Math 135A, Winter 2011.
March 2, 2011.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Calculators, books or notes are not allowed. Unless you are directed to do so, or it is required for further work, do *not* evaluate complicated expressions to give the result as a decimal number.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

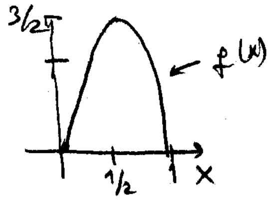
1	
2	
3	
4	
TOTAL	

1. Random variable X has the density function

$$f(x) = \begin{cases} c(x - x^2) & x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Recall $\int_0^1 x^n \, dx = \frac{1}{n+1}$ for $n > -1$.
(a) Compute the constant c.

$$1 = c \int_0^1 (x - x^2)\, dx = c\left(\frac{1}{2} - \frac{1}{3}\right) = c \cdot \frac{1}{6}$$



$\underline{\underline{c = 6}}$ 8

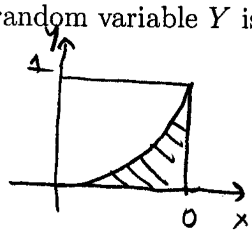
(b) Compute Var(X).

$$EX = 6 \int_0^1 (x^2 - x^3)\, dx = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

$$EX^2 = 6 \int_0^1 (x^3 - x^4)\, dx = 6 \cdot \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{6}{20}$$ 9

$$\mathrm{Var}(X) = EX^2 - (EX)^2 = \frac{6}{20} - \frac{1}{4} = \underline{\underline{\frac{1}{20}}}$$

(c) The random variable Y is uniform on $[0,1]$ and is independent of X. Compute $P(Y \le X^2)$.



$$P(Y \le X^2) = \int_0^1 dx \int_0^{x^2} 6(x - x^2)\, dy$$ 8

$$f(x,y) = \begin{cases} 6(x - x^2), & x, y \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

$$= \int_0^1 dx \cdot 6(x - x^2)\, x^2 dy = \frac{6}{20} = \underline{\underline{0.3}}$$

2. A bag contains 4 balls, numbered 1, 2, 3, and 4. Select a ball from the bag three times, *with replacement*. Let X be the number of times ball 1 is selected and Y be the number of times ball 2 is selected.

(a) Determine the joint probability mass function of (X, Y).

$$P(X = i, Y = j) = \frac{\binom{3}{i}\binom{3-i}{j} \cdot 2^{3-i-j}}{4^3}$$

$i, j = 0, 1, 2, 3$

$i + j \leq 3$

9

(b) Determine the conditional probability $P(X = 1 | Y = 1)$.

$$= \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{3 \cdot 2 \cdot 2}{3 \cdot 3^2} = \underline{\underline{\frac{4}{9}}}$$

8

(c) Are X and Y independent? Explain.

No: $P(X = 3, Y = 3) = 0$, while

$P(X = 3) = P(Y = 3) = \frac{1}{4^3}$, 8

so $P(X = 3, Y = 3) \neq P(X = 3) \cdot P(Y = 3)$.

3. A casino offers the game *Three Coins*. The player tosses *three* fair coins and wins if *exactly one* Heads appears. (*All tosses in this problem are independent.*)

(a) What is the winning probability for this game?

$$\frac{3}{8}$$

P

(b) Alice plays the game *Three Coins* 200 times. Let N be the total number of games she wins. Identify the distribution of N (i.e., its probability mass function) and determine EN and $\mathrm{Var}(N)$.

$$N \sim \text{Binomial}\left(200, \tfrac{3}{8}\right).$$

$$EN = 200 \cdot \frac{3}{8} = 75$$

$$\mathrm{Var}\, N = 200 \cdot \frac{3}{8} \cdot \frac{5}{8}$$

8

(c) What is the probability that Alice wins at least 80 games? Use a relevant approximation to give the answer as a decimal number. Also, you may approximate $\sqrt{30} \approx 5$.

$$P(N \geq 80) = P\left(\frac{N - EN}{\sqrt{\mathrm{Var}\, N}} \geq \frac{80 - EN}{\sqrt{\mathrm{Var} N}}\right)$$

$$\approx P\left(Z \geq \frac{5}{\sqrt{200 \cdot \frac{3}{8} \cdot \frac{5}{8}}}\right)$$

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$$= P\left(Z \geq \frac{5 \cdot 8}{10 \cdot \sqrt{30}}\right)$$

$$\approx P(Z \geq 0.8)$$

$$= 1 - P(Z \leq 0.8) = 1 - \Phi(0.8)$$

$$\approx 1 - 0.788 = \underline{\underline{0,212}}$$

4. A casino offers the game *Ten Coins*. The player tosses *ten* fair coins and wins if *no* Heads appears. (*All tosses in this problem are independent.*)

(a) Bob plays the game *Ten Coins* until he gets the first win. Let N be the number of games he plays. Identify the distribution of N (i.e., its probability mass function) and determine EN.

p = winning prob. in Ten Coins $= \frac{1}{2^{10}}$

N is $\underline{\text{Geometric}(p)}$, so $EN = \frac{1}{p} = 2^{10} = \underline{1024}$

8

(b) Carol plays the game *Ten Coins* $2048 = 2^{11}$ times. Using a relevant approximation, compute the probability that she wins at least three times.

no. of wins for Carol is $\text{Binomial}(2^{11}, \frac{1}{2^{10}})$

$\approx \text{Poisson}(2^{11} \cdot \frac{1}{2^{10}}) = \text{Poisson}(2)$

$$P(\geq 3 \text{ wins for Carol}) = 1 - P(\leq 2 \text{ wins})$$

9

$$= 1 - e^{-2} - 2 \cdot e^{-2} - \frac{2^2}{2!} e^{-2}$$

$$= \underline{1 - 5 \cdot e^{-2}}$$

(c) Now assume Bob and Carol each play one *Ten Coins* game per night, first Bob then Carol. Compute the probability that Carol wins the game three times before Bob wins even once. Write the answer as a simple expression.

First 3 times at least one of them wins, it has to be Carol but not Bob.

$$P(\text{Carol wins but not Bob} \mid \text{at least one of them wins})^3$$

8

$$= \left(\frac{p(1-p)}{1-(1-p)^2}\right)^3 = \left(\frac{p(1-p)}{p(2-p)}\right)^3$$

$$= \left(\frac{1-p}{2-p}\right)^3 = \underline{\left(\frac{1023}{2047}\right)^3}$$