Eigenconfigurations of Tensors

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ABSTRACT. Square matrices represent linear self-maps of vector spaces, and their eigenpoints are the fixed points of the induced map on projective space. Likewise, polynomial self-maps of a projective space are represented by tensors. We study the configuration of fixed points of a tensor or symmetric tensor.

1. Introduction

Square matrices A with entries in a field K represent linear maps of vector spaces, say $K^n \to K^n$, and hence linear maps $\psi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ of projective spaces over K. If A is nonsingular then ψ is well-defined everywhere, and the eigenvectors of A correspond to the fixed points of ψ . The eigenconfiguration of A consists of n points in \mathbb{P}^{n-1} , provided A is generic and K is algebraically closed. Conversely, every spanning configuration of n points in \mathbb{P}^{n-1} arises as the eigenconfiguration of an $n \times n$ -matrix A. However, for special matrices A, we obtain multiplicities and eigenspaces of higher dimensions $[\mathbf{AE}]$. Moreover, if $K = \mathbb{R}$ and A is symmetric then its complex eigenconfiguration consists of real points only.

This paper concerns the extension from linear to non-linear maps. Their fixed points are the eigenvectors of tensors. The spectral theory of tensors was pioneered by Lim [Lim] and Qi [Qi]. It is now a much-studied topic in applied mathematics.

For instance, consider a quadratic map $\psi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$, with coordinates

(1.1)
$$\psi_i(x_1, \dots, x_n) = \sum_{j=1}^n \sum_{k=1}^n a_{ijk} x_j x_k$$
 for $i = 1, \dots, n$.

One organizes the coefficients of ψ into a tensor $A = (a_{ijk})$ of format $n \times n \times n$.

In what follows, we assume that $A = (a_{i_1i_2\cdots i_d})$ is a *d*-dimensional tensor of format $n \times n \times \cdots \times n$. The entries $a_{i_1i_2\cdots i_d}$ lie in an algebraically closed field K of characteristic zero, usually the complex numbers $K = \mathbb{C}$. Such a tensor $A \in (K^n)^{\otimes d}$ defines polynomial maps $K^n \to K^n$ and $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ just as in the formula (1.1):

$$\psi_i(x_1,\ldots,x_n) = \sum_{j_2=1}^n \sum_{j_3=1}^n \cdots \sum_{j_d=1}^n a_{ij_2j_3\cdots j_d} x_{j_2} x_{j_3} \cdots x_{j_d} \text{ for } i=1,\ldots,n.$$

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Thus each of the *n* coordinates of ψ is a homogeneous polynomial ψ_i of degree d-1 in $\mathbf{x} = (x_1, x_2, \dots, x_n)$. The eigenvectors of *A* are the solutions of the constraint

(1.2)
$$\operatorname{rank} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \psi_1(\mathbf{x}) & \psi_2(\mathbf{x}) & \cdots & \psi_n(\mathbf{x}) \end{pmatrix} \leq 1$$

The eigenconfiguration is the variety defined by the 2×2 -minors of this matrix. For a special tensor A, the ideal defined by (1.2) may not be radical, and in that case we can study its eigenscheme. Recent work in [AE] develops this for d = 2.

We note that every *n*-tuple (ψ_1, \ldots, ψ_n) of homogeneous polynomials of degree d-1 in *n* variables can be represented by some tensor *A* as above. This representation is not unique unless we require that *A* is symmetric in the last d-1 indices. Our maps $\psi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ are arbitrary polynomial dynamical system on projective space, in the sense of **[FS]**. Thus the study of eigenconfigurations of tensors is equivalent to the study of fixed-point configurations of polynomial maps.

Of most interest to us are symmetric tensors A, i.e. tensors whose entries $a_{i_1i_2\cdots i_d}$ are invariant under permuting the d indices. These are in bijection with homogeneous polynomials $\phi = \sum a_{i_1i_2\cdots i_d} x_{i_1} x_{i_2} \cdots x_{i_d}$, and we take $\psi_j = \partial \phi / \partial x_j$. The eigenvectors of a symmetric tensor correspond to fixed points of the gradient map $\nabla \phi : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1}$, and our object of study is the variety in \mathbb{P}^{n-1} defined by

(1.3)
$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \partial \phi / \partial x_1 & \partial \phi / \partial x_2 & \cdots & \partial \phi / \partial x_n \end{pmatrix} \leq 1.$$

This paper uses the term *eigenpoint* instead of eigenvector to stress that we work in \mathbb{P}^{n-1} . In our definition of eigenpoints we include the common zeros of ψ_1, \ldots, ψ_n . These are the points where the map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ is undefined. For a symmetric tensor ϕ , they are the singular points of the hypersurface $\{\phi = 0\}$ in \mathbb{P}^{n-1} . At those points the gradient $\nabla \phi$ vanishes so condition (1.3) holds.

EXAMPLE 1.1. Let n = d = 3 and $\phi = xyz$. The corresponding symmetric $3 \times 3 \times 3$ tensor A has six nonzero entries 1/6 and the other 21 entries are 0. Here $\nabla \phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $(x : y : z) \rightarrow (yz : xz : xy)$ is the classical *Cremona transformation*. This map has four fixed points, namely (1 : 1 : 1), (1 : 1 : -1), (1 : -1 : 1) and (-1 : 1 : 1). Also, the cubic curve $\{\phi = 0\}$ has the three singular points (1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1). In total, the tensor A has seven eigenpoints in \mathbb{P}^2 .

This paper is organized as follows. In Section 2 we count the number of eigenpoints, and we explore eigenconfigurations of Fermat polynomials, plane arrangements, and binary forms. Section 3 generalizes the fact that the left eigenvectors and right eigenvectors of a square matrix are distinct but compatible. We explore this compatibility for the *d* eigenconfigurations of a *d*-dimensional tensor with n = 2.

Section 4 concerns the *eigendiscriminant* of the polynomial system (1.2) and its variant in (2.2). This is the irreducible polynomial in the n^d unknowns $a_{i_1i_2\cdots i_d}$ which vanishes when two eigenpoints come together. We give a formula for its degree in terms of n, d and ℓ . Section 5 takes first steps towards characterizing eigenconfigurations among finite subsets of \mathbb{P}^{n-1} , starting with the case n = d = 3.

In Section 6 we focus on real tensors and their dynamics on real projective space $\mathbb{P}^{n-1}_{\mathbb{R}}$. We examine whether all complex eigenpoints can be real, and we use line arrangements to give an affirmative answer for n = 3. The paper concludes with a brief discussion of attractors for the dynamical systems $\psi : \mathbb{P}^{n-1}_{\mathbb{R}} \longrightarrow \mathbb{P}^{n-1}_{\mathbb{R}}$. These are also known as the robust eigenvectors of the tensor power method [AG, Rob].

2. The count and first examples

In this section we assume that the given tensor A is generic, meaning that it lies in a certain dense open subset in the space $(K^n)^{\otimes d}$ of all $n \times \cdots \times n$ -tensors. This set will be characterized in Section 4 as the nonvanishing locus of the eigendiscriminant.

THEOREM 2.1. The number of solutions in \mathbb{P}^{n-1} of the system (1.2) equals

(2.1)
$$\frac{(d-1)^n - 1}{d-2} = \sum_{i=0}^{n-1} (d-1)^i.$$

The same count holds for eigenconfigurations of symmetric tensors, given by (1.3). In the matrix case (d = 2) we use the formula on the right, which evaluates to n.

This result appeared in the tensor literature in [CS, OO], but it had already been known in complex dynamics due to Fornaess and Sibony [FS, Corollary 3.2]. We shall present two proofs of Theorem 2.1, cast in a slightly more general context.

For certain applications (e.g. in spectral hypergraph theory $[\mathbf{LQY}]$), it makes sense to focus on positive real numbers and to take the ℓ^{th} root after each iteration of the dynamical system ψ . This leads to the following generalization of our equations:

(2.2)
$$\operatorname{rank} \begin{pmatrix} x_1^{\ell} & x_2^{\ell} & \cdots & x_n^{\ell} \\ \psi_1(\mathbf{x}) & \psi_2(\mathbf{x}) & \cdots & \psi_n(\mathbf{x}) \end{pmatrix} \leq 1.$$

We refer to the solutions as the ℓ^{th} eigenpoints of the given tensor A. For $\ell = 1$, this is the definition in the Introduction. In the nomenclature devised by Qi [**CQZ**, **Qi**], one obtains *E*-eigenvectors for $\ell = 1$ and *Z*-eigenvectors for $\ell = d-1$. The subvariety of \mathbb{P}^{n-1} defined by (2.2) is called the ℓ^{th} eigenconfiguration of the tensor A.

THEOREM 2.2. The ℓ^{th} eigenconfiguration of a generic tensor A consists of

(2.3)
$$\frac{(d-1)^n - \ell^n}{d-1 - \ell} = \sum_{i=0}^{n-1} (d-1)^i \ell^{n-1-i}$$

distinct points in \mathbb{P}^{n-1} . If $\ell = d-1$ then the formula on the right is to be used.

PROOF. Consider the $2 \times n$ -matrix in (2.2). Its rows are filled with homogeneous polynomials in $S = K[x_1, \ldots, x_n]$ of degrees ℓ and m respectively, where the ψ_i are generic. Requiring this matrix to have rank ≤ 1 defines a subscheme of \mathbb{P}^{n-1} . By the Thom-Porteous-Giambelli formula [**Fu1**, §14.4], this scheme is zero-dimensional, and its length is given by the complete homogeneous symmetric polynomial of degree n-1 in the row degrees, ℓ and m. This is precisely (2.3) if we set m = d-1.

Another approach, which also shows that the scheme is reduced, is to use vector bundle techniques. Consider the $2 \times n$ -matrix as a graded S-module homomorphism from $S(-\ell) \oplus S(-m)$ to $S^{\oplus n}$. The quotient module Q of $S^{\oplus n}$ by the submodule generated by the first row $(x_1^{\ell}, \ldots, x_n^{\ell})$ is projective. In other words, the sheafification \widetilde{Q} of Q is locally free. The scheme associated with the $2 \times n$ -matrix can therefore be thought of as the zero scheme of a generic global section of $\widetilde{Q}(m)$. Since $\widetilde{Q}(m)$ is globally generated, the scheme is reduced [Ein, Lemma 2.5]. \Box

Here is a brief remark about *eigenvalues*. If $\mathbf{x} \in K^n$ is an ℓ^{th} eigenvector of A then there exists a scalar $\lambda \in K$ such that $\psi_i(\mathbf{x}) = \lambda x_i^{\ell}$ for all i. We call (\mathbf{x}, λ) an *eigenpair*. If this holds then $(\nu \mathbf{x}, \nu^{d-1-\ell} \lambda)$ is also an eigenpair for all $\nu \in K \setminus \{0\}$. Such equivalence classes of eigenpairs correspond to the ℓ^{th} eigenpoints in \mathbb{P}^{n-1} .

The case $\ell = d - 1$ is special because every eigenpoint has an associated eigenvalue. If $\ell \neq d - 1$ then eigenpoints make sense but eigenvalues are less meaningful.

PROOF OF THEOREM 2.1. The first statement is the $\ell = 1$ case of Theorem 2.2. For the second assertion, it suffices to exhibit one symmetric tensor ϕ that has the correct number of eigenpoints. We do this for the *Fermat polynomial*

(2.4)
$$\phi(\mathbf{x}) = x_1^d + x_2^d + \dots + x_n^d$$

According to (1.2), the eigenconfiguration of ϕ is the variety in \mathbb{P}^{n-1} defined by

(2.5)
$$\operatorname{rank} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^{d-1} & x_2^{d-1} & \cdots & x_n^{d-1} \end{pmatrix} \leq 1.$$

We follow $[\mathbf{Rob}]$ in characterizing all solutions \mathbf{x} in \mathbb{P}^{n-1} to the binomial equations

$$x_i x_j (x_i^{d-2} - x_j^{d-2}) = 0$$
 for $1 \le i < j \le n$.

For any non-empty subset $I \subseteq \{1, 2, ..., n\}$, there are $(d-2)^{|I|-1}$ solutions **x** with $\operatorname{supp}(\mathbf{x}) = \{i \mid x_i \neq 0\}$ equal to *I*. Indeed, we may assume $x_i = 1$ for the smallest index *i* in *I*, and the other values are arbitrary $(d-2)^{nd}$ roots of unity. In total,

$$\sum_{I} (d-2)^{|I|-1} = \sum_{i=1}^{n} \binom{n}{i} (d-2)^{i-1} = \frac{1}{d-2} \sum_{i=1}^{n} \binom{n}{i} (d-2)^{i} 1^{n-i} = \frac{(d-2+1)^{n}-1}{d-2}$$

This equals (2.1). Here we assume $d \ge 3$. The familiar matrix case is d = 2.

EXAMPLE 2.3. Let d = 4. For each I, there are $2^{|I|-1}$ eigenpoints, with $x_i = \pm 1$ for $i \in I$ and $x_j = 0$ for $j \notin I$. The total number of eigenpoints in \mathbb{P}^{n-1} is $(3^n-1)/2$.

We note that the argument in the proof of Theorem 2.1 does not work for $\ell \geq 2$. For instance, if $\ell = d - 1$ then every point in \mathbb{P}^{n-1} is an eigenpoint of the Fermat polynomial. At present we do not know an analogue to that polynomial for $\ell \geq 2$.

PROBLEM 2.4. Given any l, d and n, exhibit explicit polynomials $\phi(\mathbf{x})$ of degree d in n variables such that (2.2) has (2.3) distinct isolated solutions in \mathbb{P}^{n-1} .

We are looking for solutions with interesting combinatorial structure. In Section 6 we shall examine the case when $\phi(\mathbf{x})$ factors into linear factors, and we shall see how the geometry of hyperplane arrangements can be used to derive an answer. A first instance was the Cremona map in Example 1.1. Here is a second example.

EXAMPLE 2.5. For n = 4 the count of the eigenpoints in (2.1) gives $d^3 - 2d^2 + 2d$. We now fix d = 5, so this number equals 85. Consider the special symmetric tensor

$$\phi(\mathbf{x}) = x_1 x_2 x_3 x_4 (x_1 + x_2 + x_3 + x_4).$$

The surface defined by ϕ consists of five planes in \mathbb{P}^3 . These intersect pairwise in ten lines. Each point on such a line is an eigenpoint because it is singular on the surface. Furthermore, there are 15 isolated eigenpoints; these have real coordinates:

$$(2:2:-1:-1), (2:-1:2:-1), (2:-1:-1:2), (-1:2:2:-1), (2:0) = (-1:2:-1:2), (-1:-1:2:2), (1:1:1:1), (\frac{1}{2}(5\pm\sqrt{13}):1:1), (1:1:\frac{1}{2}(5\pm\sqrt{13}):1:1), (1:1:\frac{1}{2}(5\pm\sqrt{13}):1), (1:1:\frac{1}{2}(5\pm\sqrt{13}):1), (1:1:\frac{1}{2}(5\pm\sqrt{13}):1), (1:1:\frac{1}{2}(5\pm\sqrt{13})), (1:1:\frac{1}{2}(5\pm\sqrt{$$

The five planes divide $\mathbb{P}^3_{\mathbb{R}}$ into 15 regions. Each region contains one point in (2.6).

Now, take a generic quintic $\phi'(\mathbf{x})$ in $\mathbb{R}[x_1, x_2, x_3, x_4]$, and consider the eigenconfiguration of $\phi(\mathbf{x}) + \epsilon \phi'(\mathbf{x})$. This consists of 85 points in \mathbb{P}^3 . These are algebraic functions of ϵ . For $\epsilon > 0$ small, we find 15 real eigenpoints near (2.6). The other 70 eigenpoints arise from the 10 lines. How many are real depends on the choice of ϕ' .

The situation is easier for n = 2, when the tensor A has format $2 \times 2 \times \cdots \times 2$. It determines two binary forms ψ_1 and ψ_2 . The eigenpoints of A are defined by

(2.7)
$$y \cdot \psi_1(x, y) - x \cdot \psi_2(x, y) = 0.$$

This is a binary form of degree d, so it has d zeros in \mathbb{P}^1 , as predicted by (2.1). Conversely, every binary form of degree d can be written as $y\psi_1 - x\psi_2$. This implies:

REMARK 2.6. Every set of d points in \mathbb{P}^1 is the eigenconfiguration of a tensor.

The discussion is more interesting when we restrict ourselves to symmetric tensors. These correspond to binary forms $\phi(x, y)$ and their eigenpoints are defined by

$$y \cdot \frac{\partial \phi}{\partial x} - x \cdot \frac{\partial \phi}{\partial y} = 0.$$

The matrix case (d = 2) shows that Remark 2.6 cannot hold as stated for symmetric tensors. Indeed, if $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $\phi = ax^2 + 2bxy + cy^2$ then $\frac{1}{2}(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}) =$ $-bx^2 + (a-c)xy + by^2$. This confirms the familiar facts that the two eigenpoints $(u_1:v_1)$ and $(u_2:v_2)$ are real when $a, b, c \in \mathbb{R}$ and they satisfy $u_1u_2 + v_1v_2 = 0$. The following result generalizes the second fact from symmetric matrices to tensors.

THEOREM 2.7. A set of d points $(u_i : v_i)$ in \mathbb{P}^1 is the eigenconfiguration of a symmetric tensor if and only if either d is odd, or d is even and the operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{d/2} \text{ annihilates the corresponding binary form } \prod_{i=1}^d (v_i x - u_i y).$$

PROOF. The only-if direction follows from the observation that the Laplace

operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ commutes with the vector field $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$. Hence, for any ϕ of degree d, we obtain zero when $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ gets applied d/2 times to $y \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial y}$. For the if direction, we examine the $(d + 1) \times (d + 1)$ -matrix that represents the endomorphism $\phi \mapsto y \frac{\partial \phi}{\partial x} - x \frac{\partial \phi}{\partial y}$ on the space of binary forms of degree d. This matrix is invertible when d is odd, and its kernel is one-dimensional when d is even. Hence the map is surjective when d is odd, and it maps onto a hyperplane when dis even. The only-if part shows that this hyperplane equals $\left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial u^2}\right)^{d/2}\right)^{\perp}$. \Box

After completion of our manuscript we learned that Theorem 2.7 was also found independently by Mauro Maccioni, as part of his PhD dissertation at Firenze, Italy.

EXAMPLE 2.8 (d = 4). Four points $(u_1:v_1), (u_2:v_2), (u_3:v_3), (u_4:v_4)$ on the line \mathbb{P}^1 arise as the eigenconfiguration of a symmetric $2 \times 2 \times 2 \times 2$ -tensor if and only if

 $3u_1u_2u_3u_4 + u_1u_2v_3v_4 + u_1u_3v_2v_4 + u_1u_4v_2v_3 + \dots + u_3u_4v_1v_2 + 3v_1v_2v_3v_4 = 0.$

This equation generalizes the orthogonality of the two eigenvectors of a symmetric 2×2 -matrix. For instance, the columns of $U = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}$ represent the eigenconfiguration of a symmetric $2 \times 2 \times 2 \times 2$ -tensor, but this does not hold for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} U$. Example 2.8 underscores the fact that the constraints on eigenconfigurations of symmetric tensors A are not invariant under projective transformations. They are only invariant under the orthogonal group O(n), like the Laplace operator in Theorem 2.7. By contrast, the constraints on eigenconfigurations of general (non-symmetric) tensors, such as Theorem 5.1, will be properties of projective geometry.

We are familiar with this issue from comparing the eigenconfigurations of real symmetric matrices with those of general square matrices. These are respectively the O(n)-orbit and the GL(n)-orbit of the standard coordinate basis.

3. Compatibility of eigenconfigurations

When defining the eigenvectors of a tensor A, the symmetry was broken by fixing the first index and summing over the last d-1 indices. There is nothing special about the first index. For any $k \in \{1, \ldots, d\}$ we can regard A as the self-map

$$\psi^{[k]}: \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$$

whose i^{th} coordinate is the following homogeneous polynomial of degree d-1 in **x**:

$$\psi_i^{[k]}(\mathbf{x}) = \sum_{j_1=1}^n \cdots \sum_{j_{k-1}=1}^n \sum_{j_{k+1}=1}^n \cdots \sum_{j_d=1}^n a_{j_1 \cdots j_{k-1} i j_{k+1} \cdots j_d} x_{j_1} \cdots x_{j_{k-1}} x_{j_{k+1}} \cdots x_{j_d}.$$

Let $\operatorname{Eig}^{[k]}(A)$ denote the subvariety of \mathbb{P}^{n-1} consisting of the fixed points of $\psi^{[k]}$. For a generic tensor A, this is a finite set of points in \mathbb{P}^{n-1} of cardinality

$$D = \frac{(d-1)^n - 1}{d-2} = \#(\operatorname{Eig}^{[k]}(A)) \text{ for } d \ge 3.$$

This raises the following question: Suppose we are given d configurations, each consisting of D points in \mathbb{P}^{n-1} , and known to be the eigenconfiguration of some tensor. Under what condition do they come from the same tensor A?

We begin to address this question by considering the case of matrices (d = 2), where D = n. Our question is as follows: given an $n \times n$ -matrix A, what is the relationship between the left eigenvectors and the right eigenvectors of A?

PROPOSITION 3.1. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be two spanning subsets of \mathbb{P}^{n-1} . These arise as the left and right eigenconfigurations of some $n \times n$ matrix A if and only if, up to relabeling, the dot products of vectors corresponding to \mathbf{w}_i and \mathbf{v}_i are zero whenever $i \neq j$.

PROOF. Let V be a square matrix whose columns are the eigenvectors of A. Then the columns of $(V^{-1})^T$ form a basis of eigenvectors for A^T .

The condition in Proposition 3.1 defines an irreducible variety, denoted $\text{EC}_{n,2}$ and called the *eigencompatibility variety* for $n \times n$ -matrices. It lives naturally in the space of pairs of unordered configurations of n points in \mathbb{P}^{n-1} . In symbols,

(3.1)
$$\operatorname{EC}_{n,2} \subset \operatorname{Sym}_n(\mathbb{P}^{n-1}) \times \operatorname{Sym}_n(\mathbb{P}^{n-1}).$$

It has middle dimension n(n-1), and it maps birationally onto either factor. We may identify $\operatorname{Sym}_n(\mathbb{P}^{n-1})$ with the Chow variety of products of n linear forms in n variables. Here, each configuration $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is represented by $\prod_{i=1}^n (\mathbf{v}_i \cdot \mathbf{x})$. The coefficients of this homogeneous polynomial serve as coordinates on $\operatorname{Sym}_n(\mathbb{P}^{n-1})$. It would be worthwhile to express Proposition 3.1 in these coordinates.

EXAMPLE 3.2 (n = 2). The eigencompatibility variety EC_{2,2} for 2×2-matrices is a surface in $(\text{Sym}_2(\mathbb{P}^1))^2$. This ambient space equals $(\mathbb{P}^2)^2$, by representing a pair of unlabeled points on the line \mathbb{P}^1 with the binary quadric that defines it. To be precise, a point $((u_0:u_1:u_2), (v_0:v_1:v_2))$ in $(\mathbb{P}^2)^2$ is identified with the binary forms

$$f(s,t) = u_0 s^2 + u_1 s t + u_2 t^2$$
 and $g(s,t) = v_0 s^2 + v_1 s t + v_2 t^2$.

We want the zeros of f(s,t) and g(s,t) to be the right and left eigenconfigurations of the same 2×2 -matrix. Proposition 3.1 tells us that this is equivalent to

$$f(s,t) = \lambda(as+bt)(cs+dt)$$
 and $g(s,t) = \mu(bs-at)(ds-ct)$.

By eliminating the parameters a, b, c, d, λ , and μ , we find that the surface $EC_{2,2}$ is essentially the diagonal in $(\mathbb{P}^2)^2$. It is defined by the determinantal condition

(3.2)
$$\operatorname{rank}\left(\begin{array}{cc} u_0 & u_1 & u_2 \\ v_2 & -v_1 & v_0 \end{array}\right) \leq 1$$

Our aim in this section is to generalize this implicit representation of $EC_{2,2}$.

Let $\operatorname{EC}_{n,d}$ denote the eigencompatibility variety of *d*-dimensional tensors of format $n \times n \times \cdots \times n$. This is defined as follows. Every generic tensor *A* has *d* eigenconfigurations. The *eigenconfiguration with index k* of the tensor *A* is the fixed locus of the map $\psi^{[k]}$. Each configuration is a set of unlabeled *D* points in \mathbb{P}^{n-1} , which we regard as a point in $\operatorname{Sym}_D(\mathbb{P}^{n-1})$. The *d*-tuples of eigenconfigurations, one for each index *k*, parametrize

(3.3)
$$\operatorname{EC}_{n,d} \subset \left(\operatorname{Sym}_D(\mathbb{P}^{n-1})\right)^d$$
.

Thus $EC_{n,d}$ is the closure of the locus of *d*-tuples of eigenconfigurations of tensors.

Already the case of binary tensors (n = 2) is quite interesting. We shall summarize what we know about this. Let A be a tensor of format $2 \times 2 \times \cdots \times 2$, with d factors. Each of its d eigenconfigurations consists of D = d points on the line \mathbb{P}^1 . The symmetric power $\operatorname{Sym}_d(\mathbb{P}^1)$ is identified with the \mathbb{P}^d of binary forms of degree d. The zeros of such a binary form is an unlabeled configuration of d points in \mathbb{P}^1 . Thus, the eigencompatibility variety for binary tensors is a subvariety

$$\mathrm{EC}_{2,d} \subset (\mathbb{P}^d)^d$$

The case d = 2 was described in Example 3.2. Here are the next few cases.

EXAMPLE 3.3 (d = 3). Points in $(\mathbb{P}^3)^3$ are triples of binary cubics

$$\begin{array}{rcl} f(s,t) &=& u_0 s^3 + u_1 s^2 t + u_2 s t^2 + u_3 t^3, \\ g(s,t) &=& v_0 s^3 + v_1 s^2 t + v_2 s t^2 + v_3 t^3, \\ h(s,t) &=& w_0 s^3 + w_1 s^2 t + w_2 s t^2 + w_3 t^3, \end{array}$$

where two binary cubics are identified if they differ by a scalar multiple. The three eigenconfigurations of a $2 \times 2 \times 2$ -tensor $A = (a_{ijk})$ are defined by the binary cubics

$$\begin{array}{lll} f(s,t) &=& \lambda \cdot \left(a_{211}s^3 - (a_{111} - a_{212} - a_{221})s^2t + (a_{222} - a_{112} - a_{121})st^2 - a_{122}t^3\right) \\ g(s,t) &=& \mu \cdot \left(a_{121}s^3 - (a_{111} - a_{122} - a_{221})s^2t + (a_{222} - a_{112} - a_{211})st^2 - a_{212}t^3\right) \\ h(s,t) &=& \nu \cdot \left(a_{112}s^3 - (a_{111} - a_{122} - a_{212})s^2t + (a_{222} - a_{121} - a_{211})st^2 - a_{221}t^3\right) \end{array}$$

Our task is to eliminate the 11 parameters a_{ijk} and λ, μ, ν from these formulas. Geometrically, our variety EC_{2.3} is represented as the image of a rational map

(3.4)
$$\mathbb{P}^7 \dashrightarrow (\mathbb{P}^3)^3, A \mapsto (f, g, h).$$

This is linear in the coefficients a_{ijk} of A and maps the tensor to a triple of binary forms. To characterize the image of (3.4), in Theorem 3.6 we introduce the matrix

(3.5)
$$\mathbf{E}_{3} = \begin{pmatrix} u_{1} - u_{3} & u_{1} - u_{3} & u_{0} - u_{2} & u_{0} - u_{2} \\ v_{1} - v_{3} & 0 & v_{0} - v_{2} & 0 \\ 0 & w_{1} - w_{3} & 0 & w_{0} - w_{2} \end{pmatrix}$$

Let I be the ideal generated by the 3×3 -minors of \mathbf{E}_3 . Its zero set has the eigencompatibility variety $\mathrm{EC}_{2,3}$ as an irreducible component. There are also three extraneous irreducible components, given by the rows of the matrix:

$$I_1 = \langle u_0 - u_2, u_1 - u_3 \rangle, \ I_2 = \langle v_0 - v_2, v_1 - v_3 \rangle, \ \text{and} \ I_3 = \langle w_0 - w_2, w_1 - w_3 \rangle.$$

The homogeneous prime ideal of $EC_{2,3}$ is found to be the ideal quotient

(3.6)
$$(I:I_1I_2I_3) = \langle 2 \times 2 \text{-minors of } \begin{pmatrix} u_0 - u_2 & v_0 - v_2 & w_0 - w_2 \\ u_1 - u_3 & v_1 - v_3 & w_1 - w_3 \end{pmatrix} \rangle.$$

We conclude that the eigencompatibility variety $EC_{2,3}$ has codimension 2 in $(\mathbb{P}^3)^3$.

EXAMPLE 3.4 (d = 4). Points in $(\mathbb{P}^4)^4$ are quadruples of binary quartics

$$u_0s^4 + u_1s^3t + u_2s^2t^2 + u_3st^3 + u_4t^4, v_0s^4 + v_1s^3t + v_2s^2t^2 + v_3st^3 + v_4t^4, w_0s^4 + w_1s^3t + w_2s^2t^2 + w_3st^3 + w_4t^4, x_0s^4 + x_1s^3t + x_2s^2t^2 + x_3st^3 + x_4t^4.$$

One can represent the homogeneous ideal of the eigencompatibility variety $EC_{2,4}$ in a similar way to Example 3.3. Let *I* be the ideal generated by the 4×4 -minors of

$$(3.7) \begin{bmatrix} u_1 - u_3 & u_1 - u_3 & u_2 - u_0 + u_4 & u_2 - u_0 - u_4 & 2u_0 - u_2 + 2u_4 & 3u_0 - 2u_2 + 3u_4 \\ v_3 - v_1 & 0 & 0 & v_0 - v_2 + v_4 & 0 & v_2 & v_2 \\ 0 & w_3 - w_1 & 0 & 0 & w_0 - w_2 + w_4 & w_2 & w_2 \\ 0 & 0 & x_3 - x_1 & 0 & 0 & x_0 + x_4 & x_2 \end{bmatrix}$$

Let I_{ij} be the ideal generated by the 2×2-minors of the 2×7-submatrix consisting of the i^{th} and j^{th} rows in (3.7). The homogeneous prime ideal of EC_{2,4} $\subset (\mathbb{P}^4)^4$ is obtained as the ideal quotient $(I : I_{12}I_{13}I_{14}I_{23}I_{24}I_{34})$. We obtain dim $(\text{EC}_{2,4}) = 12$.

EXAMPLE 3.5 (d = 5). The eigencompatibility variety EC_{2,5} has codimension 4 in $(\mathbb{P}^5)^5$, so dim $(\text{EC}_{2,5}) = 21$. We represent this variety by the 5 × 8-matrix

$$(3.8) \begin{bmatrix} -u_1 + u_3 - u_5 & v_1 - v_3 + v_5 & 0 & 0 & 0 \\ u_1 - u_3 + u_5 & 0 & w_1 - w_3 + w_5 & 0 & 0 \\ -u_1 + u_3 - u_5 & 0 & 0 & x_1 - x_3 + x_5 & 0 \\ -u_1 + u_3 - u_5 & 0 & 0 & 0 & y_1 - y_3 + y_5 \\ u_0 - u_2 + u_4 & 0 & w_0 - w_2 + w_4 & 0 & 0 \\ 0 & v_0 - v_2 + v_4 & w_0 - w_2 + w_4 & 0 & 0 \\ 0 & 0 & w_0 - w_2 + w_4 & x_0 - x_2 + x_4 & 0 \\ 0 & 0 & w_0 - w_2 + w_4 & 0 & y_0 - y_2 + y_4 \end{bmatrix}^T$$

As before, the variety of maximal minors of this 5×8 -matrix has multiple components. Our variety $EC_{2,5}$ is the main component, obtained by taking the ideal quotient by determinantal ideals that are given by proper subsets of the rows.

In what follows we derive a general result for binary tensors. This will explain the origin of the matrices (3.5), (3.7) and (3.8) that were used to represent EC_{2,d}.

Fix $V = K^n$. Tensors A live in the space $V^{\otimes d}$. For each k, the map $A \mapsto \psi^{[k]}$ factors through the linear map that symmetrizes the factors indexed by $[d] \setminus \{k\}$:

$$(3.9) V^{\otimes d} \longrightarrow \operatorname{Sym}_{d-1}(V) \otimes V$$

where $\{e_1, \ldots, e_n\}$ is a basis for V. Taking the wedge product with (x_1, x_2, \ldots, x_n) defines a further linear map

Write $\ell^{[k]}$ for the composition of (3.10) after (3.9). Thus $\ell^{[k]}(A)$ is a vector of length $\binom{n}{2}$ whose entries are polynomials of degree d that define the eigenconfiguration with index k. For instance, in Example 3.3, $f = \ell^{[1]}(A)$, $g = \ell^{[2]}(A)$, $h = \ell^{[3]}(A)$.

The kernel of $\ell^{[k]}$ consists of all tensors whose eigenconfiguration with index k is all of \mathbb{P}^{n-1} . We are interested in the space of tensors where this happens simultaneously for all indices k:

(3.11)
$$K_{n,d} = \bigcap_{k=1}^{d} \operatorname{kernel}(\ell^{[k]}).$$

The tensors in $K_{n,d}$ can be regarded as being trivial as far as eigenvectors are concerned. For instance, in the classical matrix case (d = 2), we have

$$K_{n,2} = \operatorname{kernel}(\ell^{[1]}) = \operatorname{kernel}(\ell^{[2]}),$$

and this is the 1-dimensional space spanned by the identity matrix.

In what follows we restrict our attention to binary tensors (n = 2). We regard $\ell^{[k]}$ as a linear map $\mathbb{P}^{2^d-1} \dashrightarrow \mathbb{P}^d$. The eigencompatibility variety $\mathrm{EC}_{2,d}$ is the closure of the image of the map $\mathbb{P}^{2^d-1} \dashrightarrow (\mathbb{P}^d)^d$ given by the tuple $(\ell^{[1]}, \ldots, \ell^{[d]})$. Let $\mathbf{u}^{[k]}$ be a column vector of unknowns representing points in the k^{th} factor \mathbb{P}^d .

THEOREM 3.6. There exists a $d \times e$ -matrix \mathbf{E}_d with $e = \dim(K_{2,d}) + d(d+1) - 2^d$, whose entries in the k^{th} row are \mathbb{Z} -linear forms in $\mathbf{u}^{[k]}$, such that $\mathrm{EC}_{2,d}$ is an irreducible component in the variety defined by the $d \times d$ -minors of \mathbf{E}_d . Its ideal is obtained from those $d \times d$ -minors by taking the ideal quotient (or saturation) with respect to the maximal minor ideals of proper subsets of the rows of \mathbf{E}_d .

PROOF. We shall derive this using the linear algebra method in [AST, §2]. We express $\ell^{[k]}$ as a $(d+1) \times 2^d$ -matrix, and we form the $d(d+1) \times (2^d+d)$ -matrix

(3.12)
$$\begin{pmatrix} \ell^{[1]} & \mathbf{u}^{[1]} & 0 & \cdots & 0\\ \ell^{[2]} & 0 & \mathbf{u}^{[2]} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \ell^{[d]} & 0 & 0 & \cdots & \mathbf{u}^{[d]} \end{pmatrix}$$

The left $d(d+1) \times 2^d$ -submatrix has entries in $\{-1, 0, +1\}$ and its kernel is $K_{2,d}$. The rank of that submatrix is $r = 2^d - \dim(K_{2,d})$. Using row operations, we can transform (3.12) into a matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A is an $r \times 2^d$ matrix of rank r, and C is an $e \times d$ -matrix whose k^{th} column has linear entries in the coordinates of $\mathbf{u}^{[k]}$. The variety $\text{EC}_{2,d}$ is the set of all points $(\mathbf{u}^{[1]}, \mathbf{u}^{[2]}, \ldots, \mathbf{u}^{[d]})$ in $(\mathbb{P}^d)^d$ such that the kernel of (3.12) contains a vector whose last d coordinates are non-zero. Equivalently, the kernel of C contains a vector whose d coordinates are all non-zero.

Let \mathbf{E}_d be the transpose of C. This is a $d \times e$ -matrix whose k^{th} row has entries that are \mathbb{Z} -linear in $\mathbf{u}^{[k]}$. By construction, $\text{EC}_{2,d}$ is the set of points $(\mathbf{u}^{[1]}, \ldots, \mathbf{u}^{[d]})$ in $(\mathbb{P}^d)^d$ such that $\mathbf{v} \cdot \mathbf{E}_d = 0$ for some $\mathbf{v} \in (K \setminus \{0\})^d$. This completes the proof. \Box

By our matrix representation, the codimension of $EC_{2,d}$ is at most e - d + 1, so

(3.13)
$$\dim(\mathrm{EC}_{2,d}) \geq d^2 - (e - d + 1).$$

Examples 3.2, 3.3, and 3.4 suggest that (3.13) is an equality.

CONJECTURE 3.7. The dimension of $EC_{2,d}$ equals $d^2 - (e - d + 1)$.

We do not know the dimension of $K_{2,d} = \bigcap_k \operatorname{kernel}(\ell^{[k]})$. In our examples, we saw that $\dim(K_{2,d}) = 1, 0, 3, 10$ for d = 2, 3, 4, 5 respectively. It would be desirable to better understand the common kernel $K_{n,d}$ for arbitrary n and d:

PROBLEM 3.8. Find the dimension of the space $K_{n,d}$ in (3.11).

Another problem is to understand the diagonal of $\text{EC}_{n,d}$ in the embedding (3.3). This diagonal parametrizes *simultaneous eigenconfigurations*, arising from special tensors A whose d maps $\psi^{[1]}, \ldots, \psi^{[d]}$ all have the same fixed point locus in \mathbb{P}^{n-1} . Symmetric tensors A have this property, and the issue is to characterize all others.

EXAMPLE 3.9 (n = 2). The diagonal of $\text{EC}_{2,d}$ is computed by setting $\mathbf{u}^{[1]} = \mathbf{u}^{[2]} = \cdots = \mathbf{u}^{[d]}$ in the prime ideal described in Theorem 3.6. If d is odd, then there is no constraint, by Theorem 2.7. However, for d even, the diagonal of $\text{EC}_{2,d}$ is interesting. For instance, for d = 2, equating the rows in (3.2) gives two components

$$\left\langle 2 \times 2 \text{-minors of } \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & -a_1 & a_0 \end{pmatrix} \right\rangle = \langle a_0 + a_2 \rangle \cap \langle a_0 - a_2, a_1 \rangle.$$

The first component is the known case of symmetric 2×2 -matrices. The second component is a point in $\operatorname{Sym}_2(\mathbb{P}^1)$, namely the binary form $s^2 + t^2 = (s - it)(s + it)$. This is the simultaneous eigenconfiguration of any matrix $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $b \neq 0$.

4. The eigendiscriminant

The *d*-dimensional tensors of format $n \times n \times \cdots \times n$ represent points in a projective space \mathbb{P}^N where $N = n^d - 1$. For a generic tensor $A \in \mathbb{P}^N$, the ℓ^{th} eigenconfiguration, in the sense of (2.2), consists of a finite set of reduced points in \mathbb{P}^{n-1} . We know from Theorem 2.2 that the number of these points equals

$$\rho(n, d, \ell) = \sum_{i=0}^{n-1} (d-1)^i \ell^{n-1-i}.$$

In this section we study the set $\Delta_{n,d,\ell}$ of all tensors A for which the eigenconfiguration consists of fewer than $\rho(n,d,\ell)$ points or is not zero-dimensional. This set is a subvariety of \mathbb{P}^N , called the ℓ^{th} eigendiscriminant. We also abbreviate

(4.1)
$$\gamma(n,d,\ell) = \sum_{j=2}^{n-1} (-1)^{n-1+j} (j-1) \left[\sum_{k=0}^{j} (-1)^k \binom{n}{j-k} \binom{j(d-1)-k\ell-1}{n-1} \right].$$

The following is our main result in this section:

THEOREM 4.1. The ℓ^{th} eigendiscriminant is an irreducible hypersurface with

(4.2)
$$\operatorname{degree}(\Delta_{n,d,\ell}) = 2\gamma(n,d,\ell) + 2\rho(n,d,\ell) - 2\lambda$$

We identify $\Delta_{n,d,\ell}$ with the unique (up to sign) irreducible polynomial with integer coefficients in the n^d unknowns $a_{i_1i_2\cdots i_d}$ that vanishes on this hypersurface. From now on we use the term eigendisciminant to refer to the polynomial $\Delta_{n,d,\ell}$.

The case of most interest is $\ell = 1$, which pertains to the eigenconfiguration of a tensor in the usual sense of (1.2). For that case, we write $\Delta_{n,d} = \Delta_{n,d,1}$ for the eigendiscriminant, and the formula for its degree can be simplified as follows:

COROLLARY 4.2. The eigendiscriminant is a homogeneous polynomial of degree

$$\operatorname{degree}(\Delta_{n,d}) = n(n-1)(d-1)^{n-1}.$$

The following proof is due to Manuel Kauers. We are grateful for his help.

PROOF. We set $\ell = 1$ in the expression (4.2). Our claim is equivalent to

(4.3)
$$\gamma(n,d,1) = \binom{n}{2}(d-1)^{n-1} - \frac{(d-1)^n - 1}{d-2} + 1.$$

We abbreviate the innermost summand in (4.1) as

$$s_{n,d,j}(k) = (-1)^k \binom{n}{j-k} \binom{j(d-1)-k-1}{n-1}$$

Using Gosper's algorithm [PWZ, Chapter 5], we find the multiple

$$S_{n,d,j}(k) := \frac{(j(d-1)-k)(j-k-n)}{nj(d-2)} s_{n,d,j}(k).$$

It can now be checked by hand that this satisfies

$$S_{n,d,j}(k+1) - S_{n,d,j}(k) = s_{n,d,j}(k).$$

Summing over the range k = 0, ..., j - 1 and simplifying expressions lead to

(4.4)
$$\sum_{k=0}^{j} s_{n,d,j}(k) = S_{n,d,j}(j+1) - S_{n,d,j}(0) = \frac{d-1}{d-2} \binom{n-1}{j} \binom{j(d-1)-1}{n-1}.$$

This is valid for all $j \ge 1$.

Next we introduce the expression

(4.5)
$$A(n,d) = \sum_{j=0}^{n} (-1)^{j} {n \choose j} {dj-1 \choose n} (j-1)$$

Consider $\binom{d j-1}{n}(j-1)$ as a polynomial in j of degree n+1. In the binomial basis,

$$\binom{d\,j-1}{n}(j-1) = (n+1)d^n\binom{j}{n+1} + \left((d-1)d^{n-1}\binom{n+1}{2} - d^n\right)\binom{j}{n} + \text{lower degree terms,}$$

Recall from [GKP, page 190] that

$$\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \binom{j}{k} = \begin{cases} (-1)^{n} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

This implies

$$A(n,d) = (-1)^n \left((d-1)d^{n-1} \binom{n+1}{2} - d^n \right).$$

Combining this identity with (4.1), (4.4) and (4.5), we now derive

$$\begin{split} \gamma(n,d,1) &= (-1)^{n-1} \frac{d-1}{d-2} \bigg(A(n-1,d-1) + \binom{-1}{n-1} \bigg) \\ &= (-1)^{n-1} \frac{d-1}{d-2} \bigg((-1)^{n-1} \bigg((d-2)(d-1)^{n-2} \binom{n}{2} - (d-1)^{n-1} \bigg) + (-1)^{n-1} \bigg) \\ &= \frac{d-1}{d-2} \bigg((d-2)(d-1)^{n-2} \binom{n}{2} - (d-1)^{n-1} \bigg) + \frac{d-1}{d-2} \\ &= \binom{n}{2} (d-1)^n - \frac{(d-1)^n - (d-1)}{d-2}. \end{split}$$

This equals the desired expression for $\gamma(n, d, 1)$ on the right hand side of (4.3). \Box

The proof of Theorem 4.1 involves some algebraic geometry and will be presented later in this section. We first discuss a few examples to illustrate $\Delta_{n.d.}$

EXAMPLE 4.3 (d = 2). The eigendiscriminant of an $n \times n$ -matrix $A = (a_{ij})$ is the discriminant of its characteristic polynomial. In symbols,

$$\Delta_{n,2} = \operatorname{discr}_{\lambda} \left(\operatorname{det} (A - \lambda \cdot \operatorname{Id}_n) \right).$$

This is a homogeneous polynomial of degree n(n-1) in the matrix entries a_{ij} . For instance, for a 3×3 -matrix, the eigendiscriminant is a polynomial with 144 terms:

$$\Delta_{3,2} = a_{11}^4 a_{22}^2 - 2a_{11}^4 a_{22} a_{33} + 4a_{11}^4 a_{23} a_{32} + a_{11}^4 a_{33}^2 - 2a_{11}^3 a_{12} a_{21} a_{22} + \dots + a_{23}^2 a_{32}^2 a_{33}^2$$

This polynomial vanishes whenever two of the eigenvalues of A coincide.

There is a beautiful theory behind $\Delta_{n,2}$ in the case when A is real symmetric, so the eigenconfiguration is defined over \mathbb{R} . The resulting symmetric eigendiscriminant is a nonnegative polynomial of degree n(n-1) in the $\binom{n+1}{2}$ matrix entries. Its real variety has codimension 2 and degree $\binom{n+1}{3}$, and its determinantal representation governs expressions of $\Delta_{n,2}$ as a sum of squares of polynomials of degree $\binom{n}{2}$. For further reading on this topic see [Stu, Section 7.5] and the references given there.

EXAMPLE 4.4 (n = 2). The eigendiscriminant of a *d*-dimensional tensor of format $2 \times 2 \times \cdots \times 2$ is the discriminant of the associated binary form in (2.7), i.e.

$$\Delta_{2,d} = \operatorname{disc}_{(x,y)} \left(y \cdot \psi_1(x,y) - x \cdot \psi_2(x,y) \right).$$

This is a homogeneous polynomial of degree 2d-2 in the 2^d tensor entries $a_{i_1i_2\cdots i_d}$.

EXAMPLE 4.5 (n = d = 3). The eigendiscriminant $\Delta_{3,3}$ of a $3 \times 3 \times 3$ -tensor $A = (a_{ijk})$ is a homogeneous polynomial of degree 24 in the 27 entries a_{ijk} . If we specialize A to a symmetric tensor, corresponding to a ternary cubic

$$\phi(x, y, z) = c_{300}x^3 + c_{210}x^2y + c_{201}x^2z + \dots + c_{003}z^3,$$

then $\Delta_{3,3}$ remains irreducible. The resulting irreducible polynomial of degree 24 in the ten coefficients c_{ijk} is the *eigendiscriminant of a tenary cubic*. At present we

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do not know an explicit formula for $\Delta_{3,3}$, but it is fun to explore specializations of the eigendiscriminant. For instance, if $\phi = (2x+y)(2x+z)(2y+z) + u \cdot xyz$ then

 $\begin{array}{l} \Delta_{3,3} &= 16u^{24} + 2304u^{23} + 152784u^{22} + 6097536u^{21} + 159761808u^{20} + 2779161840u^{19} + 29727588168u^{18} \\ + 124641852624u^{17} &- 1234078589016u^{16} &- 18314627517360u^{15} &- 8929524942432u^{14} + 1200933047925648u^{13} \\ + 3722203539791685u^{12} &- 63418425922462464u^{11} &- 257381788882972176u^{10} + 2676970903961440800u^{9} \\ &+ 7927655114836286496u^{8} &- 89013482239908955392u^{7} &- 13934355026171012352u^{6} \\ + 1729250356371556792320u^{5} &- 515922324901192930048u^{4} &- 11838757458480721920u^{3} \\ + 28255456641734116982784u^{2} &- 56809371779894977339392u + 37304830510913780269056, \end{array}$

and if $\phi = u \cdot x^3 + v \cdot y^3 + w \cdot z^3 + xyz$ then $\Delta_{3,3}$ is the square of polynomial

 $\begin{array}{c} 531441u^4v^4w^4 - 708588u^5v^3w^3 - 708588u^3v^5w^3 - 708588u^3v^5w^3 - 708588u^3v^3w^5 + 1062882u^4v^4w^2 + 1062882u^4v^2w^4 \\ + 1062882u^2v^4w^4 - 1810836u^3v^3w^3 - 177147u^4v^4 + 39366u^4v^2w^2 + 39366u^2v^4w^2 - 177147u^4w^4 \\ + 39366u^2v^2w^4 - 177147v^4w^4 + 314928u^3v^3w + 314928u^3vw^3 + 314928uv^3w^2 - 244944u^2v^2w^2 - 46656u^3vw \\ - 46656uv^3w - 46656uvw^3 + 23328u^2v^2 + 23328u^2w^2 + 23328v^2w^2 + 6912uvw - 2304u^2 - 2304v^2 - 2304w^2 + 256. \end{array}$

We now embark towards the proof of Theorem 4.1. Let A and B be generic tensors of the same format, and write (ψ_1, \ldots, ψ_n) and $(\omega_1, \ldots, \omega_n)$ for the vectors of degree d-1 polynomials that represent the corresponding maps $\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1}$. Let \mathcal{C} denote the subvariety of \mathbb{P}^{n-1} defined by the determinantal constraints

(4.6)
$$\operatorname{rank} \begin{pmatrix} \psi_1(\mathbf{x}) & \psi_2(\mathbf{x}) & \cdots & \psi_n(\mathbf{x}) \\ \omega_1(\mathbf{x}) & \omega_2(\mathbf{x}) & \cdots & \omega_n(\mathbf{x}) \\ x_1^{\ell} & x_2^{\ell} & \cdots & x_n^{\ell} \end{pmatrix} \leq 2.$$

Since the ψ_i and ω_j are generic, this defines a variety of codimension n-2. We find that C is a curve that is smooth and irreducible, by an argument similar to that in the proof of Theorem 2.2. The following lemma is the key to Theorem 4.1.

LEMMA 4.6. The expression in (4.1) is the genus of the curve C. In symbols,

genus(
$$\mathcal{C}$$
) = $\gamma(n, d, \ell)$.

Using this lemma, we now derive the degree of the eigendiscriminant.

PROOF OF THEOREM 4.1. We define a map $\mu : \mathcal{C} \to \mathbb{P}^1$ as follows. For any point **x** on the curve \mathcal{C} , the matrix in (4.6) has rank 2, so, up to scaling, there exists a unique row vector $(a, b, c) \in K^3$ that spans the left kernel of that $3 \times n$ -matrix. We define the image of $\mathbf{x} \in \mathcal{C}$ to be the point $\mu(\mathbf{x}) = (a : b)$ on the projective line \mathbb{P}^1 . This condition means that **x** is an eigenpoint of the tensor aA + bB. Conversely, for any $(a : b) \in \mathbb{P}^1$, the fiber $\mu^{-1}(a : b)$ consists precisely of the eigenpoints of aA + bB. Hence, since A and B are generic, the generic fiber is finite and reduced of cardinality $\rho(n, d, \ell)$. In other words, $\mu : \mathcal{C} \to \mathbb{P}^1$ is a map of degree $\rho(n, d, \ell)$.

We restrict the eigendiscriminant to our \mathbb{P}^1 of tensors. The resulting binary form $\Delta_{n,d,\ell}(aA+bB)$ is squarefree, and its degree is the left hand side in (4.2). The points $(a:b) \in \mathbb{P}^1$ where $\Delta_{n,d,\ell}(aA+bB) = 0$ are the *branch points* of the map μ . The corresponding multiplicity-two eigenpoints **x** form the *ramification divisor* on \mathcal{C} . The number of branch points of μ is the degree of the eigendiscriminants $\Delta_{n,d,\ell}$.

The Riemann-Hurwitz Formula [Fu2, Exercise 8.36] states that the number of branch points of the map $\mu : \mathcal{C} \to \mathbb{P}^1$ is $2 \cdot \text{genus}(\mathcal{C}) + 2 \cdot \text{degree}(\mu) - 2$. By the first paragraph, and by Lemma 4.6, this expression is the right hand side of (4.2). \Box

Our proof of Lemma 4.6 is fairly complicated, and we decided not to include it here. It is based on resolutions of vector bundles, like those seen in the proof of Theorem 2.2. We plan to develop this further and publish it in a later paper on discriminants arising from maximal minors of matrices with more than two rows.

What we shall do instead is to prove an alternative combinatorial formula for the genus of C that is equivalent to (4.1). This does not prove Lemma 4.6 because we presently do not know a direct argument to show that they are equal. Nevertheless, the following discussion is an illustration of useful commutative algebra techniques.

INSTEAD OF LEMMA 4.6. The Hilbert polynomial $H_{\mathcal{C}}(t)$ of the curve \mathcal{C} equals

 $H_{\mathcal{C}}(t) = \text{degree}(\mathcal{C}) \cdot t + (1 - \text{genus}(\mathcal{C})).$

Recall that C is a linear section of the variety defined by the maximal minors of a $3 \times n$ -matrix whose rows are homogeneous of degrees d-1, d-1 and ℓ . That variety is Cohen-Macaulay. We shall compute the Hilbert polynomial of the coordinate ring of C from its graded minimal free resolution over $S = K[x_1, \ldots, x_n]$.

Consider the S-linear map from $F = S^{\oplus n}$ to $G = S(d-1)^{\oplus 2} \oplus S(\ell)$ given by

(4.7)
$$\alpha = \begin{pmatrix} \psi_1(\mathbf{x}) & \psi_2(\mathbf{x}) & \cdots & \psi_n(\mathbf{x}) \\ \omega_1(\mathbf{x}) & \omega_2(\mathbf{x}) & \cdots & \omega_n(\mathbf{x}) \\ x_1^\ell & x_2^\ell & \cdots & x_n^\ell \end{pmatrix}.$$

By [Eis, Section A2H], the corresponding Eagon-Northcott complex $EN(\alpha)$ equals

$$0 \to \operatorname{Sym}_{n-3}(G^{\vee}) \otimes \bigwedge^{n} F \to \operatorname{Sym}_{n-4}(G^{\vee}) \otimes \bigwedge^{n-1} F \to \dots \to G^{\vee} \otimes \bigwedge^{4} F \to \bigwedge^{3} F \to \bigwedge^{3} G,$$

where $\operatorname{Sym}_i(G^{\vee})$ is the *i*th symmetric power of G^{\vee} and $\bigwedge^i F$ is the *i*th exterior power of F. We compute the Hilbert polynomial $H_M(t)$ of each module M in $\operatorname{EN}(\alpha)$.

Since C has codimension $n-2 = \operatorname{rank} F - \operatorname{rank} G+1$, the complex $\operatorname{EN}(\alpha) \otimes S(-2d-\ell+2)$ is a free resolution of the coordinate ring of C. In particular,

(4.8)
$$H_{\mathcal{C}}(t) = H_{\Lambda^3 G}(t - 2d - \ell + 2) + \sum_{j=3}^n (-1)^j H_{E_j}(t - 2d - \ell + 2),$$

where $E_j = \operatorname{Sym}_{j-3}(G^{\vee}) \otimes \bigwedge^j F$. Since $G^{\vee} = S(-d+1)^{\oplus 2} \oplus S(-\ell)$ and $F = S^{\oplus n}$, $\operatorname{Sym}_{j-3}(G^{\vee}) = \bigcap_{j=3}^{j-3} S((j-k-2)(-d+1) - k\ell)^{\oplus j-k+2} \text{ and } \bigwedge_{j=1}^{j} F = S^{\oplus \binom{n}{j}}$

$$\operatorname{Sym}_{j-3}(G^{\vee}) = \bigoplus_{k=0}^{\infty} S\left((j-k-3)(-d+1) - k\ell\right)^{\oplus j-k+2} \text{ and } \bigwedge^{\sim} F = S^{\oplus \binom{n}{j}}.$$

Their tensor product is the j^{th} term in $\text{EN}(\alpha)$. As a graded S-module, it equals

$$E_j = \bigoplus_{k=0}^{j-3} S\left((j-k-3)(-d+1) - k\ell\right)^{\oplus (j-k-2)\binom{n}{j}}.$$

The shifted Hilbert series of this module is the summand on the right of (4.8):

$$H_{E_j}(t-2d-\ell+2) = \sum_{k=0}^{j-3} (j-k-2) \binom{n}{j} \binom{t+(n-1)+(j-k-1)(1-d)-(k+1)\ell}{n-1}$$

We conclude that the Hilbert polynomial $H_{\mathcal{C}}(t)$ of the curve \mathcal{C} equals

$$\binom{t+n-1}{n-1} + \sum_{j=3}^{n} (-1)^j \sum_{k=0}^{j-3} (j-k-2) \binom{n}{j} \binom{t+(n-1)+(j-k-1)(1-d)-(k+1)\ell}{n-1}$$

The genus of C is obtained by substituting t = 0 and subtracting the result from 1:

$$\gamma(n,d,\ell) = \sum_{j=3}^{n} (-1)^{j-1} \sum_{k=0}^{j-3} (j-k-2) \binom{n}{j} \binom{(n-1) + (j-k-1)(1-d) - (k+1)\ell}{n-1}.$$

This formula is equivalent to (4.1).

$$\Box$$

EXAMPLE 4.7 (n = 4). The last formula seen above specializes to

$$\gamma(4,d,\ell) = 4\binom{3+2(1-d)-\ell}{3} - \left[2\binom{3+3(1-d)-\ell}{3} + \binom{3+2(1-d)-2\ell}{3}\right],$$

while the genus formula in (4.1) states

$$\gamma(4,d,\ell) = -\left[6\binom{2(d-1)-1}{3} - 4\binom{2(d-1)-\ell-1}{3} + \binom{2(d-1)-2\ell-1}{3} \right] \\ + 2\left[4\binom{3(d-1)-1}{3} - 6\binom{3(d-1)-\ell-1}{3} + 4\binom{3(d-1)-2\ell-1}{3} - \binom{3(d-1)-3\ell-1}{3} \right].$$

Both of these evaluate to the cubic polynomial

 $\gamma(4,d,\ell) = 5d^3 + 5d^2\ell + 3d\ell^2 + \ell^3 - 21d^2 - 14d\ell - 5\ell^2 + 27d + 9\ell - 10.$

Therefore, by Theorem 4.1, the degree of the ℓ^{th} eigendiscriminant for n = 4 equals

degree
$$(\Delta_{4,d,\ell}) = 3d^3 + 3d^2\ell + 2d\ell^2 + \ell^3 - 12d^2 - 8d\ell - 3\ell^2 + 15d + 5\ell - 6.$$

For $\ell = 1$, this factorizes as promised in Corollary 4.2: degree $(\Delta_{4,d,1}) = 12(d-1)^3$.

5. Seven points in the plane

Our study had been motivated by the desire to find a geometric characterization of eigenconfigurations among all finite subsets of \mathbb{P}^{n-1} . The solution for n = 2 was presented in Theorem 2.7. However, the relevant geometry is more difficult in higher dimensions. In this section we take some steps towards a characterization for n = 3. The eigenconfiguration of a general tensor A in $(K^3)^{\otimes d}$ consists of $d^2 - d + 1$ points in \mathbb{P}^2 . So, our question can be phrased like this: given a configuration $Z \in (\mathbb{P}^2)^{d^2 - d + 1}$, decide whether it is an eigenconfiguration. If yes, construct a corresponding tensor $A \in (K^3)^{\otimes d}$, and decide whether A can be chosen to be symmetric.

The first interesting case is d = n = 3. Here the following result holds.

THEOREM 5.1. A configuration of seven points in \mathbb{P}^2 is the eigenconfiguration of a $3 \times 3 \times 3$ -tensor if and only if no six of the seven points lie on a conic.

The only-if part of this theorem appears also in [OS, Proposition 2.1], where Ottaviani and Sernesi studied the degree 54 hypersurface of all *Lüroth quartics* in \mathbb{P}^2 . We note that part (i) in [OS, Proposition 2.1] is not quite correct. A counterexample is the configuration Z consisting of four points on a line and three other general points. It is precisely this gap that makes our proof of Theorem 5.1 a bit lengthy.

This proof will be presented later in this section. Example 1.1 shows that some triples among the seven eigenpoints in \mathbb{P}^2 can be collinear. Another interesting point is that being an eigenconfiguration is not a closed condition. For a general d it makes sense to pass to the Zariski closure. We define Eig_d to be the closure in $(\mathbb{P}^2)^{d^2-d+1}$ of the set of all eigenconfigurations. Readers from algebraic geometry may prefer unlabeled configurations, and they would take the closure in the Chow variety $\operatorname{Sym}_{d^2-d+1}(\mathbb{P}^2)$ or in the Hilbert scheme $\operatorname{Hilb}_{d^2-d+1}(\mathbb{P}^2)$. For simplicity of exposition, we work in the space of labeled point configurations. We also consider the variety of symmetric eigenconfigurations, denoted $\operatorname{Eig}_{d,\text{sym}}$. This is the Zariski closure in $(\mathbb{P}^2)^{d^2-d+1}$ of the set of eigenconfigurations of ternary forms ϕ of degree d. Towards the end of this section we examine the dimensions of Eig_d and Eig_d sym.

We begin by approaching our problem with a pinch of commutative algebra. Let $Z \in (\mathbb{P}^2)^{d^2-d+1}$ and write I_Z for the ideal of all polynomials in S = K[x, y, z] that vanish at all points in the configuration Z. This homogeneous radical ideal is Cohen-Macaulay because it has a free resolution of length 1 (see, for example, [**Eis**, Proposition 3.1]). By the Hilbert-Burch Theorem, the minimal free resolution of I_Z has the form

$$0 \to S^{\oplus (m-1)} \xrightarrow{\Phi} S^{\oplus m} \to I_Z \to 0.$$

The $m \times (m-1)$ -matrix Φ is the *Hilbert-Burch matrix* of Z. The minimal free resolution of I_Z is unique up to change of bases in the graded S-modules. In that sense, we write $\Phi_Z := \Phi$. The ideal I_Z is generated by the maximal minors of Φ_Z . The following proposition is due to Ottaviani and Sernesi (see **[OS**, Proposition 2.1]).

PROPOSITION 5.2. Let Z be a configuration in $(\mathbb{P}^2)^{d^2-d+1}$. Then Z is the eigenconfiguration of a tensor if and only if its Hilbert-Burch matrix has the form

(5.1)
$$\Phi_Z = \begin{pmatrix} L_1 & F_1 \\ L_2 & F_2 \\ L_3 & F_3 \end{pmatrix},$$

where L_1, L_2, L_3 are linear forms that are linearly independent over K.

This statement makes sense because the condition on Φ_Z is invariant under row operations over K. The ternary forms F_1, F_2, F_3 must all have the same degree, and the hypothesis on Z ensures that this common degree is d-1.

PROOF. We start with the only-if direction. Suppose that Z is an eigenconfiguration. Then there exist ternary forms ψ_1, ψ_2, ψ_3 of degree d-1 such that Z is defined set-theoretically by the 2 × 2-minors of

(5.2)
$$\begin{pmatrix} x & y & z \\ \psi_1(x, y, z) & \psi_2(x, y, z) & \psi_3(x, y, z) \end{pmatrix}.$$

The ideal generated by these minors is Cohen-Macaulay of codimension 2 and its degree equals the cardinality of Z. This implies that this ideal coincides with I_Z . The Hilbert-Burch Theorem ensures that the transpose of (5.2) equals Φ_Z . Since x, y, z are linearly independent, we see that Φ_Z has the form required in (5.1).

For the converse, suppose that the Hilbert-Burch matrix of Z has size 3×2 as in (5.1) with L_1, L_2, L_3 linearly independent. By performing row operations over K, we can replace L_1, L_2, L_3 by x, y, z. This means that the transpose of Φ_Z is (5.2) for some ψ_1, ψ_2, ψ_3 . Any such triple of ternary forms of degree d-1 arises from some tensor $A \in (K^3)^{\otimes d}$. By construction, Z is the eigenconfiguration of A.

Proposition 5.2 translates into an algorithm for testing whether a given $Z \in (\mathbb{P}^2)^{d^2-d+1}$ is an eigenconfiguration. The algorithm starts by computing the ideal

$$I_Z = \bigcap_{(\alpha:\beta:\gamma)\in Z} \langle x\beta - y\alpha, x\gamma - z\alpha, y\gamma - z\beta \rangle.$$

This ideal must have three minimal generators of degree d; otherwise Z is not an eigenconfiguration. If I_Z has three generators, then we compute the two syzygies. They must have degrees 1 and d-1, so the minimal free resolution of I_Z looks like

$$0 \to S(-d-1) \oplus S(-2d+1) \xrightarrow{\Phi} S(-d)^{\oplus 3} \to I_Z \to 0.$$

At this point we examine the matrix Φ . If the linear entries L_1, L_2, L_3 in the left column are linearly dependent, then Z is not an eigenconfiguration. Otherwise we perform row operations so that Φ^T looks like (5.2). The last step is to pick a tensor $A \in (K^3)^{\otimes d}$ that gives rise to the ternary forms ψ_1, ψ_2, ψ_3 in the second row of Φ^T .

The remaining task is to find a geometric interpretation of the criterion in Proposition 5.2. This was given for d = 3 in the result whose proof we now present.

PROOF OF THEOREM 5.1. Fix a configuration $Z \in (\mathbb{P}^2)^7$. Our claim states that the Hilbert-Burch matrix Φ_Z has format 3×2 as in (5.1), with L_1, L_2, L_3 linearly independent, if and only if no six of the points in Z lie on a conic.

We begin with the only-if direction. Take $p \in Z$ such that $Z \setminus \{p\}$ lies on a conic C in \mathbb{P}^2 . Fix linear forms L_1 and L_2 that cut out p. The cubics CL_1 and CL_2 vanish on Z. By Proposition 5.2, we have $I_Z = \langle CL_1, CL_2, F \rangle$ where F is another cubic. Since F vanishes at p, there exist quadrics Q_1 and Q_2 such that $F = Q_2L_1 - Q_1L_2$. The generators of the ideal I_Z are the 2×2 -minors of

$$\Psi = \left(\begin{array}{cc} L_1 & Q_1 \\ L_2 & Q_2 \\ 0 & C \end{array} \right).$$

This means that Ψ is a Hilbert-Burch matrix Φ_Z for Z. However, by Proposition 5.2, the left column in any Φ_Z must consist of linearly independent linear forms. This is a contradiction, which completes the proof of the only-if direction.

We now establish the if direction. Fix any configuration $Z \in (\mathbb{P}^2)^7$ of seven points that do not lie on a conic. We first prove that the minimal free resolution of I_Z has the following form, where c is either 0 or 1:

(5.3)
$$0 \to S(-4)^{\oplus (c+1)} \oplus S(-5) \to S(-3)^{\oplus 3} \oplus S(-4)^{\oplus c} \to I_Z \to 0,.$$

By the Hilbert-Burch Theorem, the resolution of I_Z equals

$$0 \to \bigoplus_{i=1}^{t} S(-b_i) \xrightarrow{\Phi_Z} \bigoplus_{i=1}^{t+1} S(-a_i) \to I_Z \to 0,$$

where $t, a_1, \ldots, a_{t+1}, b_1, \ldots, b_t \in \mathbb{N}$ with $a_1 \geq \cdots \geq a_{t+1}$ and $b_1 \geq \cdots \geq b_t$. We abbreviate $e_i = b_i - a_i$ and $f_i = b_i - a_{i+1}$ for $i \in \{1, \ldots, t\}$. These invariants satisfy

- (i) $f_i \ge e_i, e_{i+1},$
- (ii) $e_i, f_i \ge 1$.

Furthermore, Eisenbud shows in [Eis, Proposition 3.8] that

(iii)
$$a_i = \sum_{j=1}^{i-1} e_j + \sum_{j=i}^t f_j$$

There exist 3 linearly independent cubics that vanish on the seven points in Z. By [Eis, Corollary 3.9], the ideal I_Z has either 3 or 4 minimal generators, so $t \in \{2, 3\}$.

Suppose t = 2. Then $a_1 = a_2 = a_3 = 3$, and it follows from (iii) that

$$f_1 + f_2 = 3$$
 and $e_1 + f_2 = 3$.

So, by (i) and (ii), we obtain $e_1 = f_1 = 1$ and $f_2 = 2$. This implies $b_1 = 5$ and $b_2 = 4$. Therefore, I_Z has a minimal free resolution of type (5.3) with c = 0.

Next, suppose t = 3. Then $a_1 \ge a_2 = a_3 = a_4 = 3$. From (iii) we now get

By (ii), $e_1 = e_2 = f_2 = f_3 = 1$. Therefore, $b_2 = a_2 + e_2 = 4$ and $b_3 = a_2 + f_2 = 4$. Corollary 3.10 in [**Eis**] says that

$$\sum_{i \le j} e_i f_j = e_1 (f_1 + f_2 + f_3) + e_2 (f_2 + f_3) + e_3 f_3 = a_1 + 3 = \deg Z = 7.$$

Hence $a_1 = 4$, $b_1 = 5$, and I_Z has a minimal free resolution of type (5.3) with c = 1.

To complete the proof, we now assume that no six points of Z lie on a conic. In particular, no conic contains Z, so the minimal free resolution of I_Z equals (5.3), with $c \in \{0, 1\}$. Suppose that c = 1. The Hilbert-Burch matrix must be

$$\Phi_Z = \begin{pmatrix} L_{00} & L_{01} & Q_0 \\ L_{10} & L_{11} & Q_1 \\ L_{20} & L_{21} & Q_2 \\ 0 & 0 & L \end{pmatrix}$$

with L, L_{ij} are linear and Q_k are quadrics. Then $I_Z = \langle LQ'_0, LQ'_1, LQ'_2, Q \rangle$, where

$$Q_0' = \begin{vmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{vmatrix}, Q_1' = \begin{vmatrix} L_{00} & L_{01} \\ L_{20} & L_{21} \end{vmatrix}, Q_2' = \begin{vmatrix} L_{10} & L_{11} \\ L_{20} & L_{21} \end{vmatrix}, Q = \begin{vmatrix} L_{00} & L_{01} & Q_0 \\ L_{10} & L_{11} & Q_1 \\ L_{20} & L_{21} & Q_2 \end{vmatrix}$$

The ideal generated by L and Q contains I_Z . The intersection of the curves $\{L=0\}$ and $\{Q=0\}$ is contained in Z. Note that these curves share no positivedimensional component, since Z is zero-dimensional. Thus $\{L=Q=0\}$ consists of four points. Let L' be a linear form vanishing on two of the three other points. Then the conic $\{LL'=0\}$ contains six points of Z, which contradicts our assumption.

Hence, c = 0. The resolution (5.3) tells us that the Hilbert-Burch matrix equals

$$\Phi_Z = \begin{pmatrix} L_1 & Q_1 \\ L_2 & Q_2 \\ L_3 & Q_3 \end{pmatrix},$$

with linear forms L_i and conics Q_j . If L_1, L_2, L_3 were linearly dependent then we can take $L_3 = 0$. So, the conic Q_3 contains the six points in $Z \setminus \{L_1 = L_2 = 0\}$. Consequently, the linear forms L_1, L_2, L_3 must be linearly independent. Proposition 5.2 now implies that Z is the eigenconfiguration of some $3 \times 3 \times 3$ -tensor. \Box

After taking the Zariski closure, we have $\operatorname{Eig}_3 = (\mathbb{P}^2)^7$. We shall now discuss the subvariety $\operatorname{Eig}_{3,\text{sym}}$ of those eigenconfigurations that come from symmetric tensors. Consider the three quadrics in the second row of (5.2). We write these as

$$\begin{array}{rcl} \psi_1(\mathbf{x}) &=& a_1 x^2 + a_2 x y + a_3 x z + a_4 y^2 + a_5 y z + a_6 z^2, \\ \psi_2(\mathbf{x}) &=& b_1 x^2 + b_2 x y + b_3 x z + b_4 y^2 + b_5 y z + b_6 z^2, \\ \psi_3(\mathbf{x}) &=& c_1 x^2 + c_2 x y + c_3 x z + c_4 y^2 + c_5 y z + c_6 z^2. \end{array}$$

We shall characterize the case of symmetric tensors in terms of these coefficients.

PROPOSITION 5.3. The variety $\operatorname{Eig}_{3,\operatorname{sym}}$ is irreducible of dimension 9 in $(\mathbb{P}^2)^7$. An eigenconfiguration Z comes from a symmetric tensor ϕ as in (1.3) if and only if $a_5 - b_3 = b_3 - c_2 = 2a_4 - 2a_6 - b_2 + c_3 = 2b_6 - 2b_1 - c_5 + a_2 = 2c_1 - 2c_4 - a_3 + b_5 = 0$.

PROOF. There exists a symmetric tensor with eigenconfiguration Z if and only if there exist a cubic ϕ and a linear form $L = u_1 x + u_2 y + u_3 z$ such that

(5.4)
$$\psi_1 + Lx = \frac{\partial \phi}{\partial x}, \quad \psi_2 + Ly = \frac{\partial \phi}{\partial y}, \quad \psi_3 + Lz = \frac{\partial \phi}{\partial z}.$$

We eliminate the cubic ϕ from this system by taking crosswise partial derivatives:

$$\frac{\partial \psi_1}{\partial y} + x \frac{\partial L}{\partial y} = \frac{\partial \psi_2}{\partial x} + y \frac{\partial L}{\partial x}, \dots, \frac{\partial \psi_2}{\partial z} + y \frac{\partial L}{\partial z} = \frac{\partial \psi_3}{\partial y} + z \frac{\partial L}{\partial x}$$

This is a system of linear equations in the 21 unknowns $a_1, \ldots, a_6, b_1, \ldots, b_6, c_1, \ldots, c_6, u_1, u_2, u_3$. By eliminating the last three unknowns u_1, u_2, u_3 from that system, we arrive at the five linearly independent equations in a_i, b_j, c_k stated above. \Box

Proposition 5.3 translates into an algorithm for testing whether a given configuration Z is the eigenconfiguration of a ternary cubic ϕ . Namely, we compute the syzygies of I_Z , we check that the Hilbert-Burch matrix has the form (5.2), and then we check the five linear equations. If these hold then ϕ is found by solving (5.4).

While the equations in Proposition 5.3 are linear, we did not succeed in computing the prime ideal of $\operatorname{Eig}_{3,\text{sym}}$ in the homogeneous coordinate ring of $(\mathbb{P}^2)^7$. This is a challenging elimination problem. Some insight can be gained by intersecting $\operatorname{Eig}_{3,\text{sym}}$ with natural subfamilies of $(\mathbb{P}^2)^7$. For instance, assume that Z contains the three coordinate points, so we restrict to the subspace $(\mathbb{P}^2)^4$ defined by

$$Z = \{ (1:0:0), (0:1:0), (0:0:1), (\alpha_1:\alpha_2:\alpha_3), (\beta_1:\beta_2:\beta_3), (\gamma_1:\gamma_2:\gamma_3), (\delta_1:\delta_2:\delta_3) \}.$$

At this point it is important to recall that our problem is not projectively invariant.

THEOREM 5.4. The variety $\operatorname{Eig}_{3,\operatorname{sym}} \cap (\mathbb{P}^2)^4$ is three-dimensional, and it represents the eigenconfigurations Z of the ternary cubics in the Hesse family

(5.5)
$$\phi = ax^3 + by^3 + cz^3 + 3dxyz.$$

If a, b, c, d are real then the eigenconfiguration Z contains at least five real points.

PROOF. A ternary cubic $\phi = \sum_{i+j+k=3} c_{ijk} x^i y^j z^k$ has (1:0:0) as an eigenpoint of ϕ if and only if $c_{210} = c_{201} = 0$. Likewise, (0:1:0) is an eigenpoint if and only if $c_{120} = c_{021} = 0$, and (0:0:1) is an eigenpoint if and only if $c_{102} = c_{012} = 0$. Hence the eigenconfiguration of ϕ contains all three coordinate points if and only if ϕ is in the Hesse family (5.5). Since $\operatorname{Eig}_{3,\mathrm{sym}}$ has codimension 5 in $(\mathbb{P}^2)^7$, the intersection $\operatorname{Eig}_{3,\mathrm{sym}} \cap (\mathbb{P}^2)^4$ has codimension ≤ 5 , so its dimension is ≥ 3 . The Hesse family is 3-dimensional, and so we conclude that $\dim(\operatorname{Eig}_{3,\mathrm{sym}} \cap (\mathbb{P}^2)^4) = 3$.

The four other eigenpoints of (5.5) are $(d\chi(\chi^2 - 1) : \chi(a\chi - c) : d(\chi^2 - 1))$, where χ runs over the zeros of the polynomial

(5.6)
$$d(a^2 - d^2)\chi^4 - a(ab + cd)\chi^3 + 2(abc + d^3)\chi^2 - c(bc + ad)\chi + d(c^2 - d^2).$$

We claim that this quartic polynomial has at least two real roots for all $a, b, c, d \in \mathbb{R}$. Inside the projective space \mathbb{P}^4 of quartics $f(\chi) = k_4 \chi^4 + k_3 \chi^3 + k_2 \chi^2 + k_1 \chi + k_0$, the family (5.6) is contained in the hypersurface defined by the quadric

$$\mathcal{S} = 2k_4k_2 + k_2^2 - 4k_1k_3 + 4k_0k_4 + 2k_0k_2.$$

The discriminant of $f(\chi)$ defines a hypersurface of degree 6 in \mathbb{P}^4 . One of the open regions in the complement of the discriminant consists of quartics $f(\chi)$ with no real roots. In polynomial optimization (cf. [**BPT**, Lemma 3.3]) one represents this region by a formula of the following form, where κ is a new indeterminate:

$$f(\chi) = \begin{pmatrix} \chi^2 & \chi & 1 \end{pmatrix} \cdot \begin{pmatrix} k_0 & k_1/2 & \kappa \\ k_1 & k_2 - 2\kappa & k_3/2 \\ \kappa & k_3/2 & k_4 \end{pmatrix} \cdot \begin{pmatrix} \chi^2 \\ \chi \\ 1 \end{pmatrix},$$

The symmetric 3×3 -matrix is required to be positive definite for some $\kappa \in \mathbb{R}$. The condition of being positive definite is expressed by the leading principal minors:

$$\mathcal{P} = k_0, \ \mathcal{Q} = \det \begin{pmatrix} k_0 & k_1/2 \\ k_1/2 & k_2 - 2\kappa \end{pmatrix}, \ \mathcal{R} = \det \begin{pmatrix} k_0 & k_1/2 & \kappa \\ k_1/2 & k_2 - 2\kappa & k_3/2 \\ \kappa & k_3/2 & k_4 \end{pmatrix}.$$

It remains to be proved that there is no solution $(k_0, k_1, k_2, k_3, k_4, \kappa) \in \mathbb{R}^6$ to

 $\mathcal{P} > 0, \ \mathcal{Q} > 0, \ \mathcal{R} > 0 \text{ and } \mathcal{S} = 0.$

We showed this by computing a *sum-of-squares proof*, in the sense of [**BPT**, Chapter 3]. More precisely, using the software **SOSTools** [**SOS**], we found explicit polynomials $p, q, r, s \in \mathbb{R}[k_0, k_1, k_2, k_3, k_4, \kappa]$ with floating point coefficients such that

p, q, r are sums of squares and $p\mathcal{P} + q\mathcal{Q} + r\mathcal{R} + s\mathcal{S} = -1$.

We are grateful to Cynthia Vinzant for helping us with this computation.

We close this section by returning to tensors in $(K^3)^{\otimes d}$ for general $d \geq 3$.

THEOREM 5.5. Consider the spaces of eigenconfigurations of ternary tensors,

$$\operatorname{Eig}_{d.\operatorname{sym}} \subset \operatorname{Eig}_d \subset (\mathbb{P}^2)^{d^2 - d + 1} \quad \text{for } d \ge 3.$$

These projective varieties are irreducible, and their dimensions are

$$\dim(\text{Eig}_{d,\text{sym}}) = \frac{1}{2}(d^2 + 3d)$$
 and $\dim(\text{Eig}_d) = d^2 + 2d - 1.$

PROOF. First we show dim(Eig_d) = $d^2 + 2d - 1$. Let W be the set of 2×3 matrices (5.2). This is a $3\binom{d+1}{2}$ -dimensional vector space over K. The group

$$G = \left\{ \left(\begin{array}{cc} 1 & 0 \\ f & a \end{array} \right) \middle| a \in K \setminus \{0\} \text{ and } f \text{ is a ternary form of degree } d - 2 \right\}$$

acts on W by left multiplication. Consider $\varphi, \omega \in W$. It is immediate to see that if $\omega = g \cdot \varphi$ for some $g \in G$, then the variety defined by the 2 × 2-minors of φ equals the variety defined by the 2 × 2-minors of ω . The converse also holds because of the uniqueness of the Hilbert-Burch matrix. The set W° of elements in W whose 2 × 3-minors define $d^2 - d + 1$ distinct points is an open subset of W. Therefore,

$$\dim(\operatorname{Eig}_d) = \dim W^{\circ}/G = \dim W^{\circ} - \dim G$$
$$= 3\binom{d+1}{2} - \left[\binom{d}{2} + 1\right] = d^2 + 2d - 1.$$

Next we prove dim(Eig_{d,sym}) = $\frac{1}{2}(d^2 + 3d)$. We introduce the linear subspace

$$U := \left\{ \left(\begin{array}{cc} x & y & z \\ \partial \phi / \partial x & \partial \phi / \partial y & \partial \phi / \partial z \end{array} \right) \middle| \phi \text{ ternary form of degree } d \right\} \subset W.$$

The action of the group G on W does not restrict to U. In fact, we notice that

$$\begin{pmatrix} 1 & 0 \\ f & a \end{pmatrix} \begin{pmatrix} x & y & z \\ \partial \phi / \partial x & \partial \phi / \partial y & \partial \phi / \partial z \end{pmatrix} \in U$$

if and only if f = 0. Let $U^{\circ} = U \cap W^{\circ}$ and consider the subgroup

$$H = \left\{ \left(\begin{array}{cc} 1 & 0 \\ 0 & a \end{array} \right) \middle| a \in K \setminus \{0\} \right\} \subset G.$$

This yields $\dim(\operatorname{Eig}_{d,\operatorname{sym}}) = \dim U^{\circ}/H = 3\binom{d+2}{2} - 1 = \frac{1}{2}(d^2 + 3d)$, as desired. Our configuration spaces Eig_d and $\operatorname{Eig}_{d,\operatorname{sym}}$ are irreducible varieties because they contain the irreducible varieties W°/G and U°/H respectively as dense open subsets. \Box

6. Real eigenvectors and dynamics

In this section we focus on the real eigenpoints of a tensor A in $(\mathbb{R}^n)^{\otimes d}$. If A is generic then the number of eigenpoints in $\mathbb{P}^{n-1}_{\mathbb{C}}$ equals $((d-1)^n-1)/(d-2)$. Our hope is to show that all of them lie in $\mathbb{P}^{n-1}_{\mathbb{R}}$ for suitably chosen symmetric tensors ϕ . A second question is how many of these real eigenpoints are *robust*, in the sense that they are attracting fixed points of the dynamical system $\nabla \phi : \mathbb{P}^{n-1}_{\mathbb{R}} \dashrightarrow \mathbb{P}^{n-1}_{\mathbb{R}}$. Our results will inform future numerical work along the lines of [CDN, Table 4.12].

We begin with a combinatorial construction for the planar case (n = 3). Consider an arrangement \mathcal{A} of d distinct lines in $\mathbb{P}^2_{\mathbb{R}}$, and let ϕ be the product of d linear forms in x, y, z that define the lines in \mathcal{A} . We assume that \mathcal{A} is generic in the sense that no three lines meet in a point. Equivalently, the matroid of \mathcal{A} is a uniform rank 3 matroid on d elements. Such an arrangement \mathcal{A} has $\binom{d}{2}$ vertices in $\mathbb{P}^2_{\mathbb{R}}$, and these are the singular points of the reducible curve $\{\phi = 0\}$. The complement of \mathcal{A} in $\mathbb{P}^2_{\mathbb{R}}$ has $\binom{d}{2} + 1$ connected components, called the regions of \mathcal{A} .

We are interested in the eigenconfiguration of \mathcal{A} , by which we mean the eigenconfiguration of the symmetric tensor ϕ . Theorem 2.1 gives the expected number

(6.1)
$$d^2 - d + 1 = 1 + (d - 1) + (d - 1)^2 = 2\binom{d}{2} + 1 = \#$$
 vertices $+ \#$ regions.

The following result shows that this is not just a numerical coincidence.

THEOREM 6.1. A generic arrangement \mathcal{A} of d lines in $\mathbb{P}^2_{\mathbb{R}}$ has $d^2 - d + 1$ complex eigenpoints and they are all real. In addition to the $\binom{d}{2}$ vertices, which are singular eigenpoints, each of the $\binom{d}{2} + 1$ regions of \mathcal{A} contains precisely one real eigenpoint.

PROOF. The singular locus of the curve $\{\phi = 0\}$ consists of the vertices of the arrangement \mathcal{A} . These are the eigenpoints with eigenvalue 0. Their number is $\binom{d}{2}$.

Let L_1, L_2, \ldots, L_d be the linear forms that define the lines, so $\phi = L_1 L_2 \cdots L_d$. Consider the following optimization problem on the unit 2-sphere:

Maximize
$$\log |\phi(\mathbf{x})| = \sum_{i=1}^{d} \log |L_i(x, y, z)|$$
 subject to $x^2 + y^2 + z^2 = 1$.

The objective function takes the value $-\infty$ on the *d* great circles corresponding to \mathcal{A} . On each region of \mathcal{A} , the objective function takes values in \mathbb{R} , and is strictly concave. Hence there exists a unique local maximum $\mathbf{u}^* = (x^*, y^*, z^*)$ in the interior of each region. Such a maximum \mathbf{u}^* is a critical point of the restriction of $\phi(\mathbf{x})$ to the unit 2-sphere. The Lagrange multiplier conditions state that the vector \mathbf{u}^* is parallel to the gradient of ϕ at \mathbf{u}^* . This means that \mathbf{u}^* is an eigenvector of ϕ , and hence the pair $\pm \mathbf{u}^*$ defines a real eigenpoint of ϕ in the given region of $\mathbb{P}^2_{\mathbb{R}}$.

We proved that each of the $\binom{d}{2} + 1$ regions of \mathcal{A} contains one eigenpoint. In addition, we have the $\binom{d}{2}$ vertices. By Theorem 2.1, the total number of isolated complex eigenpoints cannot exceed $2\binom{d}{2} + 1$. This means that there are no eigenpoints in $\mathbb{P}^2_{\mathbb{C}}$ other than those already found. This completes the proof. \Box

We note that the line arrangement can be perturbed to a situation where the map $\nabla \phi : \mathbb{P}^{n-1}_{\mathbb{R}} \to \mathbb{P}^{n-1}_{\mathbb{R}}$ is regular, i.e. none of the eigenvectors has eigenvalue zero.

COROLLARY 6.2. There exists a smooth curve of degree d in the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ whose complex eigenconfiguration consists of $d^2 - d + 1$ real points.

PROOF. The eigenconfiguration of $\phi = L_1 L_2 \cdots L_d$ is 0-dimensional, reduced, and defined over \mathbb{R} . By the Implicit Function Theorem, these properties are preserved when ϕ gets perturbed to a generic ternary form ϕ_{ϵ} that is close to ϕ . \Box

It is interesting to see what happens when the matroid of \mathcal{A} is not uniform. Here the eigenconfiguration is not reduced. It arises from Theorem 6.1 by degeneration.

EXAMPLE 6.3. Let d = 6 and take \mathcal{A} to be the line arrangement defined by

$$\phi = x \cdot y \cdot z \cdot (x - y) \cdot (x - z) \cdot (y - z).$$

This is the reflection arrangement of type A_4 . Its eigenscheme is non-reduced. Each of the 12 regions contains one eigenpoint as before, and the simple vertices (1:1:0), (1:0:1), and (0:1:1) are eigenpoints of multiplicity one. However, each of the triple points (1:0:0), (0:1:0), (0:0:1), (1:1:1) is an eigenpoint of multiplicity 4. This makes sense geometrically: in a nearby generic arrangement, such a vertex splits into three vertices and one new region. We note that the scheme structure at the eigenpoint (1:0:0) is given by the primary ideal $\langle 2yz - z^2, y^2 - z^2 \rangle$.

The concavity argument concerning the optimization problem in the proof of Theorem 6.1 works in arbitrary dimensions, and we record this as a corollary.

COROLLARY 6.4. Each of the open regions of an arrangement \mathcal{A} of d hyperplanes in $\mathbb{P}^{n-1}_{\mathbb{R}}$ contains precisely one real eigenpoint of \mathcal{A} . The number of regions is

(6.2)
$$\sum_{i=0}^{n-1} \binom{d-1}{i}.$$

PROOF. The first part has the same proof as the one for n = 3 given above. The formula for the number of regions can be found in [Sta, Proposition 2.4]. \Box

Theorem 6.1 is restricted to n = 3 because hyperplane arrangements are singular in codimension 1. Hence the eigenconfiguration of a product of linear forms in $n \ge 4$ variables has components of dimension n - 3 in $\mathbb{P}^{n-1}_{\mathbb{R}}$. We conjecture that a fully real eigenconfiguration can be constructed in the vicinity of such a tensor.

CONJECTURE 6.5. Let ϕ be any product of d nonzero linear forms in $\text{Sym}_1(\mathbb{R}^n)$. Every open neighborhood of ϕ in $\text{Sym}_d(\mathbb{R}^n)$ contains a symmetric tensor ϕ_{ϵ} such that all $((d-1)^n-1)/(d-2)$ complex eigenpoints of ϕ_{ϵ} are real.

This optimistic conjecture is illustrated by the following variant of Example 1.1.

EXAMPLE 6.6 (n = d = 4). The classical Cremona transformation in \mathbb{P}^3 is $\nabla \phi$ where $\phi = xyzw$ is the product of the coordinates. The eigenconfiguration of ϕ consists of eight points, one for each sign region in \mathbb{P}^3 , and the six coordinate lines. The expected number (2.1) of complex eigenpoints is 40. Consider the perturbation

$$\begin{split} \phi_{\epsilon} &= xyzw + \epsilon \left(5x^4 + 4x^3y - 2x^2y^2 - 8xy^3 + 7y^4 + 4x^3z + 2x^2yz + 2xy^2z \\ &+ 2y^3z - 6x^2z^2 + 6xyz^2 + 7y^2z^2 - 8xz^3 + 3yz^3 + 8z^4 - 8x^3w + 2x^2yw \\ &- 3xy^2w + 5y^3w + 8x^2zw - 3y^2zw - 5xz^2w - 10yz^2w + 8z^3w - 5x^2w^2 \\ &- 6xyw^2 - 3y^2w^2 - 6xzw^2 + 3yzw^2 + 3xw^3 + 3yw^3 - 4zw^3 + 3w^4 \right). \end{split}$$

All 40 complex eigenpoints of this tensor are real, so Conjecture 6.5 holds for ϕ .

REMARK 6.7. Conjecture 6.5 is true for $n \leq 3$. For n = 3 this follows from Theorem 6.1: we can take ϕ_{ϵ} to be any perturbation of the given line arrangement ϕ . For n = 2 we take $\phi_{\epsilon} = \phi$ itself because of the following fact: if a binary form $\phi(x, y)$ is real-rooted then also $x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}$ is real-rooted. This follows from Corollary 6.4.

We put the lid on this paper with a brief discussion of the dynamical system $\psi : \mathbb{P}_{\mathbb{R}}^{n-1} \dashrightarrow \mathbb{P}_{\mathbb{R}}^{n-1}$ associated with a tensor $A \in (\mathbb{R}^n)^{\otimes d}$. Iterating this map is known as the *tensor power method*, and it is used as a tool in tensor decomposition [**AG**]. This generalizes the power method of numerical linear algebra for computing the eigenvectors of a matrix $A \in (\mathbb{R}^n)^{\otimes 2}$. One starts with some unit vector **v** and repeatedly applies the map $\mathbf{v} \mapsto \frac{A\mathbf{v}}{||A\mathbf{v}||}$. For generic inputs A and \mathbf{v} , this iteration converges to the eigenvector corresponding to the largest absolute eigenvalue.

Suppose that $\mathbf{u} \in \mathbb{P}^{n-1}_{\mathbb{R}}$ is an eigenpoint of a given tensor $A \in (\mathbb{R}^n)^{\otimes d}$. We say that \mathbf{u} is a *robust eigenpoint* if there exists an open neighborhood \mathcal{U} of \mathbf{u} in $\mathbb{P}^{n-1}_{\mathbb{R}}$ such that, for all starting vectors $\mathbf{v} \in \mathcal{U}$, the iteration of the map ψ converges to \mathbf{u} .

EXAMPLE 6.8 (Odeco Tensors). A symmetric tensor is *orthogonally decomposable* (this was abbreviated to *odeco* by Robeva [**Rob**]) if it has the form

$$\phi = \sum_{i=1}^n a_i \mathbf{v}_i^{\otimes d}$$

where $a_1, \ldots, a_n \in \mathbb{R}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis of \mathbb{R}^n . Following $[\mathbf{AG}]$, the robust eigenpoints of an odeco tensor are the basis vectors \mathbf{v}_i , and they can be computed using the tensor power method. Up to an appropriate change of coordinates, the odeco tensors are the Fermat polynomials in (2.4). The robust eigenpoints of $\phi = x_1^d + \cdots + x_n^d$ are the coordinate points $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The region of attraction of the *i*th eigenpoint \mathbf{e}_i under the iteration of the map $\nabla \phi$ is the set of all points in $\mathbb{R}^{n-1}_{\mathbb{R}}$ whose *i*th coordinate is largest in absolute value.

Odeco tensors for n = d = 3 have three robust eigenvalues. At present we do not know any ternary cubic ϕ with more than three robust eigenpoints. Theorem 6.1 might suggest that products of linear forms are good candidates. However, we ran experiments with random triples of lines in $\mathbb{P}^2_{\mathbb{R}}$, and we observed that the number of robust eigenvalues is usually one and occasionally zero. We never found a factorizable ternary cubic ϕ with two or more robust eigenpoints. The Cremona map in Example 1.1 shows that $\phi = xyz$ is a cubic with zero robust eigenpoints. Here is a similar example that points to the connection with frame theory in **[ORS]**.

EXAMPLE 6.9 (n = d = 3). We consider the factorizable ternary cubic

(6.3)
$$\phi = (2x + 2y - z)(2x - y + 2z)(-x + 2y + 2z).$$

This equals the frame decomposable tensor seen in [**ORS**, Examples 1.1 and 5.2]:

(6.4)
$$\phi = \frac{1}{24} \left((-5x + y + z)^3 + (x - 5y + z)^3 + (x + y - 5z)^3 + (3x + 3y + 3z)^3 \right).$$

Its gradient map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is given by

$$\nabla \phi = 3 \begin{pmatrix} -4x^2 + 4xy + 4xz + 2y^2 + yz + 2z^2 \\ 2x^2 + 4xy + xz - 4y^2 + 4yz + 2z^2 \\ 2x^2 + xy + 4xz + 2y^2 + 4yz - 4z^2 \end{pmatrix}.$$

This has four fixed points and three singular points, for a total of seven eigenpoints:

$$(1:1:-5), (1:-5:1), (-5:1:1), (3:3:3), (2:2:-1), (2:-1:2), (-1:2:2).$$

Note that the pairwise intersections of the lines coincide with the coefficient vectors in (6.3). By plugging $\nabla \phi$ into itself, we verify that the second iterate map equals

$$(\nabla \phi)^{\circ 2} = \nabla \phi \circ \nabla \phi = -3^{6} \phi(x, y, z) \cdot \begin{pmatrix} x & y & z \end{pmatrix}^{T}.$$

Hence $(\nabla \phi)^{\circ 2}$ is the identity map on all points in $\mathbb{P}^2_{\mathbb{R}} \setminus \{\phi = 0\}$. Every such point lies in a limit cycle of length two. The points on the curve $\{\phi = 0\}$ map to the singular points. We conclude that the ternary cubic ϕ has no robust eigenpoints.

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