These are problems with partial fractions

Note Title 10/31/2011

$$\int \frac{x^2}{15+4x-4x^2}\,dx$$

$$\frac{x^2}{15+4x-4x^2} = -\frac{x^2}{4x^2-4x-15}$$

$$4x^2-4x-15 \overline{\smash{)}\, x^2} \quad \text{quotient } \frac{1}{4}$$

$$\frac{x^2 - x - \frac{15}{4}}{x+\frac{15}{4}}$$

$$= -\left[\frac{1}{4} + \frac{x+\frac{15}{4}}{4x^2-4x-15}\right]$$

Need to have $\frac{P(x)}{Q(x)}$ $\deg P(x) < \deg Q(x)$ before doing partial frac!

$$= -\left[\frac{1}{4} + \frac{x+\frac{15}{4}}{(2x-5)(2x+3)}\right]$$

$$= -\left[\frac{1}{4} + \frac{A}{2x-5} + \frac{B}{2x+3}\right]$$

Need to determine A, B. To do so,

$$\frac{x+\frac{15}{4}}{(2x-5)(2x+3)} = \frac{A}{2x-5} + \frac{B}{2x+3}$$

$$\Longleftrightarrow \quad x+\frac{15}{4} = A(2x+3) + B(2x-5)$$

This must hold for any x. so insert

$x=\frac{5}{2}$ gives us $\frac{5}{2}+\frac{15}{4} = A\left(2\cdot\frac{5}{2}+3\right) + B\cdot 0$

so, $8A = \frac{25}{4} \Rightarrow A = \frac{25}{32}$

$x=-\frac{3}{2}$ gives us $-\frac{3}{2}+\frac{15}{4} = A\cdot 0 + B(-3-5)$

so, $-8B = \frac{9}{4} \Rightarrow B = -\frac{9}{32}$

Hence, the integral is

$$-\int\left[\frac{1}{4} + \frac{\frac{25}{32}}{2x-5} - \frac{\frac{9}{32}}{2x+3}\right]dx$$

$$= -\left[\frac{x}{4} + \frac{25}{32}\cdot\frac{1}{2}\ln|2x-5| - \frac{9}{32}\cdot\frac{1}{2}\ln|2x+3|\right] + C$$

$$= -\frac{x}{4} - \frac{25}{64}\ln|2x-5| + \frac{9}{64}\ln|2x+3| + C$$

Next,

$$\int \frac{2x^2+6}{(x^2-2x+2)^2}\,dx$$

$$\frac{2x^2+6}{(x^2-2x+2)^2} = \frac{\overset{\text{deg}=1}{Ax+B}}{\underbrace{x^2-2x+2}_{\text{deg}=2}} + \frac{\overset{\text{deg}=1}{Cx+D}}{\underbrace{(x^2-2x+2)^2}_{\text{deg}=4}}$$

$$2x^2+6 = (Ax+B)(x^2-2x+2) + Cx+D$$
$$= Ax^3 - 2Ax^2 + 2Ax + Bx^2 - 2Bx + 2B + (Cx+D)$$
$$= Ax^3 + (B-2A)x^2 + (2A-2B+C)x + 2B+D$$

Matching the coefficients on both sides gives us

$A=0$, $B-2A=2$, $2A-2B+C=0$, $2B+D=6$

$\Longleftrightarrow$ $A=0$, $B=2$, $C=4$, $D=2$

So, $$\int \frac{2x^2+6}{(x^2-2x+2)^2}\,dx = \int\left(\frac{2}{x^2-2x+2} + \frac{4x+2}{(x^2-2x+2)^2}\right)dx$$

Now this form suggests the following substitution

$$x^2-2x+2 = (x-1)^2+1$$

So $u = x-1$ or $x = u+1$ $\quad dx = du$

Then the integral becomes

$$= \int \left(\frac{2}{u^2+1} + \frac{4u+6}{(u^2+1)^2} \right) du \quad — (*)$$

Substitute $u = \tan y$ then

$$du = \sec^2 y \, dy = \underbrace{(1+\tan^2 y)}_{= 1+u^2} dy$$

$$\Longleftrightarrow \frac{du}{1+u^2} = dy$$

$$\text{So } (*) = 2 \int dy + 2 \int \frac{2\tan y + 3}{\sec^2 y} dy$$

$$= 2y + 2 \int \left(2 \cdot \frac{\sin y}{\cos y} \cdot \cos^2 y + 3 \cos^2 y \right) dy$$

$$= 2y + 2 \int \left(2 \sin y \cos y + 3 \cdot \frac{1+\cos 2y}{2} \right) dy$$

$$= 2y + 2 \int \left(\sin 2y + \frac{3}{2}(1+\cos 2y) \right) dy$$

$$= 2y + 2 \cdot \left[\frac{-1}{2} \cos 2y + \frac{3}{2} y + \frac{3}{4} \sin 2y \right] + C$$

$$= 2y - \cos 2y + 3y + \frac{3}{2} \sin 2y + C$$

$$= 5y - \cos 2y + \frac{3}{2} \sin 2y + C$$

$$= 5y - (2\cos^2 y - 1) + 3 \sin y \cdot \cos y + C$$

$$= 5y - \left(\frac{2}{\sec^2 y} - 1 \right) + 3 \cdot \frac{\sin y}{\cos y} \cdot \cos^2 y + C$$

$$= 5y - \left(\frac{2}{\sec^2 y} - 1 \right) + 3 \cdot \tan y \cdot \frac{1}{\sec^2 y} + C$$

$$= 5 \tan^{-1}(u) - \left(\frac{2}{1+u^2} - 1 \right) + \frac{3u}{1+u^2} + C$$

$$= 5\tan^{-1}(u) + \frac{3u-2}{1+u^2} + \underbrace{C+1}$$

you can replace this by new C

$$= 5\tan^{-1}(x-1) + \frac{3x-5}{1+(x-1)^2}$$

$$= 5\tan^{-1}(x-1) + \frac{3x-5}{x^2-2x+2} + C$$