MATH 116: Solution Set #4.

§4-1: #1.1, 1.5, 1.9, 1.10 §4-2: #2.4 §4-3: #3.5, 3.6

#1.1 Let $\mathcal{U} = \{(u^1,u^2) \in \mathbb{R}^2 \mid -\pi < u^1 < \pi, -\pi < u^2 < \pi\}$ and define

$$\underset{\sim}{x}(u^1,u^2) = ((2+\cos u^1)\cos u^2, (2+\cos u^1)\sin u^2, \sin u^1),$$

(a) Prove that $\underset{\sim}{x}$ is a simple surface.

Since $\sin u^1$ is 1 to 1 for $-\pi < u^1 < \pi$, and $\forall u^2$, $\cos u^2 + \sin u^2$ Clearly $\underset{\sim}{x}(u^1,u^2)$ is 1 to 1. Also, since $\sin x$ and $\cos x$ are $\in C^k$ for any k, $\underset{\sim}{x}$ is of class C^k for any k.

Clearly $\mathcal{U}$ is an open set by its definition.

Now, $\dfrac{\partial \underset{\sim}{x}}{\partial u^1} = (-\sin u^1 \cos u^2, -\sin u^1 \sin u^2, \cos u^1)$

and $\dfrac{\partial \underset{\sim}{x}}{\partial u^2} = ((2+\cos u^1)(-\sin u^2), (2+\cos u^1)\cos u^2, 0)$

$$\Rightarrow \frac{\partial \underset{\sim}{x}}{\partial u^1} \times \frac{\partial \underset{\sim}{x}}{\partial u^2} = \Big(-(2+\cos u^1)\cos u^1 \cos u^2, -(2+\cos u^1)(\sin u^2)(\cos u^1),$$

$$-(2+\cos u^1)\sin u^1 \cos^2 u^2 - (2+\cos u^1)\sin u^1 \sin^2 u^2\Big)$$

$$= -(2+\cos u^1)(\cos u^1 \cos u^2, \cos u^1 \cdot \sin u^2, \sin u^1). \text{ (since } \cos^2 + \sin^2 = 1)$$

For $\underset{\sim}{x}$ to be a simple surface, $\underset{\sim}{x}_1 \times \underset{\sim}{x}_2 \neq \vec{0}$ on $\mathcal{U}$.

Since $(2+\cos u^1) \neq 0$ $\forall u^1$, if $\underset{\sim}{x}_1 \times \underset{\sim}{x}_2 = \bar{0}$ each component must vanish, hence $u^1 = 0 \Rightarrow \underset{\sim}{x}_1 \times \underset{\sim}{x}_2 = -3(\cos u^2, \sin u^2, 0)$

But no value of u^2 gives $\cos u^2 = \sin u^2 = 0$. Therefore $\underset{\sim}{x}_1 \times \underset{\sim}{x}_2 \neq 0$ on $\mathcal{U}$ and $\underset{\sim}{x}$ is a simple surface.

(b) Compute $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ and $\hat{n}$ as functions of u^1 and u^2

#1.1(b) cont.

From before: $\underset{\sim}{X}_1 = (-\sin u^1 \cos u^2, -\sin u^1 \sin u^2, \cos u^1)$

$$\underset{\sim}{X}_2 = (2+\cos u^1)(-\sin u^2, \cos u^2, 0)$$

and
$$\hat{n} = \frac{-(2+\cos u^1)(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1)}{[2+\cos u^1]\,(\cos^2 u^1 \cos^2 u^2 + \cos^2 u^1 \sin^2 u^2 + \sin^2 u^1)^{1/2}}$$

$$= -(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1) \quad (\text{since } \cos^2 + \sin^2 = 1)$$

#1.5 Let $\underset{\sim}{X}(u^1,u^2) = (u^1+u^2, u^1-u^2, u^1u^2)$. Show that $\underset{\sim}{X}$ is simple.

Clearly $\underset{\sim}{X}$ is 1 to 1 and of class C^k, $k \geq 1$.

(If $\underset{\sim}{X}(u^1,u^2) = \underset{\sim}{X}(v^1,v^2) \Rightarrow u^1+u^2 = v^1+v^2$, $u^1-u^2 = v^1-v^2$, and $u^1u^2 = v^1v^2$
hence $u^1 = u^2+v^1-v^2 \Rightarrow 2u^2+v^1-v^2 = v^1+v^2 \Rightarrow u^2 = v^2$, $u^1 = v^1$ follows, hence $\underset{\sim}{X}$ is 1 to 1)

Also, assuming $\mathcal{U} = \mathbb{R}^2$, $\mathcal{U}$ is an open set.

Now $\underset{\sim}{X}_1 = (1, 1, u^2)$ and $\underset{\sim}{X}_2 = (1, -1, u^1)$.

So, $\underset{\sim}{X}_1 \times \underset{\sim}{X}_2 = (u^1+u^2, -u^1+u^2, -2) \neq \bar{0} \ \forall u^1, u^2$ on $\mathcal{U}$.

Therefore $\underset{\sim}{X}$ is a simple surface.

Find $\hat{n}$ and the equation of the tangent plane at $(1,2)$.

$$\hat{n} = \frac{(u^1+u^2, u^2-u^1, -2)}{((u^1+u^2)^2+(u^1-u^2)^2+4)^{1/2}} = \frac{(u^1+u^2, u^2-u^1, -2)}{(2(u^1)^2+2(u^2)^2+4)^{1/2}}, \text{ so } \hat{n}(1,2) = \frac{(3,1,-2)}{\sqrt{14}}$$

The equation of the tangent plane at $(1,2)$ is given by:

$\langle \underset{\sim}{x} - \underset{\sim}{X}_0, \hat{n} \rangle = 0$. With $\underset{\sim}{X}_0(1,2) = (3,-1,2)$ and $\hat{n} = \frac{1}{\sqrt{14}}(3,1,-2)$

$\Rightarrow \langle (x-3, y+1, z-2), (3,1,-2) \rangle = 0 \Rightarrow 3x-9+y+1-2z+4 = 0$

$3(x-3)+(y+1)-2(z-2)$

or $3x+y-2z=4$.

§4.1 #1.9 Let $\underset{\sim}{X}(\theta,v) = (\cos\theta, \sin\theta, 0) + v(\sin\tfrac{1}{2}\theta\cos\theta, \sin\tfrac{1}{2}\theta\sin\theta, \cos\tfrac{1}{2}\theta)$ with $-\pi<\theta<\pi$, $-\frac{1}{2}<v<\frac{1}{2}$. Compute $\hat{n}(\theta,0)$ and show that $\lim_{\theta\to-\pi}\hat{n}(\theta,0) = -\lim_{\theta\to\pi}\hat{n}(\theta,0)$ while $\lim_{\theta\to-\pi}\underset{\sim}{X}(\theta,0) = \lim_{\theta\to\pi}\underset{\sim}{X}(\theta,0)$.

$\underset{\sim}{X}_1 = (-\sin\theta, \cos\theta, 0) + f(\theta,v)$ where $f(\theta,0)=0$.

and $\underset{\sim}{X}_2 = (\sin\tfrac{1}{2}\theta\cos\theta, \sin\tfrac{1}{2}\theta\sin\theta, \cos\tfrac{1}{2}\theta)$.

So, $$\hat{n}(\theta,0) = \frac{\underset{\sim}{X}_1(\theta,0)\times\underset{\sim}{X}_2(\theta,0)}{|\underset{\sim}{X}_1(\theta,0)\times\underset{\sim}{X}_2(\theta,0)|} = \frac{(\cos\frac{\theta}{2}\cos\theta, \cos\frac{\theta}{2}\sin\theta, -\sin\frac{\theta}{2}(\cos^2\theta+\sin^2\theta))}{(\cos^2\frac{\theta}{2}\cos^2\theta + \cos^2\frac{\theta}{2}\sin^2\theta + \sin^2\frac{\theta}{2})^{1/2}}$$

$$= (\cos\tfrac{\theta}{2}\cos\theta, \cos\tfrac{\theta}{2}\sin\theta, -\sin\tfrac{\theta}{2}). \quad (\text{since } \cos^2+\sin^2=1)$$

Now, $\lim_{\theta\to-\pi}\hat{n}(\theta,0) = \lim_{\theta\to-\pi}(\cos\frac{\theta}{2}\cos\theta, \cos\frac{\theta}{2}\sin\theta, -\sin\frac{\theta}{2}) = (0,0,1)$

and $\lim_{\theta\to\pi}\hat{n}(\theta,0) = (0,0,-1) \Rightarrow \lim_{\theta\to-\pi}\hat{n}(\theta,0) = -\lim_{\theta\to\pi}\hat{n}(\theta,0)$.

Also, $\lim_{\theta\to-\pi}\underset{\sim}{X}(\theta,0) = \lim_{\theta\to-\pi}(\cos\theta,\sin\theta,0) = (-1,0,0)$

and $\lim_{\theta\to\pi}\underset{\sim}{X}(\theta,0) = (-1,0,0) \Rightarrow \lim_{\theta\to-\pi}\underset{\sim}{X}(\theta,0) = \lim_{\theta\to\pi}\underset{\sim}{X}(\theta,0)$.

#1.10. Let $S^2 = \{(u,v,w)\in\mathbb{R}^3 \mid u^2+v^2+w^2=1\}$ and $\mathbb{R}^2 = \{(u,v,w)\in\mathbb{R}^3 \mid w=0\}$. If $(u,v,0)\in\mathbb{R}^2$, the line determined by $(u,v,0)$ and $(0,0,1)$ intersects S^2 in a point other than $(0,0,1)$. Denote this point by $\underset{\sim}{X}(u,v)$. Compute the actual form of $\underset{\sim}{X}(u,v)$ and show that $\underset{\sim}{x}:\mathbb{R}^2\to\mathbb{R}^3$ is a simple surface.

Consider the line $\underset{\sim}{\alpha}(t)$ determined by $(u,v,0)$ and $(0,0,1)$.

$\Rightarrow \underset{\sim}{\alpha}(t) = (0,0,1) + t[(u,v,0)-(0,0,1)]$
$= (ut, vt, 1-t)$.

#1.10 cont (4)

Now $\underset{\sim}{X}(u,v)=(X^1(u,v),X^2(u,v),X^3(u,v))$ and $(x^1)^2+(x^2)^2+(x^3)^2=1$.

When these two curves intersect, $\underset{\sim}{X}(u,v)=\underset{\sim}{\alpha}(u,v,t)$, so:

$\underset{\sim}{X}(u,v)=(ut,vt,1-t)$ since $|\underset{\sim}{X}|^2=1$. we have that

$(ut)^2+(vt)^2+(1-t)^2=1$ or $(ut)^2+(vt)^2+t^2-2t=0$.

$\Rightarrow [(u^2+v^2+1)t-2]t=0$. So, either $t=0$

or $t=\frac{2}{u^2+v^2+1}$. Since $t=0$ gives the pt $(0,0,1)$ the top of the sphere, $t=\frac{2}{u^2+v^2+1}$ must be the desired pt.

Hence: $\underset{\sim}{X}(u,v)=\left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},1-\frac{2}{u^2+v^2+1}\right)$.

Show that $\underset{\sim}{X}:\mathbb{R}^2\to\mathbb{R}^3$ is a simple surface.

Since the x^1 and x^2 components of $\underset{\sim}{X}$ contain single powers of u and v, clearly $\underset{\sim}{X}(u,v)$ is 1 to 1. Since $u^2+v^2+1\neq 0$ we see that $\underset{\sim}{X}\in C^k$ for any $k\geq 1$ and obviously $\mathbb{R}^2$ is an open set.

Now, $\underset{\sim}{X}_1=\frac{1}{(u^2+v^2+1)^2}(2(u^2+v^2+1)-4u^2,-4uv,+4u)$

$$=\frac{1}{(u^2+v^2+1)^2}(2v^2-2u^2+2,-4uv,4u)$$

and $\underset{\sim}{X}_2=\frac{1}{(u^2+v^2+1)^2}(-4uv,2u^2-2v^2+2,4v)$

So, $\underset{\sim}{X}_1\times\underset{\sim}{X}_2=\frac{1}{(u^2+v^2+1)^4}\big(-16uv^2-4u(2u^2-2v^2+2),-16u^2v-4v(2v^2-2u^2+2),$
$(2v^2-2u^2+2)(2u^2-2v^2+2)-16u^2v^2\big)$

$$=\frac{1}{(u^2+v^2+1)^3}(-8u,-8v,4(1-u^2-v^2)).$$

§4-1 #1.10 cont

Now, for $\underset{\sim}{x}_1 \times \underset{\sim}{x}_2 = 0$ all components must vanish simultaneousl since $\frac{1}{(1+u^2+v^2)^3} \neq 0$ $\forall (u,v) \in \mathbb{R}^2$. But if $x_1 = x_2 = 0 \Rightarrow u = v = 0$ and hence $x_3 = 4 \neq 0$ so $\underset{\sim}{x}_1 \times \underset{\sim}{x}_2 \neq 0$ $\forall (u,v) \in \mathbb{R}^2$ and therefore $\underset{\sim}{x}$ is a simple surface.

§4-2 #2.4

Describe some possible parametrizations of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let $\mathcal{U}^+ = \mathcal{U}^- = \{(x,y) \in \mathbb{R}^2 \mid (\frac{x}{a})^2 + (\frac{y}{b})^2 < 1\}$

let $\mathcal{V}^+ = \mathcal{V}^- = \{(x,y) \in \mathbb{R}^2 \mid (\frac{x}{b})^2 + (\frac{y}{c})^2 < 1\}$

and let $\mathcal{W}^+ = \mathcal{W}^- = \{(x,y) \in \mathbb{R}^2 \mid (\frac{x}{a})^2 + (\frac{y}{c})^2 < 1\}$.

Then the ellipsoid can be covered with six patches namely:

$$\underset{\sim}{x}^+ : \mathcal{U}^+ \to \mathbb{R}^3 : \underset{\sim}{x}^+(u^1,u^2) = (u^1, u^2, c\sqrt{1-(\tfrac{u^1}{a})^2-(\tfrac{u^2}{b})^2})$$

$$\underset{\sim}{x}^- : \mathcal{U}^- \to \mathbb{R}^3 : \underset{\sim}{x}^-(u^1,u^2) = (u^1, u^2, -c\sqrt{1-(\tfrac{u^1}{a})^2-(\tfrac{u^2}{b})^2})$$

$\underset{\sim}{y}^+$, $\underset{\sim}{y}^-$, $\underset{\sim}{z}^+$, and $\underset{\sim}{z}^-$ are similar with $(\underset{\sim}{y}^{\pm})^2 = \pm b\sqrt{1-(\frac{v^1}{a})^2-(\frac{v^2}{c})^2}$ and $(\underset{\sim}{z}^{\pm})^1 = \pm a\sqrt{1-(\frac{w^1}{b})^2-(\frac{w^2}{c})^2}$ as you would expect.

The proof that where each path overlaps the appropriate composite function is a C^k coordinate transformation is similar to example 2.2 and is omitted.

§4-3

#3.5 For a coordinate patch $\underset{\sim}{x} : \mathcal{U} \to \mathbb{R}^3$ show that u^1 is arclength on the u^1-curves if and only if $g_{11} = 1$.

Assume $\underset{\sim}{X}: \mathcal{U} \to \mathbb{R}^3$ and u' is arclength on the u'-curves.

Now, the u'-curves are given by:

$$\underset{\sim}{\alpha}(u') = \underset{\sim}{X}(u', b) \text{ for some fixed } b.$$

Hence, $\underset{\sim}{X}_1(u', b) = \frac{\partial \underset{\sim}{X}}{\partial u'} = \frac{\partial \underset{\sim}{\alpha}(u')}{\partial u'}$. Now, $g_{11} = \langle \underset{\sim}{X}_1, \underset{\sim}{X}_1 \rangle$

$\Rightarrow g_{11} = \langle \frac{\partial \underset{\sim}{\alpha}(u')}{\partial u'}, \frac{\partial \underset{\sim}{\alpha}(u')}{\partial u'} \rangle = \left| \frac{\partial \alpha(u')}{\partial u'} \right|^2$ but $\alpha(u')$ is arc length parameterized hence $\left| \frac{\partial \alpha(u')}{\partial \alpha(u')} \right| = 1 \Rightarrow g_{11} = 1^2 = 1$.

Now assume $\underset{\sim}{X}: \mathcal{U} \to \mathbb{R}^3$ and $g_{11} = 1$.

Since the u'-curves are given by $\alpha(u') = \underset{\sim}{X}(u', b)$ for some fixed b we have $1 = g_{11} = \langle \underset{\sim}{X}_1, \underset{\sim}{X}_1 \rangle = \left| \frac{\partial \underset{\sim}{X}}{\partial u'} \right|^2$. But along the u'-curves $\underset{\sim}{X}(u', b) = \underset{\sim}{\alpha}(u') \Rightarrow \frac{\partial \underset{\sim}{X}}{\partial u'} = \frac{\partial \underset{\sim}{\alpha}}{\partial u'} \Rightarrow \left| \frac{\partial \underset{\sim}{X}}{\partial u'} \right|^2 = 1 = \left| \frac{\partial \underset{\sim}{\alpha}}{\partial u'} \right|^2$

hence $\left| \frac{\partial \underset{\sim}{\alpha}}{\partial u'} \right| = 1$ and $\underset{\sim}{\alpha}(u')$ is arclength parameterized.

QED

#3.6: Let x and y be cartesian coords. of the plane while r and θ are polar coords. Show that $x = r\cos\theta$ and $y = r\sin\theta$ is a C' coordinate transformation for $r > 0$.

Let $f^1(r,\theta) = r\cos\theta$, $f^2(r,\theta) = r\sin\theta$ then $(x,y) = (f^1, f^2)$

Clearly $f^1, f^2 \in C^k$ and if $r > 0$, $-\pi < \theta \leq \pi$ then $f = (f^1, f^2)$ is 1 to 1.

Then for $(r,\theta) = g(x,y) = (g^1, g^2)$, let $g^1 = \sqrt{x^2+y^2}$. Technically we should define 4 transformations $g^2_L = \tan^{-1}(\frac{y}{x})$, $x > 0$

$g^2_R = \pi - \tan^{-1}(\frac{y}{x})$ for $x < 0$, $g^2_u = \cot^{-1}(\frac{x}{y})$ $y > 0$ and $g^2_d = \pi - \cot^{-1}(\frac{x}{y})$ $y < 0$.

Which ever one is appropriate; since $(x,y) = (0,0)$ is excluded in all cases. it is clear that $g = (g^1, g^2)$ is $\in C^k$, and 1 to 1.

§4-3: #3.6 cont

Let $\underset{\sim}{X}(u^1,u^2)=(u^1,u^2)$. Then clearly $\underset{\sim}{X}_1=(1,0)$ and $\underset{\sim}{X}_2=(0,1)$ hence with $g_{ij}=\langle \underset{\sim}{X}_i,\underset{\sim}{X}_j\rangle$ we see that $(g_{ij})_{x,y}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Find the metric for polar coordinates.

Let $\underset{\sim}{X}(r,\theta)=(r\cos\theta, r\sin\theta)$

Then $\underset{\sim}{X}_1=(\cos\theta,\sin\theta)$ and $\underset{\sim}{X}_2=(-r\sin\theta, r\cos\theta)$

So, $g_{11}=\langle \underset{\sim}{X}_1,\underset{\sim}{X}_1\rangle=\cos^2\theta+\sin^2\theta=1$

and $g_{12}=g_{21}=\langle \underset{\sim}{X}_1,\underset{\sim}{X}_2\rangle=-r\cos\theta\sin\theta+r\cos\theta\sin\theta=0$

and $g_{22}=\langle \underset{\sim}{X}_2,\underset{\sim}{X}_2\rangle=r^2\sin^2\theta+r^2\cos^2\theta=r^2$

$$\Rightarrow (g)_{r,\theta}=\begin{pmatrix}1&0\\0&r^2\end{pmatrix}.$$

Extra:

(i) $$\frac{d}{dt}\underset{\sim}{X}(\gamma(t))=\frac{d}{dt}\left(x(u^1(t),u^2(t)),y(u^1(t),u^2(t)),z(u^1(t),u^2(t))\right)$$
$$=\left(\frac{\partial x}{\partial u^1}\frac{du^1}{dt}+\frac{\partial x}{\partial u^2}\frac{du^2}{dt},\ \frac{\partial y}{\partial u^1}\frac{du^1}{dt}+\frac{\partial y}{\partial u^2}\frac{du^2}{dt},\ \frac{\partial z}{\partial u^1}\frac{du^1}{dt}+\frac{\partial z}{\partial u^2}\frac{du^2}{dt}\right)$$
$$=\left(\frac{\partial x}{\partial u^1}\frac{du^1}{dt},\frac{\partial y}{\partial u^1}\frac{du^1}{dt},\frac{\partial z}{\partial u^1}\frac{du^1}{dt}\right)+\left(\frac{\partial x}{\partial u^2},\frac{\partial y}{\partial u^2},\frac{\partial z}{\partial u^2}\right)\frac{du^2}{dt}$$
$$=\frac{\partial}{\partial u^1}(\underset{\sim}{X})\frac{du^1}{dt}+\frac{\partial}{\partial u^2}(\underset{\sim}{X})\frac{du^2}{dt}.$$

(ii) Clearly if $\gamma'(t)=(a^1,a^2)$ and $\gamma(t)=(u^1(t),u^2(t))$ Then $\gamma'(t)=\left(\frac{du^1}{dt},\frac{du^2}{dt}\right)=(a^1,a^2)\Rightarrow \frac{du^1}{dt}=a^1$ and $\frac{du^2}{dt}=a^2$ hence if $\frac{d}{dt}\underset{\sim}{X}(\gamma(t))=\vec{W}$ then $\vec{W}=\frac{\partial \underset{\sim}{X}}{\partial u^1}a^1+\frac{\partial \underset{\sim}{X}}{\partial u^2}a^2$.

Extra cont.

(iii). $\underset{\sim}{X}(u^1,u^2) = (u^1, u^2, (u^1)^2 + (u^2)^2)$ let $\gamma(t) = (t, t^2)$.

Find $\underset{\sim}{X}(\gamma(t)) \equiv (\underset{\sim}{X} \circ \gamma)(t)$.

Clearly $\gamma(t) = (t, t^2) \Rightarrow u^1 = t$ and $u^2 = t^2$

hence $\underset{\sim}{X}(\gamma(t)) = (t, t^2, t^2 + t^4)$.

If $\bar{V} \equiv \gamma'(t)$ then $\bar{V} = (1, 2t)$ so $\vec{V}(1) = (1, 2)$

Also, $\bar{W} \equiv \frac{d}{dt}\underset{\sim}{X}(\gamma(t)) = \frac{\partial \underset{\sim}{X}}{\partial u^1}\frac{du^1}{dt} + \frac{\partial \underset{\sim}{X}}{\partial u^2}\frac{du^2}{dt}$.

where $\frac{du^1}{dt} = 1$ and $\frac{du^2}{dt} = 2t$ and $\frac{\partial \underset{\sim}{X}}{\partial u^1} = (1, 0, 2u^1)$

and $\frac{\partial \underset{\sim}{X}}{\partial u^2} = (0, 1, 2u^2)$

Hence, $\bar{W} = (1, 0, 2u^1)\cdot 1 + (0, 1, 2u^2)\cdot 2t$ and since $u^1 = t$, $u^2 = t^2$

$\Rightarrow \bar{W} = (1, 2t, 2t + 4t^3) \Rightarrow \bar{W}(1) = (1, 2, 6)$.

Now, $\underset{\sim}{X}_1(\gamma(1)) = (1, 0, 2)$ and $\underset{\sim}{X}_2(\gamma(1)) = (0, 1, 2)$

so clearly $a^1 \underset{\sim}{X}_1(1,1) + a^2 \underset{\sim}{X}_2(1,1) = 1(1, 0, 2) + 2(0, 1, 2)$

$= (1, 2, 6) = \bar{W}(1)$.