#4.5

Homework Solutions Math 116
Set #5

Unit speed curve: (let $\dot{} \equiv \frac{d}{ds}$, $' \equiv \frac{d}{dt}$)

$$\gamma(s) = \underset{\sim}{x}(\gamma^1, \gamma^2)$$

$$\dot{\gamma} = \underset{\sim}{x}_1 \dot{\gamma}^1 + \underset{\sim}{x}_2 \dot{\gamma}^2 = \underset{\sim}{x}_i \dot{\gamma}^i$$

$$\ddot{\gamma} = \underset{\sim}{x}_k \ddot{\gamma}^k + \underset{\sim}{x}_{ij} \dot{\gamma}^i \dot{\gamma}^j = \kappa N$$

$$\underset{\sim}{x}_{ij} = L_{ij} \vec{n} + \Gamma_{ij}^k \underset{\sim}{x}_k$$

$$\kappa N = \ddot{\gamma} = \left\{ \ddot{\gamma}^k + \Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j \right\} \underset{\sim}{x}_k + L_{ij} \dot{\gamma}^i \dot{\gamma}^j \vec{n}$$

$$\kappa_n = \langle \ddot{\gamma}, \vec{n} \rangle = L_{ij} \dot{\gamma}^i \dot{\gamma}^j$$

We find L_{ij} at $(1,1)$:

$$\underset{\sim}{x}(u^1, u^2) = (u^1, u^2, (u^1)^2 + (u^2)^2)$$

$$\underset{\sim}{x}_1 = (1, 0, 2u^1)$$

$$\underset{\sim}{x}_2 = (0, 1, 2u^2)$$

$$\underset{\sim}{x}_{11} = (0,0,2)$$

$$\underset{\sim}{x}_{12} = (0,0,0) = \underset{\sim}{x}_{21}$$

$$\underset{\sim}{x}_{22} = (0,0,2)$$

$$\vec{n} = \frac{(-2,-1,1)}{\sqrt{6}}$$

$$L_{ij} = \langle \underset{\sim}{x}_{ij}, \vec{n} \rangle$$

$$L_{11} = (0,0,2) \cdot (-2,-1,1) \frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

$$L_{12} = (0,0,0) \cdot (-2,-1,1) \frac{1}{\sqrt{6}} = 0$$

$$L_{22} = (0,0,2) \cdot (-2,-1,1) \frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

$$L_{ij} = \frac{2}{\sqrt{6}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$k_n = L_{ij}\,\dot\gamma^i\,\dot\gamma^j = L_{ij}\,\frac{d}{dt}\gamma^i\,\frac{d}{dt}\gamma^j\,\frac{1}{|\dot\gamma|}$$

$$\cancel{\dot\gamma = (2,1,6)} \qquad \cancel{|\dot\gamma| = \sqrt{41}}$$

$$\gamma^1(t) = t^2 \qquad \dot\gamma^1(t) = 2t\,\frac{dt}{ds} = \frac{2t}{|\dot\gamma|}\Big|_{t=1} = \frac{2}{\sqrt{41}}$$

$$\gamma^2(t) = t$$

$$\dot\gamma^2(t) = \frac{dt}{ds} = \frac{1}{|\dot\gamma|} = \frac{1}{\sqrt{41}}$$

$$L_{ij}\,\dot\gamma^i\,\dot\gamma^j = \left(\frac{2}{\sqrt{41}}, \frac{1}{\sqrt{41}}\right)\frac{2}{\sqrt6}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}2/\sqrt{41}\\ 1/\sqrt{41}\end{pmatrix}$$

$$= \frac{1}{41}\cdot\frac{2}{\sqrt6}\,(2,1)\cdot(2,1)$$

$$= \frac{1}{41}\,\frac{2}{\sqrt6}\,(4+1) = \frac{10}{41\sqrt6}$$

#4.5 directly:

$$\gamma(t) = (t^2, t, t^4 + t^2)$$

$$ds = \left|\frac{d}{dt}\gamma\right| dt, \quad T = \frac{d\gamma}{ds} = \frac{d\gamma}{dt}\frac{dt}{ds} = \frac{d\gamma/dt}{|\dot\gamma|}$$

$$\kappa N = \frac{d}{ds}T = \frac{d}{ds}\left(\dot\gamma \frac{dt}{ds}\right) = \ddot\gamma\left(\frac{dt}{ds}\right)^2 + \dot\gamma \frac{d^2t}{ds^2} \qquad \dot{} = \frac{d}{dt}$$

$$\kappa_n = \langle \kappa N, \vec n\rangle = \left\langle \left(\frac{dt}{ds}\right)^2 \ddot\gamma + \frac{d^2t}{ds^2}\dot\gamma, \vec n\right\rangle$$

$$= \left(\frac{dt}{ds}\right)^2 \langle \ddot\gamma, \vec n\rangle$$

$$\dot\gamma = (2t, 1, 4t^3 + 2t)$$

$$\left(\frac{ds}{dt}\right)^2 = |\dot\gamma|^2 = 4t^2 + 1 + (4t^3+2t)^2\Big|_{t=1} = 41$$

$$\ddot\gamma = (2, 0, 12t^2 + 2)\Big|_{t=1} = \overrightarrow{(2, 0, 14)}$$

$$\vec n = \begin{array}{c} \\ \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{array}\begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} \cdot \frac{1}{11} = \frac{(-2, -2, 1)}{\sqrt{6}}$$

$$K_n = \left(\frac{dt}{ds}\right)^2 \langle \ddot{\gamma}, \vec{n} \rangle \Big|_{t=1}$$

$$= \frac{1}{41} \overrightarrow{(2,0,14) \cdot (-2,-2,1) \frac{1}{\sqrt{6}}}$$

$$= \frac{-4+14}{41\sqrt{6}} = \frac{10}{41\sqrt{6}} \checkmark$$

#4.9 On unit sphere, $|\gamma(t)| = 1 = \gamma(t) \cdot \gamma(t)$.

Thus: $\dot{\gamma} \cdot \gamma + \gamma \cdot \dot{\gamma} = 0$ $\qquad \gamma = \vec{n}$

$$\Rightarrow \dot{\gamma} \cdot \gamma = 0$$

$$\Rightarrow \ddot{\gamma} \cdot \gamma + \dot{\gamma} \cdot \dot{\gamma} = 0 \Leftrightarrow \ddot{\gamma} \cdot \vec{n} + |\dot{\gamma}|^2 = 0$$

But: $K_n = \dfrac{\langle \ddot{\gamma}, \vec{n} \rangle}{\left|\frac{ds}{dt}\right|^2} \Rightarrow K_n + \dfrac{|\dot{\gamma}^2|}{|\dot{\gamma}^2|} = 0$

$K_n = -1 \checkmark$

#4.10 You may assume 5.3 pg 45

"The only unit speed curves of const curvature are circles or lines"

Assume: $K_g = \text{const}$. $K = \sqrt{K_g^2 + K_n^2} = \text{const} \Rightarrow$ circle.

#5.3 Show on S^2 the gt circles are geodesic

By (4.10), $K_g = \text{const} \Rightarrow \gamma(s)$ is circle $\Rightarrow$

if $K_g = 0$ ($\equiv$ geodesic) then $\gamma(s)$ is circle.

But on geodesic, $\ddot{\gamma}(s) = |\ddot{\gamma}(s)|\,\vec{n}$, and

on ~~circle~~ S^2, $\vec{n} = \gamma(s)$. $\therefore$

$$|\ddot{\gamma}(s)\cdot\gamma| = |\ddot{\gamma}(s)| = K$$

But $\gamma\cdot\gamma = 1 \Rightarrow \dot{\gamma}\cdot\gamma = 0 \Rightarrow \ddot{\gamma}\cdot\gamma - \dot{\gamma}\dot{\gamma} = 0$

$\Rightarrow \ddot{\gamma}\cdot\gamma = -1 \Rightarrow K = 1 \Rightarrow \gamma$ great circle.

#5.1 $\underset{\sim}{x}(t,\theta) = (r(t)\cos\theta,\ r(t)\sin\theta,\ z(t)) = \gamma(t)$

(a) ~~xxx~~ Assume: $\gamma(t)$ is parameterized by arclength; ie., $|\dot\gamma|^2 = \dot r^2 + \dot z^2 = 1$. Then

$$\ddot\gamma(t) = (\ddot r\cos\theta,\ \ddot r\sin\theta,\ \ddot z) = \kappa N$$

$$c\vec{n} = \underset{\sim}{x}_1 \times \underset{\sim}{x}_2 = (\dot r\cos\theta, \dot r\sin\theta, \dot z) \times (-r\sin\theta, r\cos\theta, 0)$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \dot r\cos\theta & \dot r\sin\theta & \dot z \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -\underset{\sim}{i}\, r\dot z \cos\theta - \underset{\sim}{j}\, r\dot z\sin\theta + \dot r r\, \underset{\sim}{k}$$

$$\Rightarrow \vec{n} = (-\dot z\cos\theta, -\dot z\sin\theta, \dot r) \qquad = r(-\dot z\cos\theta, -\dot z\sin\theta, \dot r)$$

$$\cancel{1 = |\dot\gamma|^2 = \dot\gamma\cdot\dot\gamma} \quad \cancel{\Rightarrow \ddot\gamma\cdot\dot\gamma = 0}$$

$$\cancel{0 = (\ddot r\cos\theta, \ddot r\sin\theta, \ddot z)\cdot(\dot r\cos\theta, \dot r\sin\theta, \dot z)}$$

$$\cancel{0 = \dot r\ddot r + \dot z\ddot z} \qquad \cancel{\ddot z = -\frac{\dot r\ddot r}{\dot z}}$$

$$\cancel{(\dot r\ddot r)^2 = (\dot z\ddot z)^2 = (1-\dot z^2)\ddot r^2 = \dot z^2\ddot z^2}$$

But: $\dot{r}^2 + \dot{z}^2 = 1 \Rightarrow 2\dot{r}\ddot{r} + 2\dot{z}\ddot{z} = 0$

$$\dot{r}\ddot{r} + \dot{z}\ddot{z} = 0$$

$$\ddot{z} = \frac{-\dot{r}\ddot{r}}{\dot{z}}$$

$$\therefore \ddot{\gamma} = \left(\ddot{r}\cos\theta, \ddot{r}\sin\theta, -\frac{\dot{r}\ddot{r}}{\dot{z}}\right)$$

$$= \frac{\ddot{r}}{\dot{z}}\left(\dot{z}\cos\theta, \dot{z}\sin\theta, -\dot{r}\right) \parallel \vec{n}$$

$\Rightarrow \gamma$ is geodesic. ✓

(b) Latitude: $\underset{\sim}{x}(t,\theta) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$
$= \gamma(\theta)$

$\gamma'(\theta) = (-r\sin\theta, r\cos\theta, 0)$

$|\gamma'| = r = \dfrac{ds}{d\theta} \Rightarrow ds = r\,d\theta \Rightarrow \theta = \dfrac{s}{r}$

$\gamma\left(\frac{s}{r}\right) = (r\cos\theta, r\sin\theta, z)$

$$\frac{d}{ds}\gamma\left(\frac{s}{r}\right) = \frac{1}{r}(-r\sin\theta, r\cos\theta, 0) = (-\sin\theta, \cos\theta, 0) = T$$

$$\frac{d^2}{ds^2}\gamma\left(\frac{s}{r}\right) = \frac{1}{r}(-\cos\theta, -\sin\theta, 0) = \kappa N$$

$$\vec{n} = (-\dot{z}\cos\theta, -\dot{z}\sin\theta, \dot{r})$$

$\therefore$ $N \parallel \vec{n}$ iff $\dot{r} = 0$ ✓

#6.2 $\gamma(s)$ unit speed, $S = \vec{n} \times T$. Now by Prop 6.10, γ geodesic iff maximally straight iff T is $\parallel$ along γ. But S is unit and makes const $90°$ angle with T on $\gamma(s)$ $\Rightarrow$ by Prop 6.9, S is $\parallel$ along γ iff T is $\parallel$ along γ. ✓

#6.3 (a) $N = a\vec{n} + bS$ since $N \perp T$

$$X_N = \cancel{} = bS$$

$$N - \langle N, \vec{n}\rangle \vec{n} = N - a\vec{n} = bS \quad ✓$$

(b) (i) $\Leftrightarrow$ (ii) $X_N = 0 \Leftrightarrow b = 0 \Leftrightarrow N = a\vec{n} \Leftrightarrow N$ is parallel to $\vec{n} \Leftrightarrow \gamma$ geodesic

(i) $\Rightarrow$ (iii) $X_N = 0 \Rightarrow X_N \parallel$ along γ ✓

(iii) $\Rightarrow$ (i) $X_N \parallel$ along $\gamma \Rightarrow S \parallel$ along γ

$\Rightarrow T \parallel$ along $\gamma \Rightarrow \gamma$ geodesic $\Rightarrow X_N = 0$ ✓