow of a Hamiltonian vector field. The mapping ϕ_t^* preserves the volume form μ of the symplectic manifold for all t, namely, the invariance condition $\phi_t^* \mu = \mu$ is satisfied.*

Indeed, the volume form $\mu \in \Lambda^{2m}(S)$ is expressible as

$$\mu = \frac{(-1)^{\frac{m(m-1)}{2}}}{m!} \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{m} = C\omega^m.$$

Nevertheless, on account of the relations (5.7.4) and $\phi_t^* \omega = \omega$ we obtain

$$\phi_t^* \mu = C \phi_t^* \omega \wedge \phi_t^* \omega \wedge \cdots \wedge \phi_t^* \omega = C\omega \wedge \omega \wedge \cdots \wedge \omega = \mu. \qquad \square$$

According to this theorem *the volume of the phase space is conserved under a flow generated by trajectories of a Hamiltonian vector field*. This statement is of course true for every volume elements in the phase space.

Next, let us consider a form $\Omega_k = \omega^k = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{k} \in \Lambda^{2k}(S)$, $1 \leq k \leq m$. For an arbitrary Hamiltonian vector field V_f, we obviously find that $\pounds_{V_f} \Omega_k = 0$. Hence, all the forms $\Omega_k, 1 \leq k \leq m$ remain invariant under flows generated by Hamiltonian vector fields.

11.4. CANONICAL TRANSFORMATIONS

(S, ω) is a symplectic manifold with canonical coordinates $\{q^i, p_i\}$. If a mapping $\phi : S \to S$ transforming this manifold into itself leaves the symplectic form invariant, that is, if it satisfies the condition

$$\phi^* \omega = \omega, \tag{11.4.1}$$

then it is called a ***canonical*** or ***symplectic transformation***. Accordingly, the flow of every Hamiltonian vector field produces a canonical transformation. Because of (11.4.1), we have $\phi^* \mu = \mu$ so that one obtains $\det \phi = 1$ [*see*

(5.7.11)]. Therefore, a canonical transformation will preserve a volume form and its orientation. Furthermore, it must be locally a diffeomorphism. A canonical transformation ϕ will best be expressed as a local coordinate transformations $(Q^i, P_i) \to (q^i, p_i)$ with $i = 1, \dots, m$ in the phase space specified by the functions

$$q^i = q^i(Q^1, \dots, Q^m, P_1, \dots, P_m),\ \ p_i = p_i(Q^1, \dots, Q^m, P_1, \dots, P_m).$$

Insofar as we have assumed that ϕ is a diffeomorphism, the inverse transformations

$$Q^i = Q^i(q^1, \dots, q^m, p_1, \dots, p_m),\ \ P_i = P_i(q^1, \dots, q^m, p_1, \dots, p_m)$$

will exist, at least, locally. Our expectation from such a canonical transformation would be to make the equations of motion acquire a simpler structure. The relation (11.4.1) now takes the form

$$\begin{aligned}\phi^*(dq^i \wedge dp_i) &= d(\phi^* q^i) \wedge d(\phi^* p_i) \\ &= dQ^i \wedge dP_i = dq^i \wedge dp_i.\end{aligned} \tag{11.4.2}$$

Thus, in order that ϕ turns out to be a canonical transformation, the functions Q^i and P_i must satisfy the relations

$$\begin{aligned}&\frac{\partial Q^i}{\partial q^j}\frac{\partial P_i}{\partial q^k}\,dq^j \wedge dq^k + \frac{\partial Q^i}{\partial p_j}\frac{\partial P_i}{\partial p_k}\,dp_j \wedge dp_k \\ &\qquad + \Big(\frac{\partial Q^i}{\partial q^j}\frac{\partial P_i}{\partial p_k} - \frac{\partial Q^i}{\partial p_k}\frac{\partial P_i}{\partial q^j}\Big)\,dq^j \wedge dp_k = dq^i \wedge dp_i.\end{aligned}$$

If we take into account the antisymmetry of the exterior product, the above relation leads to the following equations

$$\frac{\partial Q^i}{\partial q^j}\frac{\partial P_i}{\partial q^k} - \frac{\partial Q^i}{\partial q^k}\frac{\partial P_i}{\partial q^j} = 0, \qquad \frac{\partial Q^i}{\partial p_j}\frac{\partial P_i}{\partial p_k} - \frac{\partial Q^i}{\partial p_k}\frac{\partial P_i}{\partial p_j} = 0$$
$$\frac{\partial Q^i}{\partial q^j}\frac{\partial P_i}{\partial p_k} - \frac{\partial Q^i}{\partial p_k}\frac{\partial P_i}{\partial q^j} = \delta^k_j.$$

To treat this matter in a more general context, let us consider two $2m$-dimensional symplectic manifolds (S_1, ω_1) and (S_2, ω_2). A local diffeomorphism $\phi : S_1 \to S_2$ satisfying the relation

$$\phi^* \omega_2 = \omega_1 \tag{11.4.3}$$

is called a ***canonical*** or ***symplectic mapping***. If ϕ is a symplectic mapping, it will preserve the volume form so we must have the condition $\det \phi = 1$.

Theorem 11.4.1 (The Jacobi Theorem). *Let (S_1, ω_1) and (S_2, ω_2) be $2m$-dimensional symplectic manifolds. A diffeomorphism $\phi : S_1 \to S_2$ is a symplectic mapping if and only if Hamiltonian vector fields $V_f \in T(S_2)$ and $V_{\phi^* f} \in T(S_1)$ are to satisfy the relation*

$$(d\phi)^{-1} V_f = \phi_*^{-1} V_f = V_{\phi^* f} \quad \textit{or} \quad V_f = \phi_* V_{\phi^* f}. \tag{11.4.4}$$

for all $f \in \Lambda^0(S_2)$.

Let us first demonstrate that the relation

$$S_{\omega_1}^{-1}(\phi^*\alpha) = \phi_*^{-1} S_{\omega_2}^{-1}(\alpha) \tag{11.4.5}$$

is satisfied for all $\alpha \in \Lambda^1(S_2)$ if and only if ϕ is a symplectic mapping. Let $V = S_{\omega_2}^{-1}(\alpha) \in T(S_2)$. Utilising the relation (5.7.7), we find that

$$\phi^*\alpha = \phi^*(\mathbf{i}_V \omega_2) = \mathbf{i}_{\phi_*^{-1} V}\, \phi^*\omega_2.$$

By applying the operator $S_{\omega_1}^{-1}$ to this expression, we obtain

$$S_{\omega_1}^{-1}(\phi^*\alpha) = S_{\omega_1}^{-1} S_{\phi^*\omega_2}(\phi_*^{-1} V) = S_{\omega_1}^{-1} S_{\phi^*\omega_2}\big(\phi_*^{-1} S_{\omega_2}^{-1}(\alpha)\big).$$

Since the condition $\phi^*\omega_2 = \omega_1$ must be obeyed when ϕ is a symplectic mapping, we simply find the identity mapping $S_{\omega_1}^{-1} S_{\phi^*\omega_2} = S_{\omega_1}^{-1} S_{\omega_1} = i_{\mathfrak{V}(S_1)}$ and the relation (11.4.5) follows immediately. Conversely, if we suppose that the relation (11.4.5) is satisfied for all forms $\alpha \in \Lambda^1(S_2)$, we readily observe that the equality $S_{\omega_1}^{-1}(\phi^*\alpha) = S_{\omega_1}^{-1} S_{\phi^*\omega_2}\big(S_{\omega_1}^{-1}(\phi^*\alpha)\big)$ must result in $S_{\omega_1}^{-1} S_{\phi^*\omega_2} = I_{\mathfrak{V}(S_1)}$. This is, of course, realisable if only $\phi^*\omega_2 = \omega_1$, i.e., if ϕ is a symplectic mapping.

Let us now assume that ϕ is a symplectic mapping. In this case, the relation (11.4.5) leads to the result

$$\phi_*^{-1} V_f = \phi_*^{-1} S_{\omega_2}^{-1}(df) = S_{\omega_1}^{-1}(\phi^* df) = S_{\omega_1}^{-1}\big(d(\phi^* f)\big) = V_{\phi^* f}$$

in view of Theorem 5.8.2. Conversely, let us now assume that the equality (11.4.4) is satisfied for all $f \in \Lambda^0(S_2)$. We then successively obtain

$$d(\phi^* f) = \phi^* df = \phi^* S_{\omega_2} V_f = \phi^*(\mathbf{i}_{V_f}\omega_2) = \mathbf{i}_{\phi_*^{-1} V_f}\, \phi^*\omega_2 = \mathbf{i}_{V_{\phi^* f}}\, \phi^*\omega_2.$$

On the other hand, the same expression can also be written in the form

$$d(\phi^* f) = S_{\omega_1} V_{\phi^* f} = \mathbf{i}_{V_{\phi^* f}}\, \omega_1.$$

This implies that we obtain the relation

$$\mathbf{i}_{V_{\phi^* f}}\, \phi^*\omega_2 = \mathbf{i}_{V_{\phi^* f}}\, \omega_1$$

for each function $f \in \Lambda^0(S_2)$ and Hamiltonian vector field $V_{\phi^* f} \in T(S_1)$ from which it follows that

$$\mathbf{i}_{V_{\phi^* f}}(\phi^*\omega_2 - \omega_1) = 0.$$

Since the rank of symplectic forms must be maximal, the above equation is satisfied if and only if $\phi^*\omega_2 = \omega_1$, that is, if the diffeomorphism ϕ is a symplectic mapping.

If a diffeomorphism $\phi : S \to S$ is mapping a symplectic manifold S onto itself, then the above conditions are reduced to the ones such that the condition

$$\phi_*^{-1} V_f = V_{\phi^* f}$$

and consequently,

$$S_\omega^{-1}(\phi^*\alpha) = \phi_*^{-1} S_\omega^{-1}(\alpha)$$

must hold for all functions $f \in \Lambda^0(S)$ and forms $\alpha \in \Lambda^1(S)$. □

We can easily prove the existence of an important property related to symplectic diffeomorphisms and Poisson brackets.

Theorem 11.4.2. *Let (S_1, ω_1) and (S_2, ω_2) be $2m$-dimensional symplectic manifolds. A diffeomorphism $\phi : S_1 \to S_2$ is symplectic if and only if it preserves Poisson brackets, that is, if and only if the relation*

$$\phi^*\{f, g\} = \{\phi^* f, \phi^* g\} \tag{11.4.6}$$

is satisfied for all $f, g \in \Lambda^0(S_2)$.

In view of (11.3.7) and (11.3.10), we can write the Poisson bracket $\{f, g\} \in \Lambda^0(S_2)$ in the form

$$\{f, g\} = V_g(f) = £_{V g} f.$$

On account of (11.4.4), we have

$$V_g = \phi_* V_{\phi^* g}$$

if and only if ϕ is a symplectic mapping. Hence, making use of (5.11.17) we obtain

$$\phi^*\{f, g\} = \phi^* £_{\phi_* V_{\phi^* g}} f = £_{V_{\phi^* g}}(\phi^* f) = \{\phi^* f, \phi^* g\}.$$

This is tantamount to say that a symplectic mapping provides a homomorphism on $\Lambda^0(S_2)$ with respect to the Lie product defined by the Poisson bracket. □

It is quite straightforward to show that Theorem 11.4.2 will also be in *effect for Poisson brackets of* 1*-forms*. If we employ the relation (2.10.21), we obtain

$$\begin{aligned}\phi^*\{\alpha,\beta\} &= -\,\phi^*\mathbf{i}_{[V_\alpha,V_\beta]}(\omega_2) = -\,\phi^*\mathbf{i}_{\phi_*\phi_*^{-1}[V_\alpha,V_\beta]}(\omega_2)\\ &= -\,\phi^*\mathbf{i}_{\phi_*[\phi_*^{-1}V_\alpha,\,\phi_*^{-1}V_\beta]}(\omega_2) = -\,\mathbf{i}_{[\phi_*^{-1}V_\alpha,\,\phi_*^{-1}V_\beta]}(\phi^*\omega_2).\end{aligned}$$

On the other hand, the definition $\alpha = \mathbf{i}_{V_\alpha}(\omega_2)$ yields

$$\phi^*\alpha = \phi^*\mathbf{i}_{V_\alpha}(\omega_2) = \mathbf{i}_{\phi_*^{-1}V_\alpha}(\phi^*\omega_2).$$

If ϕ is a symplectic mapping, one must have $\phi^*\omega_2 = \omega_1$ and the relation $\phi^*\alpha = \mathbf{i}_{\phi_*^{-1}V_\alpha}(\omega_1)$ will follow. So we easily obtain

$$\phi^*\{\alpha,\beta\} = -\,\mathbf{i}_{[\phi_*^{-1}V_\alpha,\,\phi_*^{-1}V_\beta]}(\omega_1) = \{\phi^*\alpha,\phi^*\beta\}. \tag{11.4.7}$$

Conversely, if (11.4.7) is to be satisfied for all forms $\alpha,\beta\in\Lambda^1(S_2)$, then the relation

$$-\,\mathbf{i}_{[\phi_*^{-1}V_\alpha,\,\phi_*^{-1}V_\beta]}(\phi^*\omega_2) = -\,\mathbf{i}_{[\phi_*^{-1}V_\alpha,\,\phi_*^{-1}V_\beta]}(\omega_1)$$

requires that $\phi^*\omega_2 = \omega_1$. □

Let us now consider a canonical mapping $\phi : S \to S$ on a symplectic manifold (S,ω). This mapping is of course represented by transformations between local canonical coordinates. We know that in this situation both sets of canonical coordinates must satisfy the relation (11.4.2). In order to systematically investigate the implication of (11.4.2), let us first define 1-forms θ and Θ as follows

$$\theta = p_i\,dq^i, \quad \Theta = P_i\,dQ^i.$$

The relation (11.4.2) compels us to write

$$\phi^*d\theta = d\Theta \quad \text{or} \quad d(\Theta - \phi^*\theta) = 0.$$

Thus, according to the Poincaré lemma we obtain at least locally

$$\Theta - \phi^*\theta = dF, \quad F\in\Lambda^0(S).$$

where F is an arbitrary function. Hence, whenever $\phi : S\to S$ satisfies locally the expression

$$P_i\,dQ^i - p_i\,dq^i = dF, \tag{11.4.8}$$

then it becomes a canonical mapping. However, we have also to keep in mind that the function F should be so chosen that the mapping ϕ must be a diffeomorphism. For instance, if we choose a smooth function $F = F(\mathbf{p},\mathbf{q})$

in (11.4.8), the canonical mapping is determined by the equations

$$P_i \frac{\partial Q^i}{\partial q^j} = p_j + \frac{\partial F}{\partial q^j}, \quad P_i \frac{\partial Q^i}{\partial p_j} = \frac{\partial F}{\partial p_j}$$

Example 11.4.1. For $m = 1$, we define a mapping $(p, q) \to (P, Q)$ by

$$P = \frac{1}{2}(p^2 + q^2), \quad Q = \text{arctan}\, \frac{q}{p}.$$

We can then write

$$\begin{aligned} P\,dQ - p\,dq &= \frac{1}{2}(p^2 + q^2)\frac{p\,dq - q\,dp}{p^2\left(1 + \dfrac{q^2}{p^2}\right)} - p\,dq \\ &= -\frac{1}{2}(p\,dq + q\,dp) = -\frac{1}{2}\,d(pq) \end{aligned}$$

and find that

$$P\,dQ - p\,dq = dF, \quad F = -\frac{1}{2}\,pq.$$

Hence, this diffeomorphism is a canonical mapping. ■

Theorem 11.4.3. *Let (S, ω) be a $2m$-dimensional symplectic manifold. A canonical mapping $\phi : S \to S$ preserves the form of the Hamilton equations governing the motion of a dynamical system on this manifold.*

Let the canonical mapping $\phi : S \to S$ be prescribed by the coordinate transformation $(Q^i, P_i) \to (q^i, p_i)$. In the local coordinates (q^i, p_i) of the manifold S we shall assume that the Hamilton equations are specified in the symplectic form by

$$\mathbf{i}_{V_H}(\omega) = dH$$

where $H(\mathbf{p}, \mathbf{q})$ is the Hamiltonian function. On applying the pull-back operation on this equation, we obtain

$$\phi^*\mathbf{i}_{V_H}(\omega) = \mathbf{i}_{\phi_*^{-1}V_H}(\phi^*\omega) = \phi^* dH = d(\phi^* H)$$

Since ϕ is a canonical mapping, we can, of course, write $\phi^*\omega = \omega$ so we finally find that

$$\mathbf{i}_{V_{\phi^* H}}(\omega) = d(\phi^* H).$$

Therefore, by defining the function $K(\mathbf{P}, \mathbf{Q}) = \phi^* H = H \circ \phi$, we end up in the following expression

$$\mathbf{i}_{V_K}(\omega) = dK, \;\; K(\mathbf{P},\mathbf{Q}) = H\big(\mathbf{p}(\mathbf{P},\mathbf{Q})\mathbf{q}(\mathbf{P},\mathbf{Q})\big).$$

Thus, the vector field V_K associated with the function K is a Hamiltonian vector field and its trajectories satisfy the Hamilton equations

$$\dot{Q}^i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q^i}$$

corresponding to the Hamiltonian function K. □

This result brings to mind to search for an appropriate canonical transformation that simplifies the structure of the function K to a great extent so much so that the Hamilton equations take a much simpler form in the new canonical coordinates. Achievement of such a strategy entails, of course, much facile integration of differential equations. After having obtained the solution corresponding to the simplified system, we need to perform only some algebraic operations concerning canonical coordinates in order to obtain the actual solution associated with the physical system. We shall discuss this approach later in detail in Sec. 11.5 in a more general context.

11.5. NON-CONSERVATIVE MECHANICS

Let us consider a dynamical system of m degrees of freedom. We assume that the constraints between members of the system may be rheonomic, namely, they may be time-dependent. Or some parameters describing the system may be time-dependent. We further suppose that the potential function associated with the system may also be depending on time. In this situation, the kinetic energy of the system and its potential function now take in general the following forms

$$T = \frac{1}{2}g_{ij}(\mathbf{q},t)\,\dot{q}^i\,\dot{q}^j + g_i(\mathbf{q},t)\,\dot{q}^i + g(\mathbf{q},t), \;\; V = V(\mathbf{q},t).$$

Thus, the Lagrangian function becomes explicitly dependent on time:

$$L(\mathbf{q},\dot{\mathbf{q}},\mathrm{t}) = T(\mathbf{q},\dot{\mathbf{q}},\mathrm{t}) - V(\mathbf{q},t). \tag{11.5.1}$$

As a result of this both the Hamiltonian function and thus the generalised momenta become dependent explicitly on time:

$$p_i = \frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}^i}, \tag{11.5.2}$$
$$H(\mathbf{p},\mathbf{q},t) = p_i\dot{q}^i - L(\mathbf{q},\dot{\mathbf{q}},t).$$

Thus, the Hamiltonian function is now a mapping like $H : S \times \mathbb{R} \to \mathbb{R}$

where $(S, \omega = dq^i \wedge dp_i)$ is again a symplectic manifold. The generalised coordinates **q** and the generalised momenta **p** still satisfy the Hamilton equations (11.2.7). We next like to introduce a $(2m+1)$-dimensional manifold $\mathfrak{S} = S \times \mathbb{R}$ whose coordinate cover is evidently given by $\{\mathbf{p}, \mathbf{q}, t\}$. We then define the following form $\theta_H \in \Lambda^1(\mathfrak{S})$:

$$\theta_H = p_i dq^i - H dt = \theta - H dt. \tag{11.5.3}$$

We further introduce the form $\omega_H \in \Lambda^2(\mathfrak{S})$ by

$$\begin{aligned}\omega_H = &- d\theta_H = dq^i \wedge dp_i + dH \wedge dt \\ &= \omega + dH \wedge dt.\end{aligned} \tag{11.5.4}$$

Insofar as $d\omega_H = -d^2\theta_H = 0$, the 2-form ω_H, too, is closed. Due to the fact that the dimension of the manifold $\mathfrak{S}$ is $2m+1$, the rank of the form ω_H would be at most $2m$. On the other hand, if one takes $t =$ *constant*, one finds $\omega_H = \omega$ so that the rank of ω_H cannot be less than $2m$. Consequently, the rank of the form ω_H is $2m$, that is, it is maximal. But $\mathfrak{S}$ is no longer a symplectic manifold because its dimension is an odd number. Nevertheless, although $(\mathfrak{S}, \omega_H)$ is not a symplectic manifold, it is straightforward to see that its restriction on a submanifold $t =$ *constant* is a symplectic manifold that is diffeomorphic to the manifold (S, ω).

We now define a vector field $\mathcal{V}_H \in T(\mathfrak{S})$ depending on a Hamiltonian function H in the following way

$$\mathcal{V}_H = \frac{\partial}{\partial t} + V_H \tag{11.5.5}$$

where the *Hamiltonian vector field* generated by the Hamiltonian function $H(\mathbf{p}, \mathbf{q}, t)$ is again given by

$$V_H(\mathbf{p}, \mathbf{q}, t) = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}.$$

Since $V_H(H) = 0$, it follows from (11.5.5) that

$$\mathcal{V}_H(H) = \frac{\partial H}{\partial t}. \tag{11.5.6}$$

Because we obviously have $\mathbf{i}_{\mathcal{V}_H}(\omega) = \mathbf{i}_{V_H}(\omega)$, we conclude that

$$\begin{aligned}\mathbf{i}_{\mathcal{V}_H}(\omega_H) &= \mathbf{i}_{\mathcal{V}_H}(\omega) + \mathbf{i}_{\mathcal{V}_H}(dH \wedge dt) = \mathbf{i}_{V_H}(\omega) + \mathcal{V}_H(H)dt - dH \\ &= \frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt - dH = 0.\end{aligned}$$

As a matter of fact, we can readily show that *each vector field* $\mathcal{V} \in T(\mathfrak{S})$ *satisfying the relation* $\mathbf{i}_{\mathcal{V}}(\omega_H) = 0$ *can only be a multiple of the vector field* $\mathcal{V}_H$ *by a scalar function*. Indeed, let us consider an arbitrary vector

$$\mathcal{V} = \xi^i \frac{\partial}{\partial q^i} + \eta_i \frac{\partial}{\partial p_i} + \tau \frac{\partial}{\partial t}$$

where $\xi^i, \eta_i, \tau \in \Lambda^0(\mathfrak{S})$. The condition

$$\begin{aligned}\mathbf{i}_{\mathcal{V}}(\omega_H) &= \xi^i\, dp_i - \eta_i\, dq^i + V(H)\, dt - \tau\, dH \\ &= \Big(\xi^i - \tau\frac{\partial H}{\partial p_i}\Big) dp_i - \Big(\eta_i + \tau\frac{\partial H}{\partial q^i}\Big) dq^i + \Big(\xi^i \frac{\partial H}{\partial q^i} + \eta_i \frac{\partial H}{\partial p_i}\Big) dt = 0\end{aligned}$$

now requires that the components of the vector $\mathcal{V}$ must satisfy

$$\xi^i = \tau\frac{\partial H}{\partial p_i}, \quad \eta_i = -\tau\frac{\partial H}{\partial q^i}, \quad \xi^i \frac{\partial H}{\partial q^i} + \eta_i \frac{\partial H}{\partial p_i} = 0.$$

It is obvious that the last expression vanishes identically. Thus, the desired vector field is obtained as follows

$$\mathcal{V} = \tau\Big(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}\Big) = \tau\, \mathcal{V}_H$$

where $\tau \in \Lambda^0(\mathfrak{S})$ is an arbitrary function. It is easily observed that trajectories of such a vector field determine the time evolution of the dynamical system under various initial conditions. If we denote the parameter of a trajectory by s, we can write

$$\frac{dq^i}{ds} = \tau\frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{ds} = -\tau\frac{\partial H}{\partial q^i}, \quad \frac{dt}{ds} = \tau.$$

However, once we eliminate the parameter s, we again arrive at the usual Hamilton equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

This result reveals then the possibility of determining the vector field $\mathcal{V}_H$ uniquely by imposing the conditions

$$\mathbf{i}_{\mathcal{V}}(\omega_H) = 0, \quad \mathbf{i}_{\mathcal{V}}(dt) = \tau = 1. \tag{11.5.7}$$

Therefore, the equations (11.5.7) can now be regarded as equivalent to the Hamilton equations associated with a time dependent Hamiltonian function $H(\mathbf{p}, \mathbf{q}, t)$. On account of the satisfaction of the relation $\mathbf{i}_{\mathcal{V}}(\omega_H) = 0$ by a non-zero vector $\mathcal{V}$, we realise that the form ω_H happens to be *a degenerate* 2*-form*. This conclusion should, of course, be expected.

We would like now to evaluate the Lie derivative of the form ω_H with respect to the vector field $\mathcal{V}_H$. Because of the relations $d\omega_H = 0$ and $\mathbf{i}_{\mathcal{V}_H}(\omega_H) = 0$, we find that

$$\pounds_{\mathcal{V}_H}\omega_H = d\mathbf{i}_{\mathcal{V}_H}(\omega_H) + \mathbf{i}_{\mathcal{V}_H}(d\omega_H) = 0.$$

Hence, the form ω_H will remain invariant under the flow on the manifold $\mathfrak{S}$ which is brought into being by the vector field $\mathcal{V}_H$. This result will naturally imply that the forms $\Omega_k = \omega_H^k = \underbrace{\omega_H \wedge \omega_H \wedge \cdots \wedge \omega_H}_{k}, 1 \leq k \leq m$ remain invariant as well under the same flow.

Inasmuch as the vector field $\mathcal{V}_g$ corresponding to a function $g \in \Lambda^0(\mathfrak{S})$ has been given by

$$\mathcal{V}_g = \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t},$$

we then obtain

$$\begin{aligned}\pounds_{\mathcal{V}_g}(f) = \mathcal{V}_g(f) &= \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial t} + \{f, g\}\end{aligned}$$

for a function $f \in \Lambda^0(\mathfrak{S})$. This result means that the necessary and sufficient condition in order that a given function f remains invariant, or in other words, constant under the flow generated by a vector field $\mathcal{V}_g$ is the satisfaction of the following equation

$$\frac{\partial f}{\partial t} + \{f, g\} = 0. \tag{11.5.8}$$

In this case, when we consider the motion of a dynamical system described by the flow produced by a vector field associated with a given Hamiltonian function H, a function $f \in \Lambda^0(\mathfrak{S})$ verifying the equation

$$\pounds_{\mathcal{V}_H}(f) = \frac{\partial f}{\partial t} + \{f, H\} = 0 \tag{11.5.9}$$

must satisfy the relation

$$f\big(\mathbf{q}(t), \mathbf{p}(t), t\big) = f\big(\mathbf{q}_0(t_0), \mathbf{p}_0(t_0), t_0\big) = \textit{constant}.$$

Therefore, it corresponds to an integral of the motion of the system. On the other hand, when H is time dependent, we of course obtain

$$\frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t} \neq 0.$$

Hence, such a Hamiltonian function is no longer an integral of the motion. In other words, the conservation of energy loses its validity in such systems.

Let us now specify a diffeomorphism $\phi : \mathfrak{S}_1 \to \mathfrak{S}_2$ by the following transformations between local coordinates (Q^i, P_i, T) and (q^i, p_i, t):

$$q^i = q^i(\mathbf{Q}, \mathbf{P}, T), \quad p_i = p_i(\mathbf{Q}, \mathbf{P}, T), \quad t = T \tag{11.5.10}$$

Suppose that $H = H(\mathbf{q}, \mathbf{p}, t) \in \Lambda^0(\mathfrak{S}_2)$, $K = K(\mathbf{Q}, \mathbf{P}, t) \in \Lambda^0(\mathfrak{S}_1)$. If the relation

$$\phi^* \big(dq^i \wedge dp_i + dH(\mathbf{q}, \mathbf{p}, t) \wedge dt\big) = dQ^i \wedge dP_i + dK(\mathbf{Q}, \mathbf{P}, t) \wedge dt$$

is satisfied, then we say that ϕ is a ***canonical transformation***. It is clear in this case that the mapping ϕ^{-1} is also a canonical transformation. The short version of the foregoing expression can, of course, be written as follows

$$\phi^* \omega_H = \omega_K. \tag{11.5.11}$$

Since $\omega_1 = dQ^i \wedge dP_i$, $\omega_2 = dq^i \wedge dp_i$ and $\phi^* t = t$, we readily deduce from (11.5.11) that the relation

$$\phi^* \omega_2 = \omega_1 + d\mathcal{F} \wedge dt$$

must be satisfied if ϕ is a canonical transformation. Here, $\mathcal{F}$ is an arbitrary smooth function defined by

$$\mathcal{F} = K - \phi^* H \in \Lambda^0(\mathfrak{S}_1).$$

Thus, we can write $K = \phi^* H + \mathcal{F}$. We can immediately realise that *every canonical transformation preserves the form of the Hamilton equations*. A vector field $\mathcal{V}_H \in T(\mathfrak{S}_2)$ on the manifold $\mathfrak{S}_2$ satisfying the conditions $\mathbf{i}_{\mathcal{V}_H}(\omega_H) = 0$ and $\mathbf{i}_{\mathcal{V}_H}(dt) = 1$ gives rise to the Hamilton equations on $\mathfrak{S}_2$:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

On the other hand, we know that a vector field $\mathcal{V} \in T(\mathfrak{S}_1)$ satisfying the conditions $\mathbf{i}_{\mathcal{V}}(\omega_K) = 0$, $\mathbf{i}_{\mathcal{V}}(dt) = 1$ is a uniquely determined vector field $\mathcal{V}_K$ generated by a Hamiltonian vector field. *Trajectories of this vector field will also satisfy the Hamilton equations on* $\mathfrak{S}_1$*:*

$$\frac{dQ^i}{dt} = \frac{\partial K}{\partial P_i}, \quad \frac{dP_i}{dt} = -\frac{\partial K}{\partial Q^i}. \tag{11.5.12}$$

The connection between the vectors $\mathcal{V}_H$ and $\mathcal{V}_K$ associated with a canonical transformation can easily be found. Consider a vector field $\mathcal{V}_H \in T(\mathfrak{S}_2)$ satisfying the conditions $\mathbf{i}_{\mathcal{V}_H}(\omega_H) = 0$ and $\mathbf{i}_{\mathcal{V}_H}(dt) = 1$. If $\phi : \mathfrak{S}_1 \to \mathfrak{S}_2$ is a canonical transformation, then the pull-back operator $\phi^* : \Lambda(\mathfrak{S}_2) \to \Lambda(\mathfrak{S}_1)$ yields the equalities

$$\begin{aligned} 0 &= \phi^*\mathbf{i}_{\mathcal{V}_H}(\omega_H) = \mathbf{i}_{\phi_*^{-1}\mathcal{V}_H}(\phi^*\omega_H) = \mathbf{i}_{\phi_*^{-1}\mathcal{V}_H}(\omega_K), \\ 1 &= \phi^*\mathbf{i}_{\mathcal{V}_H}(dt) = \mathbf{i}_{\phi_*^{-1}\mathcal{V}_H}(\phi^*dt) = \mathbf{i}_{\phi_*^{-1}\mathcal{V}_H}(d\phi^*t) = \mathbf{i}_{\phi_*^{-1}\mathcal{V}_H}(dt). \end{aligned}$$

When we are given the form ω_K, we know that a vector field $\mathcal{V}_K \in T(\mathcal{S}_1)$ satisfying the conditions $\mathbf{i}_{\mathcal{V}_K}(\omega_K) = 0$ and $\mathbf{i}_{\mathcal{V}_K}(dt) = 1$ will be determined uniquely. Hence, the above relations show unequivocally that the connection between $\mathcal{V}_H$ and $\mathcal{V}_K$ is provided by

$$\mathcal{V}_K = \phi_*^{-1}\mathcal{V}_H \quad \text{or} \quad \mathcal{V}_H = \phi_*\mathcal{V}_K.$$

Since ϕ is a diffeomorphism, $\phi_* : T(\mathfrak{S}_1) \to T(\mathfrak{S}_2)$ is an isomorphism.

In order to illuminate the local structure of canonical transformations $\phi : \mathfrak{S} \to \mathfrak{S}$ that maps the manifold $\mathfrak{S}$ onto itself and to disclose unambiguously the interrelation between Hamiltonian functions H and K, we can make use of the expression (11.5.4). From the relation

$$-\,\phi^*d\theta_H = -\,d\phi^*\theta_H = -\,d\theta_K$$

we find that the equation $d(\phi^*\theta_H - \theta_K) = 0$ has to be satisfied. Thus, according to the Poincaré lemma canonical transformations must obey at least locally to the condition $\phi^*\theta_H - \theta_K = dF$ where $F \in \Lambda^0(\mathfrak{S})$ is an arbitrary function. This expression can be written explicitly as

$$\phi^*(p_i dq^i - H dt) = P_i dQ^i - K dt + dF \tag{11.5.13}$$

The function F is called a ***generating function*** because it is instrumental in designating a canonical transformation. In order to specify a transformation between old and new canonical coordinates, the function F must depend on $4m + 1$ variables $\mathbf{q}, \mathbf{p}, \mathbf{Q}, \mathbf{P}, t$. But, owing to equations (11.5.10), we are allowed to choose only $2m + 1$ independent variables. Therefore, we can consider only four different alternatives characterising a canonical transformation between old and new coordinates that are listed below:

$$\{\mathbf{q}, \mathbf{Q}, t\}, \ \{\mathbf{q}, \mathbf{P}, t\}, \ \{\mathbf{p}, \mathbf{Q}, t\}, \ \{\mathbf{p}, \mathbf{P}, t\}.$$

We shall now discuss these choices separately.

(i). Let us chose $F = F(\mathbf{q}, \mathbf{Q}, t)$. When we insert this function into (11.5.13) and arrange the resulting terms, we obtain

$$\Big(p_i - \frac{\partial F}{\partial q^i}\Big)dq^i - \Big(P_i + \frac{\partial F}{\partial Q^i}\Big)dQ^i + \Big(K - H - \frac{\partial F}{\partial t}\Big)dt = 0.$$

Hence, the canonical transformation is specified by the equations

$$p_i = \frac{\partial F}{\partial q^i}, \quad P_i = -\frac{\partial F}{\partial Q^i}, \quad K = H + \frac{\partial F}{\partial t} \tag{11.5.14}$$

In case $\det\Big(\dfrac{\partial^2 F}{\partial Q^i \partial q^j}\Big) \neq 0$, then $(11.5.14)_2$ yields $q^i = q^i(\mathbf{Q}, \mathbf{P}, t)$ through the inverse function theorem. On introducing these relations into equations $(11.5.14)_1$ we are led to $p_i = p_i(\mathbf{Q}, \mathbf{P}, t)$. Substituting functions so obtained into the functions $H(\mathbf{q}, \mathbf{p}, t)$ and $F(\mathbf{q}, \mathbf{Q}, t)$ we can determine the Hamiltonian function $K(\mathbf{Q}, \mathbf{P}, t)$.

For instance, let us choose $F = a_{ij}(t)\, q^i Q^j$, $\det[a_{ij}] \neq 0$. Then, the canonical transformation becomes

$$p_i = a_{ij}(t)\, Q^j,\ P_i = -a_{ji}(t)\, q^j \text{ or } q^i = -b^{ij}(t) P_j,\ \mathbf{B} = (\mathbf{A}^{\mathrm{T}})^{-1}.$$

If we take $\mathbf{A} = \mathbf{I}$, this transformation merely interchanges the generalised coordinates and generalised momenta.

(ii). Let us choose $F = -P_i Q^i + F_1(\mathbf{q}, \mathbf{P}, t)$. (11.5.13) gives then

$$\Big(p_i - \frac{\partial F_1}{\partial q^i}\Big)dq^i + \Big(Q^i - \frac{\partial F_1}{\partial P_i}\Big)dQ^i + \Big(K - H - \frac{\partial F_1}{\partial t}\Big)dt = 0$$

from which it follows that

$$p_i = \frac{\partial F_1}{\partial q^i}, \quad Q^i = \frac{\partial F_1}{\partial P_i}, \quad K = H + \frac{\partial F_1}{\partial t}. \tag{11.5.15}$$

If $\det\Big(\dfrac{\partial^2 F_1}{\partial P_i \partial q^j}\Big) \neq 0$, then $(11.5.15)_2$ yields the relation $q^i = q^i(\mathbf{Q}, \mathbf{P}, t)$ and $(11.5.15)_1$ provides $p_i = p_i(\mathbf{Q}, \mathbf{P}, t)$. Finally, the transformed Hamiltonian function $K(\mathbf{Q}, \mathbf{P}, t)$ follows from $(11.5.15)_3$.

For instance, let us choose $F_1 = a_i^j(t)\, q^i P_j$, $\det[a_i^j] \neq 0$. The canonical transformation becomes

$$p_i = a_i^j(t)\, P_j,\ Q^i = a_j^i(t)\, q^j \quad \text{or} \quad q^i = b_j^i(t) Q^j,\ \mathbf{B} = (\mathbf{A}^{\mathrm{T}})^{-1}.$$

If we take $\mathbf{A} = \mathbf{I},$ this transformation does not change at all the canonical variables.

(iii). Let us chose $F = p_i q^i + F_2(\mathbf{p}, \mathbf{Q}, t)$. Then, (11.5.13) results in

$$-\Big(q^i + \frac{\partial F_2}{\partial p_i}\Big)dp_i - \Big(P_i + \frac{\partial F_2}{\partial Q^i}\Big)dQ^i + \Big(K - H - \frac{\partial F_2}{\partial t}\Big)dt = 0$$

and the canonical transformation is prescribed by the equations

$$q^i = -\frac{\partial F_2}{\partial p_i}, \quad P_i = -\frac{\partial F_2}{\partial Q^i}, \quad K = H + \frac{\partial F_2}{\partial t}. \qquad (11.5.16)$$

If $\det\Big(\dfrac{\partial^2 F_2}{\partial Q^i \partial p_j}\Big) \neq 0$, the expression $(11.5.16)_2$ determines the functions $p_i = p_i(\mathbf{Q}, \mathbf{P}, t)$, and $(11.5.16)_1$ yields the functions $q^i = q^i(\mathbf{Q}, \mathbf{P}, t)$. By employing these expressions, we deduce the transformed Hamiltonian function from $(11.5.16)_3$.

For instance, if we choose $F_2 = a^i_j(t)\, p_i Q^j$, $\det[a^i_j] \neq 0$, then the canonical transformation is found to be

$$q^i = -a^i_j(t)\, Q^j,\ P_i = -a^j_i(t)\, p_j \quad \text{or} \quad p_i = -b^j_i(t) P_j,\ \mathbf{B} = (\mathbf{A}^{\mathrm{T}})^{-1}.$$

If we take $\mathbf{A} = \mathbf{I}$, then this transformation changes only the signs of the canonical variables.

(iv). Let us choose $F = p_i q^i - P_i Q^i + F_3(\mathbf{p}, \mathbf{P}, t)$. Then, it follows from (11.5.13) that

$$-\Big(q^i + \frac{\partial F_3}{\partial p_i}\Big)dp_i + \Big(Q^i - \frac{\partial F_3}{\partial P_i}\Big)dP_i + \Big(K - H - \frac{\partial F_3}{\partial t}\Big)dt = 0.$$

Consequently, the canonical transformation is specified by the equations

$$q^i = -\frac{\partial F_3}{\partial p_i}, \quad Q^i = \frac{\partial F_3}{\partial P_i}, \quad K = H + \frac{\partial F_3}{\partial t}. \qquad (11.5.17)$$

Indeed, if $\det\Big(\dfrac{\partial^2 F_3}{\partial P_i \partial p_j}\Big) \neq 0$, then $(11.5.17)_2$ leads to $p_i = p_i(\mathbf{Q}, \mathbf{P}, t)$ and

$(11.5.17)_1$ gives $q^i = q^i(\mathbf{Q}, \mathbf{P}, t)$. We obtain the transformed Hamiltonian function $K(\mathbf{Q}, \mathbf{P}, t)$ from the equation $(11.5.17)_3$.

For instance, if we choose $F_3 = a^{ij}(t)\, p_i P_j$, $\det[a^{ij}] \neq 0$, then the canonical transformation becomes

$$q^i = -a^{ij}(t)\, P_j,\ Q^i = a^{ji}(t)\, p_j \quad \text{or} \quad p_i = b_{ij}(t) Q^j,\ \mathbf{B} = (\mathbf{A}^{\mathrm{T}})^{-1}.$$

If we take $\mathbf{A} = \mathbf{I}$, this transformation interchanges the generalised coordinates and generalised momenta with a change of sign in one set of variables.

The equations (11.5.14-17) specify also the canonical transformations in conservative mechanics. However, in this situation the functions F, F_1, F_2, F_3 are either independent of time or may only be certain particular functions of time. The transformation

$$K(\mathbf{Q},\mathbf{P}) = H\big(\mathbf{q}(\mathbf{Q},\mathbf{P}),\mathbf{p}(\mathbf{Q},\mathbf{P})\big)$$

gives then the Hamiltonian function.

After having established the structure of the canonical transformations, Jacobi thought quite an ingenious idea for that time which seems to be rather natural to us now and he had asked this question: whether is it possible to determine a canonical transformation in such a manner that the transformed Hamiltonian function K turns out to be a constant that can be taken zero without loss of generality? If we can make such a choice leading to

$$K(\mathbf{Q},\mathbf{P},t) = 0,$$

then the corresponding Hamilton equations (11.5.12) take their simplest possible form

$$\frac{dQ^i}{dt} = 0, \quad \frac{dP_i}{dt} = 0.$$

Thus, $2m$ integrals of motion are simply obtained as follows

$$Q^i(\mathbf{q},\mathbf{p},t) = a^i = \mathit{constant}, \quad P_i(\mathbf{q},\mathbf{p},t) = b_i = \mathit{constant}$$

in terms of new canonical variables. On the other hand, such a canonical transformation can be prescribed by selecting the generating function F as to satisfy the equation

$$\frac{\partial F(\mathbf{q},\mathbf{Q},t)}{\partial t} + H(\mathbf{q},\mathbf{p},t) = 0.$$

In this case, however, the coordinates Q^i are constant so that the function F depends only on variables $q^1, q^2, \dots, q^m, t$ and the constants $Q^1 = a^1$, $Q^2 = a^2, \dots, Q^m = a^m$. The generalised momenta p_i are then given by $(11.5.14)_1$. Consequently, the function F must satisfy the ***Hamilton-Jacobi differential equation***

$$\frac{\partial F(\mathbf{q},\mathbf{a},t)}{\partial t} + H(\mathbf{q}, \frac{\partial F}{\partial \mathbf{q}}, \mathbf{a}, t) = 0. \qquad (11.5.18)$$

where $\mathbf{a} = (a^1, a^2, \dots, a^m)$.

Although this equation is always attributed today to two mathematicians W. R. Hamilton and C. G. J. Jacobi, it has been actually published first by Hamilton in 1834. Here, we have obviously employed the abbreviated notations

$$\begin{aligned}&F(\mathbf{q},t)=F(q^1,q^2,\dots,q^m,t)\\&H(\mathbf{q},\frac{\partial F}{\partial \mathbf{q}},t)=H(q^1,q^2,\dots,q^m,\frac{\partial F}{\partial q^1},\frac{\partial F}{\partial q^2},\dots,\frac{\partial F}{\partial q^m},t).\end{aligned}$$

The Hamilton-Jacobi equation is a first order, generally non-linear, partial differential equation with $m+1$ independent variables q^i,t. Therefore, the function F depends on $m+1$ integration constants $a^1,a^2,\dots,a^m,a^{m+1}$. But, it is evident that the function $F+a^{m+1}$ satisfies likewise the equation (11.5.18). Since transformation equations involve only some derivatives of F, this constant will have no effect in this approach. Hence, it can be discarded. Thus, by using the representation $\mathbf{a}=\{a^1,a^2,\dots,a^m\}$, the function F that is the solution of the equation (11.5.18) may be expressible in the form

$$F=F(\mathbf{q},\mathbf{a},t)$$

where we obviously have $Q^i=a^i$. Hence, the following equations are deduced from the relations (11.5.14)

$$\begin{aligned}p_i&=\frac{\partial F}{\partial q^i}=p_i(\mathbf{q},\mathbf{a},t),\\P_i&=-\frac{\partial F}{\partial Q^i}=-\frac{\partial F}{\partial a^i}=P_i(\mathbf{q},\mathbf{a},t)=b_i.\end{aligned}\tag{11.5.19}$$

The initial conditions of the dynamical system corresponding to generalised positions and velocities may be given in the following way

$$\mathbf{q}(t_0)=\mathbf{q}_0=\textit{constant},\quad \mathbf{p}(t_0)=\mathbf{p}_0=\textit{constant}.$$

Since we have assumed that $\det\Big(\partial^2F/\partial a^i\partial q^j\Big)\neq 0$, insertion of the initial conditions into $(11.5.19)_1$ leads to the determination of the constants $\mathbf{a}$ that are arbitrary at the outset as

$$\mathbf{a}=\mathbf{a}(\mathbf{q}_0,\mathbf{p}_0,t_0).$$

Substituting the constants a_i so obtained together with initial conditions into $(11.5.19)_2$, we determine the constants b^i:

$$\mathbf{b}=\mathbf{P}\big(\mathbf{q}_0,\mathbf{a}(\mathbf{q}_0,\mathbf{p}_0,t_0),t_0\big).$$

Recalling again the condition $\det\left(\partial^2 F/\partial a^i \partial q^j\right) \neq 0$, we can in principle construct the inverse function from $(11.5.19)_2$, to arrive eventually at the following equations

$$\mathbf{q} = \mathbf{q}(t; \mathbf{a}, \mathbf{b})$$

that describe the evolution of the system with time. It is thus evident that in case we can determine the function $F(\mathbf{q}, t)$, sometimes called the ***Hamilton principal function***, satisfying the first order partial differential equation (11.5.18), then the expressions describing the motion of the system are found by almost algebraic manipulations. Therefore, this method seems, at first glance, to be a much more effective approach to determine the motion of a system than trying to solve directly the Hamilton equations. In reality, it is highly unlikely to be able solve directly the Hamilton-Jacobi equation, except in very few cases. The standard technique of characteristics to solve this non-linear partial differential equation requires again to obtain the solution of the Hamilton equations [*see* Example 9.2.3]. Therefore, it does not bring about a fresh approach. Nonetheless the discussion of the Hamilton-Jacobi equation may provide rather significant qualitative information about the behaviour of a dynamical system.

In order to comprehend better the meaning of the function F, let us calculate its derivative with respect to time along the trajectory of the system. When (11.5.14) and (11.5.18) are taken into account, it is easily found that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^i}\frac{dq^i}{dt} = p_i \dot{q}^{\,i} - H = L.$$

Hence, this time rate of change of F yields the Lagrangian function.

If the Hamiltonian function H does not explicitly depend on time the equation (11.5.18) leads of course to the result

$$\frac{\partial^2 F(\mathbf{q}, \mathbf{Q}, t)}{\partial t^2} = 0$$

since $\partial H/\partial t = 0$. A simple integration then gives

$$F(\mathbf{q}, \mathbf{Q}, t) = \, - E(\mathbf{q}, \mathbf{Q})t + W(\mathbf{q}, \mathbf{Q}), \quad \mathbf{Q} = \mathbf{a}.$$

Consequently, the generalised momenta become

$$p_i = \frac{\partial F}{\partial q^i} = \, - \frac{\partial E}{\partial q^i}t + \frac{\partial W}{\partial q^i}.$$

Hence, H will be explicitly independent of time t if only $\partial E/\partial q^i = 0$. According to the definition of the function F, we have to take $\mathbf{Q} = \mathbf{a} =$ *constant*. Hence, we must write $E(\mathbf{Q}) = a^m =$ *constant*. Thus, F will now be expressible in the form

$$F(\mathbf{q}, \mathbf{a}, t) = -Et + W(q^1, \dots, q^m; a^1, \dots, a^{m-1}, E) \quad (11.5.20)$$

whence we can deduce the following relations

$$\begin{aligned} p_i &= \frac{\partial W}{\partial q^i}, & i &= 1, 2, \dots, m \quad (11.5.21)\\ b_i &= -\frac{\partial W}{\partial a^i}, & i &= 1, 2, \dots, m-1\\ b_m &= -\frac{\partial F}{\partial E} = t - \frac{\partial W}{\partial E}. \end{aligned}$$

In this case, the Hamilton-Jacobi equation reduces to the non-linear partial differential equation

$$H(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}) = E \quad (11.5.22)$$

that helps determine the function W that is called sometimes the ***Hamilton characteristic function***.

Example 11.5.1. ***Harmonic Oscillator***. Let us denote by q the coordinate of the 1-dimensional configuration manifold associated with the rectilinear harmonic motion of a particle with mass m. Then, the kinetic and potential energies are prescribed in the following manner

$$T = \frac{1}{2}m\dot{q}^2, \quad V = \frac{1}{2}kq^2.$$

Introducing the definition $\omega^2 = k/m$, we see that the Lagrangian function and the generalised momentum that is equal to the ordinary momentum in this case are given by

$$\begin{aligned} L &= \frac{m}{2}(\dot{q}^2 - \omega^2 q^2),\\ p &= \frac{\partial L}{\partial \dot{q}} = m\dot{q}. \end{aligned}$$

Hence, the Hamiltonian function takes the form

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) = E.$$

and the equation (11.5.20) becomes

$$\left(\frac{dW}{dq}\right)^2 + m^2\omega^2 q^2 = 2mE.$$

When we choose the $+$ sign in front of the square root, we obtain the differential equation

$$\frac{dW}{dq} = m\omega\sqrt{\frac{2E}{m\omega^2} - q^2}$$

whose solution is easily found as follows

$$\begin{aligned} W(q;E) &= m\omega\int\sqrt{\frac{2E}{m\omega^2} - q^2}\,dq + a \\ &= \frac{m\omega}{2}\left[q\sqrt{\frac{2E}{m\omega^2} - q^2} + \frac{2E}{m\omega^2}\arctan\frac{q}{\sqrt{\dfrac{2E}{m\omega^2} - q^2}}\right] + a. \end{aligned}$$

Therefore, $(11.5.21)_3$ yields

$$b = -\frac{\partial F}{\partial E} = t - \frac{\partial W}{\partial E} = t - \frac{1}{\omega}\arctan\frac{q}{\sqrt{\dfrac{2E}{m\omega^2} - q^2}}$$

and we finally obtain by using inverse trigonometric functions

$$q = \sqrt{\frac{2E}{m\omega^2}}\sin\omega(t-b).$$

The constants E and b are to be determined from the initial conditions. ■

Example 11.5.2. ***Central-Force Motion***. Let us denote the polar coordinates of the 2-dimensional configuration manifold that is associated with the central motion of a particle with mass m by $q^1 = r, q^2 = \theta$. Then, its Lagrangian function can be written as follows

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

Hence, the generalised momenta become

$$p_1 = p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}\,,\ \ p_2 = p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

and the Hamiltonian function is prescribed by the expression

$$H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2m}\Big(p_r^2 + \frac{p_\theta^2}{r^2}\Big) + V(r).$$

Equation (11.5.22) now takes the form

$$\frac{1}{2m}\Big[\Big(\frac{\partial W}{\partial r}\Big)^2 + \frac{1}{r^2}\Big(\frac{\partial W}{\partial \theta}\Big)^2\Big] + V(r) = E$$

with $W = W(r, \theta; a^1, E)$. Since the function H does not depend on θ explicitly, the Hamilton equations yield $\dot{p}_\theta = 0$, and $p_\theta = a^1 = constant$. Thus, we are allowed to write from (11.5.21)

$$p_r = \frac{\partial W}{\partial r}, \qquad p_\theta = \frac{\partial W}{\partial \theta} = a^1,$$
$$b_1 = -\frac{\partial W}{\partial a^1}, \quad b_2 = t - \frac{\partial W}{\partial E}.$$

It then follows from the second equation above

$$W(r, \theta; a^1, E) = a^1\theta + w(r; a^1, E).$$

If we take into consideration the $+$ sign in front of the square root, we observe that the function w must satisfy the following differential equation

$$\frac{dw}{dr} = \sqrt{2m\big[E - V(r)\big] - \Big(\frac{a^1}{r}\Big)^2}$$

whose solution is easily obtainable in the form

$$w(r; a^1, E) = \int_{r_0}^{r} \sqrt{2m\big[E - V(s)\big] - \Big(\frac{a^1}{s}\Big)^2}\, ds$$

where r_0 is yet an arbitrary constant. Upon introducing this relation into $(11.5.21)_{2\text{-}3}$, we find that

$$b_1 = -\theta + \int_{r_0}^{r} \frac{a^1 ds}{s^2\sqrt{2m\big[E - V(s)\big] - (a^1/s)^2}},$$
$$b_2 = t - \int_{r_0}^{r} \frac{m\, ds}{\sqrt{2m\big[E - V(s)\big] - (a^1/s)^2}}.$$

If the initial conditions are such that $r = r_0$ and $\theta = \theta_0$ for $t = t_0$, we get

$$b_1 = -\theta_0, \quad b_2 = t_0.$$

Hence, in terms of the parameter r, the equations describing the motion of the particle are expressible as

$$\theta-\theta_0=\int_{r_0}^{r}\frac{a^1ds}{s\sqrt{2ms^2\big[E-V(s)\big]-(a^1)^2}},$$
$$t-t_0=\int_{r_0}^{r}\frac{ms\,ds}{\sqrt{2ms^2\big[E-V(s)\big]-(a^1)^2}}.$$

We have to take $V(r)=k/r$ in order to discuss perhaps the most important application of the central-force motion. When $k<0$, this potential corresponds to the Newton law of gravitational attraction [the English mathematician and physicist Sir Isaac Newton (1643-1727)]. If the constant k may be taken either negative or positive, this potential represents the Coulomb law describing the force between point electric charges that can be attractive or repulsive [the French engineer and physicist Charles Augustin de Coulomb (1736-1806)]. When $V(r)=k/r$, we readily find that

$$\theta-\theta_0=\int_{r_0}^{r}\frac{a^1ds}{s\sqrt{2mE\,s^2-2mks-(a^1)^2}},$$
$$t-t_0=\int_{r_0}^{r}\frac{ms\,ds}{\sqrt{2mE\,s^2-2mks-(a^1)^2}}.$$

If we make the substitution $s=1/\sigma$ in the first equation above and rename the constant a^1 by l, we obtain

$$\theta-\theta_0=-\int_{1/r_0}^{1/r}\frac{d\sigma}{\sqrt{\dfrac{2mE}{l^2}-\dfrac{2mk}{l^2}\sigma-\sigma^2}}$$
$$=-\arccos\frac{\dfrac{\sigma l^2}{mk}-1}{\sqrt{1+\dfrac{2El^2}{mk^2}}}\Bigg|_{1/r_0}^{1/r}.$$

Let us define an angle θ_1 by $\theta_1=\arccos\Big[\Big(\dfrac{l^2}{mkr_0}-1\Big)\Big/\sqrt{1+\dfrac{2El^2}{mk^2}}\Big]$. Then, we finally reach to the conclusion

$$\frac{1}{r}=\frac{mk}{l^2}\Big[1+\sqrt{1+\frac{2El^2}{mk^2}}\cos(\theta-\theta_0-\theta_1)\Big].$$

Introducing the definition $p = l^2/mk$ and $e = \sqrt{1 + (2El^2/mk^2)}$ we arrive at the standard equation describing conics in polar coordinates

$$r = \frac{p}{1 + e\cos(\theta - \theta')}, \quad \theta' = \theta_0 + \theta_1.$$

On defining a function

$$f(\sigma) = \sqrt{\frac{2mE}{l^2} - \frac{2mk}{l^2}\sigma - \sigma^2}$$

we determine the function $t = t(r)$ in a similar way as follows

$$t(r) - t_0 = t - t_0 = -\frac{m}{l}\int_{1/r_0}^{1/r} \frac{d\sigma}{\sigma^2 f(\sigma)} = T(\sigma)\big|_{1/r_0}^{1/r}$$

where the function $T(\sigma)$ is given by

$$T(\sigma) = \frac{2E^{1/2} l f(\sigma) + (2m)^{1/2} k\sigma \log \dfrac{2mE - km\sigma + (2mE)^{1/2} l f(\sigma)}{\sigma}}{4E^{3/2}}$$

The function $T(\sigma)$ determines the time taken by the particle on its trajectory traversing from the radial distance r_0 to the radial distance r. ■

11.6. ELECTROMAGNETISM

Let us consider the 4-dimensional manifold $\mathbb{R}^4$. Its coordinates will be denoted by $x^\mu, \mu = 1, 2, 3, 4$. $x^i, i = 1, 2, 3$ correspond to spatial coordinates while $x^4 = t$ denotes the time coordinate. Electromagnetic fields on this manifold, representing a material medium or the vacuum, are governed by the ***Maxwell equations*** [the English mathematician and physicist James Clerk Maxwell (1831-1879)] that are given, in rationalised M.K.S. units, by

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0}, \quad \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J}, \quad \nabla \cdot \mathbf{D} = \rho \end{aligned} \tag{11.6.1}$$

where the vectors **E**, **H**, **B** and **D** specify, respectively, *the electric field*, *the magnetic field*, *the magnetic induction* and *the electric displacement field.* The vector **J** is *the free electric current density* whereas the scalar ρ is *the free electric charge density*. The divergence of the equation $(11.6.1)_3$ yields an equation corresponding to *the conservation of electric charge*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{11.6.2}$$

Actually, the vector fields **E**, **H**, **B**, **D** and **J** cannot be utterly independent of one another. There are relations among them called *constitutive equations* reflecting the physical properties of the medium. The simplest physically meaningful relations of that kind can be given by

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = \sigma \mathbf{E}$$

where the three physical constants ϵ, μ, σ are known, respectively, as *the dielectric* and *the magnetic permittivities* and *the electric conduction coefficient*. The values of these constants in the *vacuum* are

$$\epsilon = \epsilon_0 = 8.854187817620 \times 10^{-12}\, \text{F/m (Farad/metre)}$$
$$\mu = \mu_0 = 4\pi \times 10^{-7}\, \text{N/A}^2\ (\text{Newton/ampere}^2)$$

and $\sigma = 0$. These constants satisfy the relation $c = 1/\sqrt{\epsilon_0 \mu_0}$ where the physical constant c is the speed of light in the vacuum. The most recent value of c is 299792.458 km/sec.

The equation $(11.6.1)_1$ is known as the ***Faraday induction law*** [the autodidact English physicist and chemist Michael Faraday (1791-1867)] that shows that a mechanical energy causing a magnetic induction in a region to change with time can be converted to the electrical energy. The equation $(11.6.1)_2$ is the ***Gauss law*** implying that magnetic charges (monopoles) do not exist in nature in the realm of the classical physics. The equation $(11.6.1)_3$ is a somewhat modified version of the ***Ampère law*** expressing the fact that electric currents create magnetic fields. Equation $(11.6.1)_4$, when written in the form $\nabla \cdot \mathbf{E} = \rho/\epsilon$, is originally obtained from the ***Coulomb law*** that specifies the repulsive or attractive force between two electric point charge as the expression $q_1 q_2/\epsilon r^2$ by exactly following the path leading to the Gauss law. In the original version of the Ampère law the term $\partial \mathbf{D}/\partial t$ which will be called later *the displacement current* does not exist. However, the governing equations at that form are not consistent because they violate the equation (11.6.2) associated with the conservation of charge that can also be derived independently. The genius of Maxwell has caused the creation of a consistent theory of electromagnetism. He has cleverly introduced a displacement vector **D** to recover the conservation of charge. It has been realised, however, that only in particular, but practically very important, cases this vector could be identified as $\epsilon \mathbf{E}$. This theory of electromagnetism was perhaps the greatest scientific achievement in the

19th century and has paved the way for incredible technological developments in the 20th century.

Let us now express equations (11.6.1) in terms of their components in Cartesian coordinates by paying attention to the dictates of the summation convention as follows

$$e^{ijk}E_{k,j} + \frac{\partial B^i}{\partial t} = 0,\ B^i_{,i} = 0;\ e^{ijk}H_{k,j} - \frac{\partial D^i}{\partial t} = J^i,\ D^i_{,i} = \rho.$$

Next, we wish to introduce the 4×4 *antisymmetric matrices* $\mathbb{F} = [F^{\mu\nu}]$ and $\mathbb{H} = [H^{\mu\nu}]$ by the following entries

$$\begin{aligned} F^{ij} &= -F^{ji} = e^{jik}E_k, \quad F^{4i} = -F^{i4} = B^i, \\ H^{ij} &= -H^{ji} = e^{jik}H_k, \quad H^{4i} = -H^{i4} = -D^i. \end{aligned} \tag{11.6.3}$$

In matrix notation, we can, of course, write

$$\mathbb{F} = \begin{bmatrix} 0 & -E_3 & E_2 & -B^1 \\ E_3 & 0 & -E_1 & -B^2 \\ -E_2 & E_1 & 0 & -B^3 \\ B^1 & B^2 & B^3 & 0 \end{bmatrix},$$
$$\mathbb{H} = \begin{bmatrix} 0 & -H_3 & H_2 & D^1 \\ H_3 & 0 & -H_1 & D^2 \\ -H_2 & H_1 & 0 & D^3 \\ -D^1 & -D^2 & -D^3 & 0 \end{bmatrix}.$$

Hence, the Maxwell equations are now expressible in the form

$$\begin{aligned} \frac{\partial F^{ji}}{\partial x^j} + \frac{\partial F^{4i}}{\partial x^4} &= 0, \quad \frac{\partial F^{i4}}{\partial x^i} = 0; \\ \frac{\partial H^{ji}}{\partial x^j} + \frac{\partial H^{4i}}{\partial x^4} &= J^i, \quad \frac{\partial H^{i4}}{\partial x^i} = \rho. \end{aligned}$$

Let us now define a 4-vector $\{J^\mu\} = \{J^i, J^4 = \rho\}$ and note that $F^{44} = 0$ and $H^{44} = 0$. Then, it is straightforward to see that the Maxwell equations can be written concisely as follows

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = 0, \quad \frac{\partial H^{\mu\nu}}{\partial x^\mu} = J^\nu. \tag{11.6.4}$$

Let $\mu = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ be the volume form in the manifold $\mathbb{R}^4$. By using the familiar bases $\mu_{\mu\nu}$ induced by this volume form, we can introduce the two 2-forms $\mathcal{F} \in \Lambda^2(\mathbb{R}^4)$ and $\mathcal{H} \in \Lambda^2(\mathbb{R}^4)$ by employing the antisymmetric coefficients $F^{\mu\nu}$ and $H^{\mu\nu}$ through

$$\mathcal{F} = \frac{1}{2}F^{\mu\nu}\mu_{\mu\nu},$$
$$\mathcal{H} = \frac{1}{2}H^{\mu\nu}\mu_{\mu\nu}$$

where $\mu_{\mu\nu} = \mathbf{i}_{\partial_\mu} \circ \mathbf{i}_{\partial_\nu}(\mu)$ and a vector field $\mathcal{J} \in T(\mathbb{R}^4)$ by

$$\mathcal{J} = J^i \frac{\partial}{\partial x^i} + \rho \frac{\partial}{\partial x^4} = J^\mu \frac{\partial}{\partial x^\mu}. \tag{11.6.5}$$

As is easily seen, we can now write

$$\begin{aligned} d\mathcal{F} &= \frac{1}{2}F^{\mu\nu}_{,\gamma}\, dx^\gamma \wedge \mu_{\mu\nu} \\ &= \frac{1}{2}F^{\mu\nu}_{,\gamma}(\delta^\gamma_\mu\, \mu_\nu - \delta^\gamma_\nu\, \mu_\mu) = F^{\mu\nu}_{,\mu}\, \mu_\nu. \end{aligned}$$

Hence, the equations $(11.6.4)_1$ are equivalent to the exterior equation

$$d\mathcal{F} = 0. \tag{11.6.6}$$

On the other hand, because of the relation

$$\mathbf{i}_{\mathcal{J}}(\mu) = J^\nu \mu_\nu \in \Lambda^3(\mathbb{R}^4)$$

the equations $(11.6.4)_2$ become equivalent to the exterior equation

$$d\mathcal{H} = \mathbf{i}_{\mathcal{J}}(\mu). \tag{11.6.7}$$

Let us now express the form $\mathcal{F}$ with respect to the natural basis dx^μ as follows:

$$\mathcal{F} = \frac{1}{2}F^{\mu\nu}\mu_{\mu\nu} = \frac{1}{2}\mathcal{F}_{\alpha\beta}\, dx^\alpha \wedge dx^\beta.$$

On the other hand, for $m = 4$ and $k = 2$ the relation (5.5.10) yields the following expression

$$\mu_{\mu\nu} = \frac{1}{2}\, e_{\nu\mu\alpha\beta}\, dx^\alpha \wedge dx^\beta.$$

Therefore, we find that

$$\begin{aligned} \mathcal{F}_{\alpha\beta} &= -\mathcal{F}_{\beta\alpha} = \frac{1}{2}\, e_{\nu\mu\alpha\beta} F^{\mu\nu} \\ &= -\frac{1}{2}\, e_{\alpha\beta\mu\nu}\, F^{\mu\nu}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\mathcal{F}_{ij} &= -e_{ijk4}F^{k4} = e_{ijk4}B^k = e_{ijk}B^k,\\ \mathcal{F}_{i4} &= -\frac{1}{2}\,e_{i4jk}F^{jk} = \frac{1}{2}\,e_{ijk}e^{jkl}E_l = \delta_i^l E_l = E_i\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\mathcal{F}_{12} = -F^{34} = B^3,\ \mathcal{F}_{13} = F^{24} = -B^2,\ \mathcal{F}_{14} = -F^{23} = E_1,\\ &\mathcal{F}_{23} = -F^{14} = B^1,\quad \mathcal{F}_{24} = F^{13} = E_2,\quad \mathcal{F}_{34} = -F^{12} = E_3.\end{aligned}$$

Thus, the antisymmetric matrix $\boldsymbol{\mathcal{F}} = [\mathcal{F}_{\alpha\beta}]$ is given by

$$\boldsymbol{\mathcal{F}} = \begin{bmatrix} 0 & B^3 & -B^2 & E_1\\ -B^3 & 0 & B^1 & E_2\\ B^2 & -B^1 & 0 & E_3\\ -E_1 & -E_2 & -E_3 & 0\end{bmatrix}. \tag{11.6.8}$$

Similarly, the form $\mathcal{H}$ can be rewritten as

$$\mathcal{H} = \frac{1}{2}\mathcal{H}_{\alpha\beta}\,dx^\alpha\wedge dx^\beta,\ \ \mathcal{H}_{\alpha\beta} = -\frac{1}{2}\,e_{\alpha\beta\mu\nu}\,H^{\mu\nu}$$

and one finds that

$$\mathcal{H}_{ij} = -e_{ijk}D^k,\quad \mathcal{H}_{i4} = H_i.$$

Hence, the antisymmetric matrix $\boldsymbol{\mathcal{H}} = [\mathcal{H}_{\alpha\beta}]$ is given by

$$\boldsymbol{\mathcal{H}} = \begin{bmatrix} 0 & -D^3 & D^2 & H_1\\ D^3 & 0 & -D^1 & H_2\\ -D^2 & D^1 & 0 & H_3\\ -H_1 & -H_2 & -H_3 & 0\end{bmatrix}. \tag{11.6.9}$$

With these representations, Equation (11.6.6) leads to

$$\begin{aligned}d\mathcal{F} &= \frac{1}{2}\mathcal{F}_{\alpha\beta,\gamma}\,dx^\gamma\wedge dx^\alpha\wedge dx^\beta\\ &= \frac{1}{3!}\mathcal{F}_{\gamma\alpha\beta}\,dx^\gamma\wedge dx^\alpha\wedge dx^\beta = 0\end{aligned}$$

and we arrive at the equations [*see p.* 265]

$$\mathcal{F}_{\gamma\alpha\beta} = \frac{\partial\mathcal{F}_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial\mathcal{F}_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial\mathcal{F}_{\gamma\alpha}}{\partial x^\beta} = 0 \tag{11.6.10}$$

that are counterparts of Equations $(11.6.4)_1$. Similarly, on considering the definition (5.5.8), Equation (11.6.7) leads to

$$\begin{aligned} d\mathcal{H} &= \frac{1}{3!}\mathcal{H}_{\gamma\alpha\beta}\, dx^\gamma \wedge dx^\alpha \wedge dx^\beta \\ &= \frac{1}{3!}e_{\nu\gamma\alpha\beta}J^\nu dx^\gamma \wedge dx^\alpha \wedge dx^\beta \end{aligned}$$

from which follow the equations

$$\mathcal{H}_{\gamma\alpha\beta} = \frac{\partial\mathcal{H}_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial\mathcal{H}_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial\mathcal{H}_{\gamma\alpha}}{\partial x^\beta} = e_{\nu\gamma\alpha\beta}J^\nu \qquad (11.6.11)$$

that are counterparts of Equations $(11.6.4)_2$.

It is immediately seen that the equation (11.6.7) elicits the condition

$$d\mathbf{i}_{\mathcal{J}}(\mu) = J^\nu_{\ ,\nu}\,\mu = 0$$

that is none other than the equation (11.6.2) for the conservation of charge:

$$J^\nu_{\ ,\nu} = J^i_{\ ,i} + \frac{\partial\rho}{\partial t} = 0.$$

The exterior equations (11.6.6) and (11.6.7) are obviously coordinate free versions of Maxwell equations. We shall now try to establish a general solution of these equations. Since the manifold $\mathbb{R}^4$ is star-shaped with respect to each of its points, the use the homotopy operator with the centre $\mathbf{x}_0 = \mathbf{0}$ enables us to represent the forms $\mathcal{F}$ and $\mathcal{H}$ as follows

$$\begin{aligned} \mathcal{F} &= dH(\mathcal{F}) + H(d\mathcal{F}) = dH(\mathcal{F}) \\ \mathcal{H} &= dH(\mathcal{H}) + H(d\mathcal{H}) = dH(\mathcal{H}) + H\big(\mathbf{i}_{\mathcal{J}}(\mu)\big). \end{aligned}$$

Let us now introduce 1-forms

$$H(\mathcal{F}) = \Phi \in \Lambda^1(\mathbb{R}^4), \quad H(\mathcal{H}) = \Psi \in \Lambda^1(\mathbb{R}^4).$$

In accordance with (6.3.1), we obtain for $k = 3$ the following 2-form

$$H\big(\mathbf{i}_{\mathcal{J}}(\mu)\big) = x^\mu \int_0^1 \mathbf{i}_{\partial_\mu}\big(J^\nu(s\mathbf{x})\mu_\nu\big)s^2 ds = x^\mu\Big(\int_0^1 s^2 J^\nu(s\mathbf{x})\, ds\Big)\mu_{\mu\nu}$$

If we define a linear operator $A : \Lambda^0(\mathbb{R}^4) \to \Lambda^0(\mathbb{R}^4)$ by the rule

$$A(f)(\mathbf{x}) = \int_0^1 s^2 f(s\mathbf{x})\, ds, \;\; f \in \Lambda^0(\mathbb{R}^4)$$

we find at once that

$$H\big(\mathbf{i}_{\mathcal{J}}(\mu)\big) = x^{\mu} A(J^{\nu})\, \mu_{\mu\nu} = x^{[\mu} A(J^{\nu]})\, \mu_{\mu\nu}.$$

Therefore, the general structure of exterior forms determining electromagnetic fields takes the shape

$$\begin{aligned}\mathcal{F} &= d\Phi, \\ \mathcal{H} &= d\Psi + x^{[\mu} A(J^{\nu]})\, \mu_{\mu\nu}.\end{aligned} \tag{11.6.12}$$

It is clear that the forms Φ and Ψ cannot be prescribed uniquely unless we impose some restrictions. As a matter of fact, for arbitrary smooth functions f, g, the exterior forms $\Phi + df$ and $\Psi + dg$ satisfy perfectly again the equations (11.6.12).

Let us now consider 1-forms $\Phi = \Phi_\mu(\mathbf{x})\, dx^\mu$ and $\Psi = \Psi_\mu(\mathbf{x})\, dx^\mu$. If we explicitly write the equation $(11.6.12)_1$, we get

$$\frac{1}{2}\mathcal{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu = \Phi_{\mu,\nu}\, dx^\nu \wedge dx^\mu = -\Phi_{[\mu,\nu]}\, dx^\mu \wedge dx^\nu$$

from which we find that

$$\mathcal{F}_{\mu\nu} = -2\Phi_{[\mu,\nu]} = \Phi_{\nu,\mu} - \Phi_{\mu,\nu}.$$

We thus obtain

$$\mathcal{F}_{ij} = e_{ijk} B^k = \Phi_{j,i} - \Phi_{i,j}.$$

From this relation, we can easily deduce that

$$e^{ijl} e_{ijk} B^k = 2B^l = e^{ijl}(\Phi_{j,i} - \Phi_{i,j}) = 2e^{lij}\Phi_{j,i}$$

and finally $B^{\mathrm{i}} = e^{ijk}\Phi_{k,j}$. Let us now regard three functions (Φ_1, Φ_2, Φ_3) defined on the manifold $\mathbb{R}^4$ as the components of a 3-vector field $\mathbf{\Phi}(\mathbf{x}, t)$. Thus, we come to the conclusion that we can write

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{\Phi}$$

by using the familiar curl operator. On the other hand, the remaining relations yield

$$\mathcal{F}_{i4} = E_i = \Phi_{4,i} - \Phi_{i,4}$$

that can be expressed obviously as

$$\mathbf{E} = \boldsymbol{\nabla}\phi - \frac{\partial \mathbf{\Phi}}{\partial t}$$

when we introduce the scalar function $\Phi_4 = \phi(\mathbf{x}, t)$. If we repeat the same operations for $(11.6.12)_2$, then we easily obtain

$$\frac{1}{2}\mathcal{H}_{\mu\nu}\,dx^\mu\wedge dx^\nu=\Psi_{[\nu,\mu]}\,dx^\mu\wedge dx^\nu+\frac{1}{2}x^{[\alpha}A(J^{\beta]})\,e_{\beta\alpha\mu\nu}\,dx^\mu\wedge dx^\nu$$

and hence we find that

$$\mathcal{H}_{\mu\nu}=\Psi_{\nu,\mu}-\Psi_{\mu,\nu}-e_{\mu\nu\alpha\beta}x^\alpha A(J^\beta).$$

Consequently, we reach to the relations

$$\mathcal{H}_{ij}=\,-\,e_{ijk}D^k=\Psi_{j,i}-\Psi_{i,j}-e_{ijk}x^kA(J^4)+e_{ijk}x^4A(J^k).$$

If we note that $e^{ijl}e_{ijk}=2\delta^l_k$, the components D^i are then determined by the relations

$$D^i=\,-\,e^{ijk}\Psi_{k,j}-x^iA(\rho)+tA(J^i).$$

Therefore, the representation of the vector field **D** becomes

$$\mathbf{D}=\,-\,\boldsymbol{\nabla}\times\boldsymbol{\Psi}-A(\rho)\,\mathbf{x}+tA(\mathbf{J})$$

where we define the 3-vector $\boldsymbol{\Psi}(\mathbf{x},t)=(\Psi_1,\Psi_2,\Psi_3)$. Finally the remaining expressions lead to the relations

$$\mathcal{H}_{i4}=H_i=\Psi_{4,i}-\Psi_{i,4}-e_{i4jk}x^jA(J^k)$$

from which we infer that $H_i=\psi_{,i}-\Psi_{i,4}-e_{ijk}x^jA(J^k)$ or

$$\mathbf{H}=\boldsymbol{\nabla}\psi-\frac{\partial\boldsymbol{\Psi}}{\partial t}-\mathbf{x}\times A(\mathbf{J})$$

where we have introduced the scalar function $\Psi_4=\psi(\mathbf{x},t)$. If we collect all the results that have been obtained so far, we then express the general solution of the Maxwell equations in the following form

$$\mathbf{B}=\boldsymbol{\nabla}\times\boldsymbol{\Phi},\quad\mathbf{E}=\boldsymbol{\nabla}\phi-\frac{\partial\boldsymbol{\Phi}}{\partial t}\tag{11.6.13}$$
$$\mathbf{D}=\,-\,\boldsymbol{\nabla}\times\boldsymbol{\Psi}-A(\rho)\,\mathbf{x}+tA(\mathbf{J}),\;\;\mathbf{H}=\boldsymbol{\nabla}\psi-\frac{\partial\boldsymbol{\Psi}}{\partial t}-\mathbf{x}\times A(\mathbf{J})$$

in terms of arbitrary vector fields $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, and arbitrary scalar fields ϕ and ψ depending on independent variables $(\mathbf{x},t)$. As we have mentioned above these fields cannot be prescribed uniquely. Indeed, when $\lambda(\mathbf{x},t)$ and $\Lambda(\mathbf{x},t)$ are arbitrary scalar functions, we immediately observe that the relations (11.6.13) remains unchanged if we replace the vector-valued functions $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, and scalar-valued functions ϕ and ψ by

$$\mathbf{\Phi}+\nabla\lambda,\ \phi+\frac{\partial\lambda}{\partial t};\ \ \mathbf{\Psi}+\nabla\Lambda,\ \psi+\frac{\partial\Lambda}{\partial t}.$$

As we have mentioned before, in a physical medium the field vectors will not be all independent and they will be interconnected by some constitutive relations. Naturally, these relations affect the structure of the equations (11.6.1) to a great extent. As an example, let us choose the constitutive relations $\mathbf{D}=\epsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$. Although these relations are quite simple, they are considerably important as far as practical applications are concerned. When we insert these relations into the Maxwell equations, they become

$$\begin{aligned}\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}&=\mathbf{0},\quad \nabla\cdot\mathbf{B}=0,\\ \nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}&=\mu\mathbf{J},\quad \nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon}\end{aligned}\tag{11.6.14}$$

where we define $c=1/\sqrt{\epsilon\mu}$. c is a constant of the dimension of velocity. In terms of the components these equations take the form

$$e^{ijk}E_{k,j}+\frac{\partial B^i}{\partial t}=0,\ B^i_{,i}=0;\ e^{ijk}B_{k,j}-\frac{1}{c^2}\frac{\partial E^i}{\partial t}=\mu J^i,\ E^i_{,i}=\frac{\rho}{\epsilon}.$$

We see now that the same field vectors **E** and **B** appear in both group of equations. However, the positions of upper and lower indices, that were employed to comply with the summation convention, evoke covariant and contravariant components of vectors. To explore this possibility, we shall try to equip the manifold $\mathbb{R}^4$ with an indefinite metric to make it an incomplete Riemannian manifold. We shall now introduce the indefinite ***Lorentz metric*** by the relation [*see* Exercise **7.8**]

$$g_{\lambda\mu}dx^\lambda dx^\mu=(dx^1)^2+(dx^2)^2+(dx^3)^2-c^2(dx^4)^2\tag{11.6.15}$$

where $x^4=t$. Hence, the metric tensor and its inverse are given by the following matrices, respectively

$$[g_{\lambda\mu}]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-c^2\end{bmatrix},\quad [g^{\lambda\mu}]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-\dfrac{1}{c^2}\end{bmatrix}$$

and we have $g=|\det[g_{\lambda\mu}]|=c^2$. Now, by using this metric we may determine covariant components of an antisymmetric tensor whose contravariant components are given by $F^{\mu\nu}$ as follows

$$F_{\alpha\beta} = -F_{\beta\alpha} = g_{\alpha\mu}g_{\beta\nu}F^{\mu\nu}.$$

Thus, we easily find that

$$F_{ij} = g_{i\mu}g_{j\nu}F^{\mu\nu} = e_{ijk}E^k,$$
$$F_{4i} = g_{4\mu}g_{i\nu}F^{\mu\nu} = -c^2 B_i$$

Next, we define a new antisymmetric tensor by its contravariant components through the relation

$$\widetilde{F}^{\alpha\beta} = \frac{1}{2c}\,\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu} = \frac{1}{2c^2}\,e^{\alpha\beta\mu\nu}F_{\mu\nu}$$

whence we readily deduce that

$$\begin{aligned}\widetilde{F}^{ij} &= \frac{1}{2c^2}\,e^{ij\mu\nu}F_{\mu\nu} = \frac{1}{2c^2}\,e^{ijk4}F_{k4} = e^{ijk}B_k \qquad (11.6.16)\\ \widetilde{F}^{4j} &= \frac{1}{2c^2}\,e^{4j\mu\nu}F_{\mu\nu} = \frac{1}{2c^2}\,e^{4jkl}F_{kl} = -\frac{1}{2c^2}\,e^{jkl}F_{kl}\\ &= -\frac{1}{2c^2}\,e^{jkl}e_{lkm}E^m = \frac{1}{c^2}\,\delta^j_m E^m = \frac{1}{c^2}E^j.\end{aligned}$$

The components of the divergence of the tensor $\widetilde{F}^{\mu\nu}$, which is a 4-vector, are clearly given now by the following expressions

$$\begin{aligned}\frac{\partial\widetilde{F}^{\mu i}}{\partial x^\mu} &= \frac{\partial\widetilde{F}^{ji}}{\partial x^j} + \frac{\partial\widetilde{F}^{4i}}{\partial x^4}\\ &= e^{jik}\frac{\partial B_k}{\partial x^j} + \frac{1}{c^2}\frac{\partial E^i}{\partial t}\\ &= -\left(e^{ijk}\frac{\partial B_k}{\partial x^j} - \frac{1}{c^2}\frac{\partial E^i}{\partial t}\right)\\ \frac{\partial\widetilde{F}^{\mu 4}}{\partial x^\mu} &= \frac{\partial\widetilde{F}^{i4}}{\partial x^i} = -\frac{1}{c^2}\frac{\partial E^i}{\partial x^i}.\end{aligned}$$

Consequently, the admitted forms of the constitutive relations transform the Maxwell equations into the form

$$\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = 0, \quad \frac{\partial\widetilde{F}^{\alpha\beta}}{\partial x^\alpha} = -\mu J^\beta. \qquad (11.6.17)$$

The existence of the constitutive relations make it possible to write the field equations (11.6.13) in much more interesting and meaningful forms. In order to simplify the necessary manipulations, we prefer to take $\mathbf{J} = \mathbf{0}$ and

$\rho = 0$. In that case, it is clear that the resolving functions must satisfy the relations

$$-\boldsymbol{\nabla}\times\boldsymbol{\Psi} = \epsilon\Big(\boldsymbol{\nabla}\phi - \frac{\partial\boldsymbol{\Phi}}{\partial t}\Big), \qquad (11.6.18)$$
$$\boldsymbol{\nabla}\times\boldsymbol{\Phi} = \mu\Big(\boldsymbol{\nabla}\psi - \frac{\partial\boldsymbol{\Psi}}{\partial t}\Big).$$

When we evaluate the divergences of these two equations, we are led, respectively, to the equations

$$\nabla^2\phi - \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}) = 0, \qquad (11.6.19)$$
$$\nabla^2\psi - \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\boldsymbol{\Psi}) = 0$$

where ∇^2 denotes the Laplace operator in the configuration manifold. Let us next evaluate the curls of the equations (11.6.18):

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Psi} = \epsilon\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{\Phi}),$$
$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Phi} = -\mu\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{\Psi}).$$

On recalling the well-known and easily verifiable following vectorial identity involving curl, gradient and divergence operators

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\Psi} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\Psi}) - \boldsymbol{\nabla}^2\boldsymbol{\Psi},$$

where $\boldsymbol{\nabla}^2$ denotes the Laplace operator on 3-vector fields, utilising the equations (11.6.18) and introducing again the constant $c^2 = 1/\epsilon\mu$, we easily find that

$$\boldsymbol{\nabla}^2\boldsymbol{\Psi} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\Psi}}{\partial t^2} = \boldsymbol{\nabla}f,$$
$$\boldsymbol{\nabla}^2\boldsymbol{\Phi} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\Phi}}{\partial t^2} = \boldsymbol{\nabla}g$$

where the functions f and g are given by

$$f = \boldsymbol{\nabla}\cdot\boldsymbol{\Psi} - \frac{1}{c^2}\frac{\partial\psi}{\partial t},$$
$$g = \boldsymbol{\nabla}\cdot\boldsymbol{\Phi} - \frac{1}{c^2}\frac{\partial\phi}{\partial t}.$$

Differentiating these functions with respect to time and making use of the equations (11.6.19), we obtain

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial f}{\partial t},$$
$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial g}{\partial t}.$$

We would like now to remove the arbitrariness in the selection of functions $(\boldsymbol{\Psi}, \psi)$ and $(\boldsymbol{\Phi}, \phi)$ by imposing that the ***Lorenz conditions*** $f = 0$ and $g = 0$ [after Danish physicist and mathematician Ludvig Valentin Lorenz (1829-1891)] should be satisfied. To this end, it suffices to choose

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Psi} - \frac{1}{c^2}\frac{\partial\psi}{\partial t} = 0,$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{\Phi} - \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0.$$

In this case, we can easily verify that all field quantities can be prescribed by considering only one pair of functions, say, for instance $(\boldsymbol{\Phi}, \phi)$. Indeed, (11.6.13) now takes the form

$$\mathbf{B} = \boldsymbol{\nabla}\times\boldsymbol{\Phi}, \qquad \mathbf{E} = \boldsymbol{\nabla}\phi - \frac{\partial\boldsymbol{\Phi}}{\partial t},$$
$$\mathbf{D} = \epsilon\Big(\boldsymbol{\nabla}\phi - \frac{\partial\boldsymbol{\Phi}}{\partial t}\Big), \quad \mathbf{H} = \frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{\Phi}.$$

These functions have to satisfy the scalar-valued and vector-valued wave equations

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0,$$
$$\boldsymbol{\nabla}^2\boldsymbol{\Phi} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\Phi}}{\partial t^2} = \mathbf{0}.$$

ϕ and $\boldsymbol{\Phi}$ are called, respectively, *scalar* and *vector potentials* of electromagnetic fields. The number c denotes naturally the velocity of propagation of electromagnetic waves in such a medium.

11.7. THERMODYNAMICS

Let us consider a thermodynamic system $\mathbb{T}$ occupying a finite region in $\mathbb{R}^3$. The variables that describe the behaviour of the system will be called as the ***state variables***. We distinguish three different sets of substate variables:

the external variables $\{q^i, i = 1, 2, \dots, n\}$, *the internal variables* $\{\nu^\alpha, \alpha = 1, 2, \dots, N\}$ and *the empirical temperature* or solely *the temperature* T. Thus, we may assume that the system $\mathbb{T}$ is incorporated in an $(n + N + 1)$-dimensional Euclidean manifold. The ***external agencies*** acting upon the system $\mathbb{T}$ is called the set of ***external forces*** $\{F_i(\mathbf{q}, \boldsymbol{\nu}, T), i = 1, 2, \dots, n\}$. The ***infinitesimal work*** that will occur during some infinitesimal changes in the external variables will be denoted by the 1-form

$$W = F_i(\mathbf{q}, \boldsymbol{\nu}, T)\, dq^i. \tag{11.7.1}$$

We adopt the convention that the signs of forces F_i are positive when the forces do work on the system $\mathbb{T}$ whereas are negative if the system $\mathbb{T}$ do work on the external agencies. We say that the external forces are *conservative* if they satisfy the relations

$$\frac{\partial F_i}{\partial q^j} = \frac{\partial F_j}{\partial q^i}.$$

In the present context, we shall consider this particular situation. In this case when we restrict the form W to the submanifold given by $\boldsymbol{\nu} = \mathit{constant}$, $T = \mathit{constant}$, we obtain

$$\begin{aligned} dW|_{\boldsymbol{\nu},T} &= F_{i,j}\, dq^j \wedge dq^i \\ &= \frac{1}{2}(F_{i,j} - F_{j,i}) dq^j \wedge dq^i = 0 \end{aligned}$$

where we employed the notation $F_{i,j} = \dfrac{\partial F_i}{\partial q^j}$. Hence the form $W|_{\boldsymbol{\nu},T}$ is closed and it is exact since the manifold is star-shaped so that we can write

$$W|_{\boldsymbol{\nu},T} = d\Psi|_{\boldsymbol{\nu},T}.$$

The function $\Psi(\mathbf{q}, \boldsymbol{\nu}, T)$ which we shall call *isothermal work function* can be evaluated as follows by the homotopy operator

$$\Psi(\mathbf{q}, \boldsymbol{\nu}, T) = \Psi_0(\boldsymbol{\nu}, T) + \int_0^1 (q^i - q^i_0)\, F_i\big[\mathbf{q}_0 + t(\mathbf{q} - \mathbf{q}_0), \boldsymbol{\nu}, T\big]\, dt \tag{11.7.2}$$

on resorting to Theorem 6.3.1(i) and the relation (6.3.1). $\mathbf{q}_0$ is an arbitrary point. From this relation, we immediately deduce that

$$F_i = \frac{\partial \Psi}{\partial q^i}. \tag{11.7.3}$$

Thus, the 1-form $d\Psi$ is now expressible as

$$\begin{aligned} d\Psi &= \frac{\partial \Psi}{\partial q^i}\, dq^i + \frac{\partial \Psi}{\partial \nu^\alpha}\, d\nu^\alpha + \frac{\partial \Psi}{\partial T}\, dT \\ &= F_i\, dq^i + N_\alpha d\nu^\alpha - N\, dT \end{aligned}$$

where we have defined

$$N_\alpha(\mathbf{q}, \boldsymbol{\nu}, T) = \frac{\partial \Psi}{\partial \nu^\alpha}, \quad N(\mathbf{q}, \boldsymbol{\nu}, T) = -\frac{\partial \Psi}{\partial T} \tag{11.7.4}$$

The trivial condition $d^2\Psi = 0$ then leads at once to the celebrated ***Maxwell reciprocity relations***

$$\begin{aligned} &\frac{\partial F_i}{\partial q^j} = \frac{\partial F_j}{\partial q^i}, \quad \frac{\partial F_i}{\partial \nu^\alpha} = \frac{\partial N_\alpha}{\partial q^i}, \quad \frac{\partial F_i}{\partial T} = -\frac{\partial N}{\partial q^i}, \\ &\frac{\partial N_\alpha}{\partial \nu^\beta} = \frac{\partial N_\beta}{\partial \nu^\alpha}, \quad \frac{\partial N_\alpha}{\partial T} = \frac{\partial N}{\partial \nu^\alpha}. \end{aligned}$$

Then, the work form (11.7.1) can be written in the following manner

$$W = d\Psi + N\, dT - N_\alpha d\nu^\alpha \tag{11.7.5}$$

whence we obtain

$$dW = dN \wedge dT - dN_\alpha \wedge d\nu^\alpha.$$

The expression (11.7.5) implies that if all components N_α do not vanish, then the Darboux class of the form W is at least 4. If the external forces are dependent on some of the internal variables, then the Maxwell reciprocity relations clearly show that at least some of the coefficients N_α should not vanish as a consequence of this property.

If we denote the heat input to the system $\mathbb{T}$ by the *heat* 1*-form* Q, then *the first law of thermodynamics* states that in quasistatic situations in which the time change of the system is so slow that its kinetic energy can be neglected, one can write

$$dE = W + Q$$

where $E(\mathbf{q}, \boldsymbol{\nu}, T)$ is called the ***internal energy function***. Therefore, when we make use of (11.7.5), we obtain

$$Q = d(E - \Psi) - N\, dT + N_\alpha d\nu^\alpha$$

and consequently

$$dQ = -dN \wedge dT + dN_\alpha \wedge d\nu^\alpha = -dW.$$

In his case, the Darboux class of the form Q is at least 4, that is, in view of Theorem 6.6.3 it does not possess the inaccessibility property. Hence, the form Q is in general not completely integrable and Theorem 5.13.4 requires that $Q \wedge dQ \neq 0$. On the other hand, we naturally find

$$dQ|_{\boldsymbol{\nu}} = -\,dN \wedge dT$$

on the submanifold $\boldsymbol{\nu} = constant$. Experimental information show that we cannot get our system occupying a state on the submanifold $\boldsymbol{\nu} = constant$ to reach to another state on the same submanifold without exchanging heat. In other words, two states on the submanifold $\boldsymbol{\nu} = constant$ cannot be connected by an *adiabatic path* in this manifold. This means that $dQ|_{\boldsymbol{\nu}} \neq 0$. Let us now discuss the condition under which the form $Q|_{\boldsymbol{\nu}}$ becomes completely integrable. The relation

$$Q|_{\boldsymbol{\nu}} \wedge dQ|_{\boldsymbol{\nu}} = 0$$

yields

$$d(E - \Psi) \wedge dN \wedge dT = 0$$

This, in turn, indicates that $E - \Psi$ is functionally dependent on variables N and T. Thus, we can write

$$E(\mathbf{q}, \boldsymbol{\nu}, T) = \Psi(\mathbf{q}, \boldsymbol{\nu}, T) + f\big(N(\mathbf{q}, \boldsymbol{\nu}, T), \boldsymbol{\nu}, T\big). \tag{11.7.6}$$

So we find that

$$Q = df(N, \boldsymbol{\nu}, T) - N\,dT + N_\alpha d\nu^\alpha. \tag{11.7.7}$$

Let us now consider two thermodynamic systems $\mathbb{T}_1$ and $\mathbb{T}_2$ having exactly the same temperature and internal variables. In this case, we can obviously write

$$N_1(\mathbf{q}_1, \boldsymbol{\nu}, T) = -\frac{\partial \Psi_1}{\partial T}, \quad N_2(\mathbf{q}_2, \boldsymbol{\nu}, T) = -\frac{\partial \Psi_2}{\partial T}.$$

If we let these two systems to interact, our physical experience tells us that the common isothermal work function should be expressed in the following fashion

$$\Psi_{12}(\mathbf{q}_1, \mathbf{q}_2, \boldsymbol{\nu}, T) = \Psi_1(\mathbf{q}_1, \boldsymbol{\nu}, T) + \Psi_2(\mathbf{q}_2, \boldsymbol{\nu}, T) + \psi(\mathbf{q}_1, \mathbf{q}_2, \boldsymbol{\nu}).$$

This relation implies that isothermal work function must be a *semi-additive* function (in thermodynamics, it is frequently taken $\psi = 0$. In such a case, Ψ will become a strictly additive function). This is tantamount that interaction forces in the composite system are independent of the temperature. It is now clear that we can write

$$N_{12} = -\frac{\partial \Psi_{12}}{\partial T} = -\frac{\partial \Psi_1}{\partial T} - \frac{\partial \Psi_2}{\partial T} = N_1 + N_2.$$

The composition rule of the internal energy will follow from the physical assumption that the function $E - \Psi$ is strictly additive and it is found as

$$E_{12}(\mathbf{q}_1, \mathbf{q}_2, \boldsymbol{\nu}, T) = E_1(\mathbf{q}_1, \boldsymbol{\nu}, T) + E_2(\mathbf{q}_2, \boldsymbol{\nu}, T) + \psi(\mathbf{q}_1, \mathbf{q}_2, \boldsymbol{\nu}).$$

Then, (11.7.6) provides the functional relation

$$\begin{aligned} f(N_{12}, \boldsymbol{\nu}, T) &= f(N_1 + N_2, \boldsymbol{\nu}, T) \\ &= f(N_1, \boldsymbol{\nu}, T) + f(N_2, \boldsymbol{\nu}, T) \end{aligned} \tag{11.7.8}$$

that must be held by the function f. With the definition $u = N_1 + N_2$, this relation leads to

$$\frac{\partial f}{\partial N_1} = \frac{\partial f}{\partial N_2} = \frac{\partial f}{\partial u}.$$

Since N_1 and N_2 are independent variables, we finally obtain

$$f(N, \boldsymbol{\nu}, T) = g(\boldsymbol{\nu}, T)N + g_1(\boldsymbol{\nu}, T).$$

However, (11.7.8) now implies that $g_1(\boldsymbol{\nu}, T) = 0$ and we thus reach to the conclusion

$$f(N, \boldsymbol{\nu}, T) = g(\boldsymbol{\nu}, T)N.$$

Consequently, we can write

$$\begin{aligned} &E(\mathbf{q}, \boldsymbol{\nu}, T) = \Psi(\mathbf{q}, \boldsymbol{\nu}, T) + g(\boldsymbol{\nu}, T)\, N(\mathbf{q}, \boldsymbol{\nu}, T) \\ &Q(\mathbf{q}, \boldsymbol{\nu}, T) = d\big[g(\boldsymbol{\nu}, T)\, N(\mathbf{q}, \boldsymbol{\nu}, T)\big] - N(\mathbf{q}, \boldsymbol{\nu}, T)\, dT + N_\alpha d\nu^\alpha. \end{aligned} \tag{11.7.9}$$

We shall now try to reduce these functional relations for E and Q into simpler forms. To this end, we want to introduce a ***thermodynamic temperature*** depending on the empirical temperature and internal variables by the following expression

$$\theta(\boldsymbol{\nu}, T) = \theta_0(\boldsymbol{\nu}) \exp\Big[\int_{T_0}^{T} \frac{d\tau}{g(\boldsymbol{\nu}, \tau)}\Big]. \tag{11.7.10}$$

Obviously, θ will satisfy the relation

$$\frac{\partial \theta}{\partial T} = \frac{\theta}{g}.$$

On the other hand, we can extract, in principle, from (11.7.10) the inverse function $T = T(\boldsymbol{\nu}, \theta)$ connecting the empirical temperature to the thermodynamic temperature. Let us now introduce the functions

$$\begin{aligned}\psi(\mathbf{q}, \boldsymbol{\nu}, \theta) &= \Psi\big(\mathbf{q}, \boldsymbol{\nu}, T(\boldsymbol{\nu}, \theta)\big),\\ e(\mathbf{q}, \boldsymbol{\nu}, \theta) &= E\big(\mathbf{q}, \boldsymbol{\nu}, T(\boldsymbol{\nu}, \theta)\big)\end{aligned}$$

and the quantity

$$\eta(\mathbf{q}, \boldsymbol{\nu}, \theta) = -\frac{\partial \psi}{\partial \theta}. \tag{11.7.11}$$

From the chain rule of differentiation, we obtain

$$N = -\frac{\partial \Psi}{\partial T} = -\frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial T} = \frac{\theta \eta}{g}$$

that yields the relation $gN = \theta\eta$. Then, the equation $(11.7.9)_2$ leads us to the expression

$$\begin{aligned}Q &= d(\theta\eta) - N dT + N_\alpha d\nu^\alpha\\ &= \theta\, d\eta + \eta\, d\theta - N dT + N_\alpha d\nu^\alpha\\ &= \theta\, d\eta + \eta\Big(\frac{\partial \theta}{\partial T}\, dT + \frac{\partial \theta}{\partial \nu^\alpha}\, d\nu^\alpha\Big) - \frac{\theta\eta}{g}\, dT + N_\alpha d\nu^\alpha.\end{aligned}$$

Let us now consider the expression

$$n_\alpha = \eta\, \frac{\partial \theta}{\partial \nu^\alpha} + N_\alpha.$$

From the chain rule again, we can write

$$N_\alpha = \frac{\partial \psi}{\partial \nu^\alpha} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial \nu^\alpha} = \frac{\partial \psi}{\partial \nu^\alpha} - \eta\, \frac{\partial \theta}{\partial \nu^\alpha}$$

so that we find

$$n_\alpha(\mathbf{q}, \boldsymbol{\nu}, \theta) = \frac{\partial \psi}{\partial \nu^\alpha}. \tag{11.7.12}$$

Thus the equations (11.7.9) take now the forms

$$\begin{aligned}e(\mathbf{q}, \boldsymbol{\nu}, \theta) &= \psi(\mathbf{q}, \boldsymbol{\nu}, \theta) + \theta\, \eta(\mathbf{q}, \boldsymbol{\nu}, \theta) \qquad (11.7.13)\\ &= \psi - \theta\frac{\partial \psi}{\partial \theta},\\ Q &= \theta\, d\eta + n_\alpha\, d\nu^\alpha.\end{aligned}$$

We call the function η as the ***entropy*** and the function ψ as the ***Helmholtz free energy function*** [after German mathematician, physicist and medical doctor Hermann Ludwig Ferdinand von Helmholtz (1821-1894)]. When we take $\boldsymbol{\nu} = constant$, it follows from $(11.7.13)_2$ that

$$Q|_{\boldsymbol{\nu}} = \theta\, d\eta$$

Hence, the heat 1-form Q is completely integrable in this case as it should be expected.

If we choose $\theta_0 > 0$ in (11.7.10), then the condition $\theta > 0$ is satisfied. We can then also write $\inf \theta = 0$. The temperature verifying this condition will be called the ***absolute temperature***. Moreover $(11.7.13)_1$ gives

$$c_{\,\mathbf{q},\boldsymbol{\nu}} = \frac{\partial e}{\partial \theta} = -\,\theta \frac{\partial^2 \psi}{\partial \theta^2}.$$

The quantity $c_{\,\mathbf{q},\boldsymbol{\nu}}$ is known as the ***specific heat*** under the restrictions $\mathbf{q} =$ *constant* and $\boldsymbol{\nu} =$ *constant*. We know that $c_{\,\mathbf{q},\boldsymbol{\nu}}$ is positive in real materials. This implies that the following inequality must be satisfied

$$\frac{\partial^2 \psi}{\partial \theta^2} < 0$$

due to the fact that $\theta > 0$.

XI. EXERCISES

11.1. Let (S, ω) be a symplectic manifold. Show that every vector field $V \in T(S)$ satisfying the condition $£_V\, \omega = 0$ is a Hamiltonian vector field.

11.2. The Hamiltonian function of the *Toda lattice* [after Japanese physicist Morikazu Toda (1917-2010)] involving three particles, which we encounter in the solid state physics, is given by the following function on the symplectic manifold $S = T^*(\mathbb{R}^3)$

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + e^{q^1 - q^2} + e^{q^2 - q^3} + e^{q^3 - q^1}.$$

(a) Write the Hamilton equations governing the motion of the system. (b) In order that the function $f \in \Lambda^0(S)$ be an integral of the motion, we know that the condition $\{f, H\} = 0$ should be satisfied. Searching for the integrals of the partial differential equation so obtained for the function f, show that the following functions are integrals of the motion

$$f_1(\mathbf{q}, \mathbf{p}) = H,$$
$$f_2(\mathbf{q}, \mathbf{p}) = p_1 + p_2 + p_3$$

$$f_3(\mathbf{q},\mathbf{p}) = \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + p_1\big(e^{q^1-q^2} + e^{q^3-q^1}\big)$$
$$+ p_2\big(e^{q^2-q^3} + e^{q^1-q^2}\big) + p_3\big(e^{q^3-q^1} + e^{q^2-q^3}\big).$$

11.3. Show that the function

$$g(\mathbf{q},\mathbf{p}) = p_1p_2p_3 - p_1e^{q^2-q^3} - p_2e^{q^3-q^1} - p_3e^{q^1-q^2}$$

is also an integral of the motion for the Toda lattice.

11.4. Is the transformation

$$Q = \log(q^{-1}\sin p), \quad P = q\cot p$$

canonical?

11.5. Determine the structure of the function $f(q^1,\dots,q^n)$ so that the mapping

$$Q^i = f(q^1,\dots,q^n)\sin p_i,$$
$$P_i = f(q^1,\dots,q^n)\cos p_i,\ \ i = 1,\dots,n$$

becomes a canonical transformation in the phase space.

11.6. Find the structure of the constant matrix $\mathbf{A} = [a_{ij}]$ so that the mapping

$$Q^i = q^i, \quad P_i = p_i + a_{ij}\,q^j$$

becomes a canonical transformation in the phase space. Determine the generating function $F_1(\mathbf{q},\mathbf{P})$ corresponding to this case.

11.7. We can build a symplectic structure in the non-conservative mechanics by introducing an *energy variable* E. Let us denote the coordinate cover in the manifold $\mathbb{R}^{2n+2}$ by $(\mathbf{q},\mathbf{p},E,t)$. We then define a symplectic form as follows

$$\omega = dq^i \wedge dp_i + dE \wedge dt.$$

Let $H(\mathbf{q},\mathbf{p},t)$ be the Hamiltonian function of the system. We next consider a function

$$P(\mathbf{q},\mathbf{p},E,t) = H(\mathbf{q},\mathbf{p},t) - E.$$

Show that along the integral curves of a Hamiltonian vector field $V_P \in T(\mathbb{R}^{2n+2})$ defined by the relation $\mathbf{i}_{V_P}\omega = dP$ and associated with the function P, the following equations are satisfied

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dE}{dt} = \frac{\partial H}{\partial t}.$$

11.8. Let the functions f and g be two integrals of the motion in a non-conservative system so that they satisfy the following equations

$$\frac{\partial f}{\partial t} + \{f,H\} = 0,$$

$$\frac{\partial g}{\partial t} + \{g, H\} = 0.$$

Show that the Poisson bracket $\{f, g\}$ is also an integral of the motion (this result is known as the ***Poisson theorem***).

11.9. Utilising the Poisson theorem prove that if the function $f(\mathbf{q}, \mathbf{p}, t)$ is an integral of the motion in a conservative system $\big(H = H(\mathbf{q}, \mathbf{p})\big)$, then all derivatives $\dfrac{\partial^n f}{\partial t^n}$, $n=1, 2, \dots$ are also integrals of the motion.

11.10. The energy balance in a perfect fluid can be expressed in the form

$$de = \theta\, d\eta - p\, d\nu$$

where p is the pressure, ν is the specific volume. Find the relations to which the exterior derivatives of the forms de and $\dfrac{de}{\theta}$ give rise under the following assumptions:

$$(a)\; e = e(\theta, \nu),\;\; p = p(\theta, \nu),$$
$$(b)\; \theta = \theta(\eta, \nu),\;\; p = p(\eta, \nu),$$
$$(c)\; \eta = \eta(\theta, \nu),\;\; p = p(\theta, \nu).$$