MATH 116 Solution Set #1.

§1.1 #3,4,5 ; §1.2 #1,2 ; §1.3 #1,3,5 ; §1.4 #1 ; §1.5 #1.

§1.1:

#1.3. $\bar{U}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}, \bar{U}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}, \bar{U}_3=\begin{pmatrix}1\\-2\\3\end{pmatrix} \Rightarrow (a^\beta_j)=\begin{pmatrix}1&0&1\\0&1&-2\\1&1&3\end{pmatrix}$ for standard basis ($\hat{e}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, etc.). since $\bar{g}_{\alpha\beta}=\langle\hat{e}_\alpha,\hat{e}_\beta\rangle=\delta_{\alpha\beta}$ we have

$$(g_{ij})=(a^\alpha_i)^{T}(g_{\alpha\beta})(a^\beta_j)=\begin{pmatrix}1&0&1\\0&1&1\\1&-2&3\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&-2\\1&1&3\end{pmatrix}=\begin{pmatrix}2&1&4\\1&2&1\\4&1&14\end{pmatrix}.$$

#1.4 Let $\langle p,q\rangle=\int_{-1}^{1}p(x)q(x)\,dx$ for $p(x)\in\mathbb{R}[x]$.

(i) Clearly, $\langle p,q\rangle=\int_{-1}^{1}p(x)q(x)\,dx=\int_{-1}^{1}q(x)p(x)\,dx=\langle q,p\rangle$ since polynomials in $\mathbb{R}[x]$ are commutative.

(ii) Clearly, $\langle p, rq+sh\rangle=\int_{-1}^{1}p(x)[rq(x)+sh(x)]\,dx$
$=\int_{-1}^{1}rp(x)q(x)\,dx+\int_{-1}^{1}p(x)sh(x)\,dx=r\int_{-1}^{1}p(x)q(x)\,dx+s\int_{-1}^{1}p(x)h(x)\,dx=r\langle p,q\rangle+s\langle p,h\rangle$

(iii) Consider $\langle p,p\rangle=\int_{-1}^{1}p^2(x)\,dx$. Let $m=\inf_{x\in I}p^2(x)\geq 0$ since $p^2(x)\geq 0$ hence $\langle p,p\rangle=\int_{-1}^{1}p^2(x)\,dx\geq\int_{-1}^{1}m\,dx=2m\geq 0$ hence $\langle p,p\rangle\geq 0$.

Clearly if $p(x)\equiv 0$ on $I=[-1,1]$ then $\langle p,p\rangle=\int_{-1}^{1}0^2\,dx=0$.

Now, assume $p(x)\not\equiv 0$ on I, then $\exists\, x_0\in I$ such that $p^2(x_0)=a>0$. But $p(x)\in\mathbb{R}[x]\Rightarrow p(x)$ is continuous, hence $p^2(x)$ is continuous hence $\exists$ an interval, say $[x_1,x_2]$ such that $p^2(x)>0$ $\forall x\in[x_1,x_2]$. But then $\langle p,p\rangle$ $\neq\int_{-1}^{1}p^2(x)\,dx=\int_{-1}^{x_1}p^2(x)\,dx+\int_{x_1}^{x_2}p^2(x)\,dx+\int_{x_2}^{1}p^2(x)\,dx$ and since $\int_{-1}^{x_1}p^2(x)\,dx\geq 0$, $\int_{x_2}^{1}p^2(x)\,dx\geq 0$ and $\int_{x_1}^{x_2}p^2(x)\,dx>0$ it is clear $\langle p,p\rangle>0$.

Now assume $\int_{-1}^{1}p^2(x)\,dx=0$ with $p(x)\not\equiv 0$. By an argument similar to the above, we can show that $\int_{-1}^{1}p^2(x)\,dx>0$

#1.4 cont

a contradiction. Hence $\langle p,p\rangle \geq 0$ where $\langle p,p\rangle = 0$ if and only if $p(x)=0$.

#1.5 Consider $g(t) = \langle \bar{u}+t\bar{v}, \bar{u}+t\bar{v}\rangle = |\bar{u}|^2 + 2t\langle \bar{u},\bar{v}\rangle + t^2|\bar{v}|^2 \geq 0$ since $\langle , \rangle$ is an inner product.
Now if $|\bar{v}|=0$ then $\bar{v}=0 \Rightarrow |\langle \bar{u},\bar{v}\rangle| = 0 = |\bar{u}||\bar{v}|$ and clearly $\bar{u}$ and $\bar{v}$ are L.D. So assume $|\bar{v}|\neq 0$.
Now since $g(t)\geq 0\ \forall t$, it must have at most one real root, hence the discriminant must be ≤ 0.

$\Rightarrow (2\langle \bar{u},\bar{v}\rangle)^2 - 4|\bar{u}|^2|\bar{v}|^2 \leq 0 \Rightarrow 4|\langle \bar{u},\bar{v}\rangle|^2 - 4|\bar{u}|^2|\bar{v}|^2 \leq 0$

$\Rightarrow |\langle \bar{u},\bar{v}\rangle| \leq |\bar{u}||\bar{v}|$. Now, $\bar{u},\bar{v}$ are L.D. iff $\exists t_0$ st $\bar{u}+t_0\bar{v}=0$ iff $g(t_0)=0$ iff disc $=0$ iff $|\langle \bar{u},\bar{v}\rangle| = |\bar{u}||\bar{v}|$.
hence the Cauchy-Schwarz Inequality is proved.

§1.2

#2.1 Let $T = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ w.r.t. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ($\hat{e}_1$ and $\hat{e}_2$)

(a) Find T' w.r.t. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow (A^\alpha_i) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ w.r.t. $\hat{e}_1, \hat{e}_2$

and $A^{-1} = (A^i_\alpha) = \frac{1}{\det A}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ hence

$$T' = A^{-1}TA = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$$

(b) A for $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ and $A^{-1} = \frac{1}{-5}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

So, $T' = A^{-1}TA = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$.

#2.2 $\det\begin{pmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{pmatrix} = (\lambda+5)(\lambda-4)+18 = \lambda^2+\lambda-2 = (\lambda+2)(\lambda-1) \Rightarrow \lambda_1 = 1, \lambda_2 = -2$

So, for λ_1 we have $\begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -6x_1+3x_2=0 \Rightarrow x_2 = 2x_1 \Rightarrow \vec{v}_{\lambda_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

and for λ_2: $\begin{pmatrix} -3 & 3 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3x_1+3x_2=0 \Rightarrow \vec{v}_{\lambda_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

§1.3

#3.1 Prove Lemma 3.3. Most of these are straight forward from the definition and the "abuse of notation" $\bar{U}\times\bar{V} = \det\begin{pmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ a^1 & a^2 & a^3\\ b^1 & b^2 & b^3\end{pmatrix}$. So I will skip most of them

(c) $\bar{U}\times\bar{V}=\bar{0}$ iff $\bar{U}$ and $\bar{V}$ are L.D. Assume $\bar{U}$ and $\bar{V}$ are L.D. then $\exists r\in\mathbb{R}$ such that $r\neq 0$ and $\bar{U}=r\bar{V}$, then $\bar{U}\times\bar{V} = (r\bar{V}\times\bar{V})$ $= r(\bar{V}\times\bar{V}) = -r(\bar{V}\times\bar{V}) = -\bar{V}\times\bar{U}$. But $\Rightarrow 2r(\bar{V}\times\bar{V})=0$ since $r\neq 0$ $(\bar{V}\times\bar{V})=0$ hence $\bar{U}\times\bar{V}=0$.

Now, if $\bar{U}\times\bar{V}=0$ then $\det\begin{vmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ a^1 & a^2 & a^3\\ b^1 & b^2 & b^3\end{vmatrix}=0$ iff 2 rows are L.D. by linear algebra. But $\{\hat{e}_i\}$ are vectors and $\{a^i\}$ and $\{b^i\}$ are scalars hence the only two rows that can be L.D. are $\{a^i\}$ and $\{b^i\}$ $\Rightarrow$ $\bar{U}$ and $\bar{V}$ are L.D.

(e) $\bar{U}\times\bar{V}\perp$ to $\bar{U}$ and $\bar{V}$ under usual dot product in $\mathbb{R}^3$.

Let $\bar{U}=\sum a^i\hat{e}_i$ and $\bar{V}=\sum b^i\hat{e}_i$. Then $\bar{U}\times\bar{V} = (a^2b^3-a^3b^2)\hat{e}_1$ $+(a^3b^1-a^1b^3)\hat{e}_2+(a^1b^2-a^2b^1)\hat{e}_3$. Hence $\langle\bar{U},\bar{U}\times\bar{V}\rangle$

$= a^1(a^2b^3-a^3b^2)+a^2(a^3b^1-a^1b^3)+a^3(a^1b^2-a^2b^1)$

$= a^1a^2b^3-a^1a^3b^2+a^2a^3b^1-a^1a^2b^3+a^1a^3b^2-a^2a^3b^1 = 0$.

Similarly $\langle\bar{V},\bar{U}\times\bar{V}\rangle=0$ hence by defn of orthogonality...

(f) $|\bar{U}\times\bar{V}| = |\bar{U}||\bar{V}|\sin\theta$ where θ is the angle between $\bar{U}$ and $\bar{V}$.

$|\bar{U}\times\bar{V}| = [\langle\bar{U}\times\bar{V},\bar{U}\times\bar{V}\rangle]^{1/2} = [\langle(\bar{U}\times\bar{V})\times\bar{U},\bar{V}\rangle]^{1/2}$ by lemma 3.4

$= [\langle-(\bar{U}\times(\bar{U}\times\bar{V})),\bar{V}\rangle]^{1/2} = [-\langle(\langle\bar{U},\bar{V}\rangle\bar{U}-\langle\bar{U},\bar{U}\rangle\bar{V}),\bar{V}\rangle]^{1/2}$

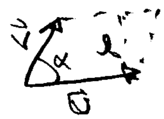
$= [\langle\bar{U},\bar{U}\rangle\langle\bar{V},\bar{V}\rangle-\langle\bar{U},\bar{V}\rangle^2]^{1/2} = [|\bar{U}|^2|\bar{V}|^2-(|\bar{U}||\bar{V}|\cos\theta)^2]^{1/2}$

$= |\bar{U}||\bar{V}|(1-\cos^2\theta)^{1/2} = |\bar{U}||\bar{V}|(\sin^2\theta)^{1/2} = |\bar{U}||\bar{V}|\sin\theta$

since $0\leq\theta\leq\pi$.

#3.3 Prove Lemma 3.5.

The volume of a parallelopiped is $V = b \cdot h$ where h is orthogonal to the base. Consider the base:



It has area $= |\bar u| \cdot \ell$, but $\ell = |\bar v| \sin\alpha$

hence $b = |\bar u||\bar v|\sin\alpha \equiv |\bar u \times \bar v|$. Now, $h = \langle \bar w, \hat e_n \rangle$ where $\hat e_n$ is a unit normal to the base. But $\frac{\bar u \times \bar v}{|\bar u \times \bar v|}$ is just such a normal, hence $h = \langle \bar w, \frac{\bar u \times \bar v}{|\bar u \times \bar v|} \rangle$

hence $\text{Vol} = b \cdot h = |\bar u \times \bar v| \langle \bar w, \frac{\bar u \times \bar v}{|\bar u \times \bar v|} \rangle = \langle \bar w, \bar u \times \bar v \rangle = [\bar w, \bar u, \bar v]$.

#3.5 Let $\{\bar a, \bar b, \bar c\}$ be an orthonormal basis of $\mathbb{R}^3$, with $\bar a \times \bar b = \bar c$. Then $\bar b \times \bar c = \bar b \times (\bar a \times \bar b) = \langle \bar b, \bar b \rangle \bar a - \langle \bar b, \bar a \rangle \bar b$

but $\langle \bar b, \bar b \rangle = 1$ and $\langle \bar b, \bar a \rangle = 0$ since $\{\bar a, \bar b, \bar c\}$ is orthonormal,

hence $\bar b \times \bar c = 1 \cdot \bar a - 0 \cdot \bar b = \bar a$. (Similarly $\bar c \times \bar a = \bar b$).

§1.4

#4.1 Let $\bar a = (2,1,-3)$, $\bar b = (1,0,1)$, $\bar c = (0,-1,3)$.

(a) $\vec\alpha(t) = \bar a + t\bar b = (2,1,-3) + t(1,0,1)$

(b) $\vec\alpha(t) = \bar b + t(\bar c - \bar b) = (1,0,1) + t(-1,-1,2)$.

(c) $\langle \bar x - \bar b, \bar a \rangle = 0 \Rightarrow \langle (x,y,z) - (1,0,1), (2,1,-3) \rangle = 0$

$\Rightarrow 2(x-1) + 1(y-0) - 3(z-1) = 0 \Rightarrow 2x + y - 3z = -1$.

(d) $\langle \bar x - \bar c, \bar a \times \bar b \rangle = 0$ where $\bar a \times \bar b = (1,-5,-1)$

$\Rightarrow \langle (x, y+1, z-3), (1,-5,-1) \rangle = 0 \Rightarrow x - 5y - z = 2$

(e) $\langle \bar x - \bar a, \bar x - \bar a \rangle = 2^2 \Rightarrow (x-2)^2 + (y-1)^2 + (z+3)^2 = 4$.

#5.1 Let $\bar{f}, \bar{g}: \mathbb{R} \to V$ where V has an inner product $\langle , \rangle$.
Let $\{\vec{v}_i\}_{i=1}^n$ be a basis for V, then
$\bar{f} = \sum_{i=1}^n f_i \vec{v}_i$ and $\bar{g} = \sum_{j=1}^n g_j \vec{v}_j$ hence $\langle \bar{f}, \bar{g} \rangle = \sum_{i=1}^n \sum_{j=1}^n f_i g_j \langle \vec{v}_i, \vec{v}_j \rangle$

hence $\frac{d}{dt}\langle \bar{f}, \vec{g} \rangle = \frac{d}{dt}\left(\sum_{i,j} f_i g_j \langle \vec{v}_i, \vec{v}_j \rangle\right) = \sum_{i,j}\left(\frac{df_i}{dt} g_j + f_i \frac{dg_j}{dt}\right) \langle \vec{v}_i, \vec{v}_j \rangle$

since $\langle \vec{v}_i, \vec{v}_j \rangle$ is a constant (for each i and j) and by the product rule for derivatives.

hence $\frac{d}{dt}\langle \bar{f}, \bar{g} \rangle = \sum_{i,j} \frac{df_i}{dt} g_j \langle \vec{v}_i, \vec{v}_j \rangle + \sum_{i,j} f_i \frac{dg_j}{dt} \langle \vec{v}_i, \vec{v}_j \rangle$

$$= \left\langle \frac{d\bar{f}}{dt}, \bar{g} \right\rangle + \left\langle \bar{f}, \frac{d\bar{g}}{dt} \right\rangle.$$

Now, if $|\bar{f}| = \text{const.}$, then $(\langle \bar{f}, \bar{f} \rangle)^{1/2} = \text{const}$
hence $\langle \bar{f}, \bar{f} \rangle = \text{const}$. So, $\frac{d}{dt}\langle \bar{f}, \bar{f} \rangle = \frac{d}{dt}(\text{const}) = 0$
$\Rightarrow \left\langle \frac{d\bar{f}}{dt}, \bar{f} \right\rangle + \left\langle f, \frac{d\bar{f}}{dt} \right\rangle = 2\left\langle \frac{d\bar{f}}{dt}, \bar{f} \right\rangle = 0 \Rightarrow \left\langle \frac{d\bar{f}}{dt}, \bar{f} \right\rangle = 0$
and therefore by definition, $\bar{f}$ and $\frac{d\bar{f}}{dt}$ are orthogonal.