(4.8.5) Given $x(u,v) = (u, v, u^3 + v^3)$, which points have $K > 0$ and $K < 0$?

Soln:

- $K = \det(L^i{}_j)$, with $L^i{}_j = g^{ie} L_{ej}$, $L_{ej} := \langle x_{ej}, n \rangle$,
  $x_{ej} := \frac{\partial^2 x}{\partial u^e \partial u^j}$, $(g^{ie}) = (g_{ie})^{-1} = \frac{1}{\det(g_{ie})} \begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix}$,
  $g_{ie} = \langle x_i, x_e \rangle$

- Calculate those quantities:

$$X_1 = \frac{\partial x}{\partial u} = (1, 0, 3u^2) \quad \& \quad X_2 = \frac{\partial x}{\partial v} = (0, 1, 3v^2)$$

$$\Rightarrow \; g_{11} = 1 + 9u^4, \quad g_{12} = 9u^2 v^2 = g_{21}, \quad g_{22} = 1 + 9v^4$$

$$\Rightarrow \; (g^{ij}) = \frac{1}{1 + 9u^4 + 9v^4} \begin{pmatrix} 1 + 9v^4 & -9u^2 v^2 \\ -9u^2 v^2 & 1 + 9u^4 \end{pmatrix}$$

- $L_{12} = \left\langle \frac{\partial^2 x}{\partial u \partial v}, n \right\rangle = L_{21} = 0$

$$L_{11} = \langle x_{11}, n \rangle = \frac{6u}{\sqrt{1 + 9u^4 + 9v^4}}, \quad L_{22} = \langle x_{22}, n \rangle = \frac{6v}{\sqrt{1 + 9u^4 + 9v^4}}$$

where we used $n = \frac{x_1 \times x_2}{|x_1 \times x_2|} = \frac{1}{\sqrt{1 + 9u^4 + 9v^4}} \begin{pmatrix} -3u^2 \\ -3v^2 \\ 1 \end{pmatrix}$

$$\Rightarrow \; L^1{}_1 = g^{11}L_{11} + g^{12}\cancelto{0}{L_{12}} = \frac{(1+9v^4)\,6u}{(1+9u^4+9v^4)^{3/2}}$$

$$L^1{}_2 = g^{12}L_{22} + g^{11}\cancelto{0}{L_{12}} = \frac{-54\,u^2v^3}{(1+9u^4+9v^4)^{3/2}}$$

$$L^2{}_1 = g^{21}L_{11} + g^{22}\cancelto{0}{L_{12}} = \frac{-54\,u^3v^2}{(1+9u^4+9v^4)^{3/2}}$$

$$L^2{}_2 = g^{22}L_{22} + g^{21}\cancelto{0}{L_{12}} = \frac{6v\,(1+9u^4)}{(1+9u^4+9v^4)^{3/2}}$$

$$\Rightarrow \; K = L^1{}_1 L^2{}_2 - L^1{}_2 L^2{}_1 =$$

$$= \frac{36\,uv\,(1+9v^4)(1+9u^4) - (54)^2(uv)^5}{(1+9u^4+9v^4)^3} =$$

$$= \frac{36\,uv}{(1+9u^4+9v^4)^2}$$

Thus $K = 0 \iff 36\,uv = 0$

We conclude: $K > 0 \quad \forall (u,v)$ with $u,v > 0$ or $u,v < 0$

and $K < 0 \quad \forall (u,v)$ with $u > 0 > v$ or $v > 0 > u$. $\square$