Math 135A, Winter 2011.
March 17, 2011.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 6 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 7 pages (including this one) with 6 problems.

1	
2	
3	
4	
5	
6	
TOTAL	

1. A pair (X, Y) of random variables has the joint density

$$f(x,y) = \begin{cases} cy & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\int_0^1 x^n\,dx = \frac{1}{n+1}$ for $n > -1$.

(a) Compute the constant c.

$$1 = c\int_0^1 dx \int_0^x y\,dy = c\int_0^1 \frac{x^2}{2}\,dx = \frac{c}{6}$$

$$\underline{\underline{c = 6}}$$

(b) Compute $E(XY)$.

$$E(XY) = 6\int_0^1 dx \int_0^x xy \cdot y\,dy$$

$$= 6\int_0^1 x\,dx \cdot \frac{x^3}{3} = 2\int_0^1 x^4\,dx = \underline{\underline{\frac{2}{5}}}$$

(c) Compute the marginal density of X.

$$f_X(x) = 6\int_0^x y\,dy = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) Compute the density of $Z = X^2$.

for $z \in [0,1]$: $P(Z \le z) = P(X^2 \le z) = P(X \le \sqrt{z})$

$$= \int_0^{\sqrt{z}} 3x^2\,dx$$

$$f_Z(z) = \begin{cases} 3z \cdot \frac{1}{2\sqrt{z}} = \frac{3}{2}\sqrt{z} & z \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

2. A group consists of 10 people: 5 Swedes, 3 Norwegians, and 2 Finns. They are seated in a *row* of 10 chairs. (Imagine the chairs numbered, from left to right, $1, 2, \ldots, 10$.)

(a) Compute the probability that all the Swedes sit together (i.e., occupy adjacent chairs).

$$\frac{5!\,6!}{10!}$$

(b) Compute the probability that all the Swedes *and also* all the Norwegians sit together.

$$\frac{5!\,3!\,4!}{10!}$$

(c) A Swede is *happy* if there is another Swede sitting next to him *and* on his left. Compute the expected number of happy Swedes. Give the result as a simple fraction.

$I_i = I\{\text{Swede } i \text{ is happy}\}$

$$EI_i = \underset{\substack{\uparrow\\ \text{can't sit}\\ \text{on seat 1}}}{\frac{9}{10}} \cdot \underset{\substack{\nwarrow\\ \text{left seat}\\ \text{occupied by}\\ \text{another Swede}}}{\frac{4}{9}} = \frac{2}{5}$$

Answer: $E(I_1 + \cdots + I_5) = 5 \cdot \frac{2}{5} = \underline{\underline{2}}$

(d) Let N be the number of Swedes that sit to the left of the leftmost Norwegian. (E.g., if the leftmost Norwegian sits on seat 4, then N is the number of Swedes on seats 1, 2, 3.) Compute EN. Give the result as a simple fraction.

Now $N = I_1 + \cdots + I_5$

$I_i = I\{\text{Swede } i \text{ has no Norw. on his left}\}$

$= I\{\text{Swede } i \text{ is leftmost in the ordering of himself and 3 Norwegians}\}$

$EI_i = \frac{1}{4}$, so $\underline{\underline{EN = \frac{5}{4}}}$

3. Roll a fair die 10 times.

(a) Compute the probability that at least one of the six numbers is rolled exactly three times.

$A_i = \{i \text{ rolled } 3 \text{ times}\}, \quad i=1,\dots,6$

$$P(A_1 \cup \dots \cup A_6) = 6 \cdot \binom{10}{3} \frac{5^7}{6^{10}} - \binom{6}{2}\binom{10}{3}\binom{7}{3} \frac{4^4}{6^{10}}$$

$$+ \binom{6}{3}\binom{10}{3}\binom{7}{3}\binom{4}{3} \frac{3}{6^{10}}$$

(b) Let X be the number of times an even number is rolled, and Y the number of times 1 is rolled. Compute the joint probability mass function of (X, Y). (Write a formula for $P(X = i, Y = j)$ for all relevant i and j rather than a large table.)

$$P(X=i, Y=j) = \binom{10}{i}\binom{10-i}{j} \frac{3^i \cdot 2^{10-i-j}}{6^{10}}$$

if $i \geq 0, \ j \geq 0, \ i+j \leq 10$

(c) Are X and Y independent? Explain.

No, $P(X=10) = \left(\frac{1}{2}\right)^{10}$, $P(Y=10) = \left(\frac{1}{6}\right)^{10}$

$$0 = P(X=10, Y=10) \neq P(X=10)\,P(Y=10)$$

4. Alice and Bob play the following game. A deck contains 4 red and 4 black cards. On every round, four cards are selected from the shuffled deck without replacement. If exactly 2 red cards are selected, the game is over and Alice wins. If exactly 1 red card is selected, the game is over and Bob wins. Otherwise, the game is repeated until one of them wins.

(a) Compute the probability that Alice wins on the first round of this game, and the probability that Bob wins on the first round.

$$P(\text{A wins on 1st round}) = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{36}{70}$$

$$P(\text{B wins on 1st round}) = \frac{\binom{4}{1}\binom{4}{3}}{\binom{8}{4}} = \frac{16}{70}$$

(b) Compute the probability that Alice wins this game.

$$\frac{36}{16+36} = \frac{9}{13}$$

(c) Compute the expected number of rounds the game is played.

$$P(\text{somebody wins on 1st round}) = \frac{52}{70} = \frac{26}{35}$$

so the answer is the expected value of Geometric$\left(\frac{26}{35}\right)$, or $\underline{\underline{\frac{35}{26}}}$.

5. A casino offers the following game. The player tosses *three* fair coins and wins \$5 if exactly 3 Heads appear, wins or loses nothing if exactly 2 Heads appear, wins \$1 if exactly one Heads appears, and, finally, loses \$4 (i.e., wins −\$4) if no Heads appear. (*All tosses in this problem are independent.*)
(a) Alice plans to play this game 4 times. Compute the expectation and variance of her winnings.

X_1 = wins on 1st game

$$EX_1 = 5 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + (-4) \cdot \frac{1}{8} = \frac{1}{2}$$

$$EX_1^2 = 25 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 16 \cdot \frac{1}{8} = \frac{44}{8} = \frac{11}{2},$$

so $$\mathrm{Var}(X_1) = \frac{11}{2} - \frac{1}{4} = \frac{21}{4}$$

<u>Answer</u>: $E(X_1 + \cdots + X_4) = \underline{\underline{2}}$

$\mathrm{Var}(X_1 + \cdots + X_4) = \underline{\underline{21}}$

(b) Bob plays the game 2100 times. Approximate the probability that he wins at least \$1050, and the probability that he wins at least \$1155 ($1155 = 1050 + 21 \cdot 5$).

$n = 2100$

$$P(X_1 + \cdots + X_n \geq x)$$

$$= P\left(\frac{X_1 + \cdots + X_n - n \cdot \frac{1}{2}}{\sqrt{n \cdot \frac{21}{4}}} \geq \frac{x - n \cdot \frac{1}{2}}{\sqrt{n \cdot \frac{21}{4}}} \right)$$

$$\approx P\left(Z \geq \frac{x - n \cdot \frac{1}{2}}{\sqrt{n \cdot \frac{21}{4}}} \right)$$

When $x = 1050 = \frac{n}{2}$, this prob. is about $\underline{\underline{0.5}}$.

When $x = 1155$:

$$= P\left(Z \geq \frac{21 \cdot 5}{\sqrt{2100 \cdot \frac{21}{4}}} \right) = P\left(Z \geq \frac{\cancel{21 \cdot 5}}{\cancel{21 \cdot 10} \cdot \frac{1}{2}} \right)$$

$$= P(Z \geq 1) = 1 - P(Z \leq 1) \approx 1 - 0.841$$

$$= \underline{\underline{0.159}}$$

6. A casino offers the game *200 Balls* for the next 19900 "guests." A bag contains 200 balls, 2 are black and 198 are white. A player selects two balls from the bag at random, without replacement, and wins a free night at the hotel if both selected balls are black. The balls are then put back into the bag for the next player.

(a) Let X be the number of winners. Determine the *exact* distribution (i.e., the probability mass function) of X.

X is Binomial $(19900, p)$,

where $p = P(\text{win}) = \frac{1}{\binom{200}{2}} = \frac{1}{19900}$

(b) Using a relevant approximation, compute the probability that there are no more than three winners.

$X \approx \text{Poisson}\left(19900 \cdot \frac{1}{19900}\right) = \text{Poisson}(1)$

$P(X \leq 3) \approx e^{-1} + e^{-1} + \frac{1}{2}e^{-1} + \frac{1}{6}e^{-1} = \underline{\underline{\frac{8}{3e}}}$

(c) Using a relevant approximation, compute the probability that there are at least two winners among the first $9950 = 19900/2$ players, but there are no more winners among the second 9950 players.

X_1 = no. of wins among first 9950 players
$\approx \text{Poisson}(\frac{1}{2})$

X_2 = no. of wins among second 9950 players
$\approx \text{Poisson}(\frac{1}{2})$

X_1 and X_2 are independent, so

$P(X_1 \geq 2, X_2 = 0) = P(X_1 \geq 2)\, P(X_2 = 0)$
$= (1 - e^{-1/2} - \frac{1}{2}e^{-1/2}) \cdot e^{-1/2}$
$= \underline{\underline{\left(1 - \frac{3}{2\sqrt{e}}\right) \cdot \frac{1}{\sqrt{e}}}}$