Math 116 Solution Set # 2

§2-1 # 11, 13, 14, 17   §2-2 # 21, 22, 23, 28   §2-3 # 31, 34, 37

#11 (a). Let $\bar\alpha(t) = (\sin 3t\cos t, \sin 3t\sin t, 0)$

Then $\frac{d\bar\alpha(t)}{dt} = (3\cos 3t\cos t - \sin 3t\sin t, 3\cos 3t\sin t + \sin 3t\sin t, 0)$

Now assume $\frac{d\bar\alpha(t)}{dt} = \bar 0$ for some t, hence each component vanishes $\Rightarrow 3\cos 3t\cos t = \sin 3t\sin t$ and $3\cos 3t\sin t = -\sin 3t\sin t$ hence $\cos 3t\cos t = -\cos 3t\sin t$. Clearly if $t = \frac{1}{3}(n\pi + \pi/2)$ then $\cos 3t = 0$ but then $\sin 3t\sin t \neq 0$ hence $\frac{d\bar\alpha}{dt}(t) \neq 0$. So assume $t \neq \frac{1}{3}(n\pi + \pi/2) \Rightarrow \cos t = -\sin t \Rightarrow t = m\pi + \frac{3}{4}\pi$

but then, $3\cos 3t\sin t + \sin 3t\sin t \neq 0$ hence in all cases $\frac{d\bar\alpha(t)}{dt} \neq 0$ for any t, and $\bar\alpha(t)$ is regular.

(b) Find the tangent line to $\vec\alpha(t)$ at $t = \pi/3$

$$\frac{d\alpha}{dt}(\pi/3) = (3(-1)(\tfrac{1}{2}) - (0), 3(-1)(\tfrac{\sqrt3}{2}) + 0, 0) = (-3/2, -3\sqrt3/2, 0)$$

also, $\bar\alpha(\pi/3) = (0, 0, 0)$

hence $\ell = \bar W \in \mathbb{R}^3 \mid \vec W = \lambda(-3/2, -3\sqrt3/2, 0)$ for $\lambda \in \mathbb{R}^1$.

#13 (a). $\bar\alpha(t) = (r\cos t, r\sin t, ht)$ hence $\frac{d\bar\alpha(t)}{dt} = (-r\sin t, r\cos t, h)$

so $\left|\frac{d\bar\alpha(t)}{dt}\right| = (r^2\sin^2 t + r^2\cos^2 t + h^2)^{1/2} = \sqrt{r^2+h^2}$.

hence $T(t_0) = (-r\sin t_0, r\cos t_0, h)/\sqrt{r^2+h^2}$ and so the tangent line at $t = t_0$ is: $\ell = \{\bar W \in \mathbb{R}^3 \mid \bar W = (r\cos t_0, r\sin t_0, ht_0) + \frac{\lambda}{\sqrt{r^2+h^2}}(-r\sin t_0, r\cos t_0, h)\}$ for $\lambda \in \mathbb{R}$

(b) Let $\bar U = (0,0,1)$, then $\cos\theta = \langle \bar U, \frac{d\bar\alpha}{dt}\rangle / |\bar U|\left|\frac{d\bar\alpha}{dt}\right| = \frac{(0+0+h)}{\sqrt{1^2}\cdot\sqrt{r^2+h^2}}$

hence $\theta = \cos^{-1}\left(\frac{h}{\sqrt{r^2+h^2}}\right) \neq f(t)$.

1.4 Let $F:(-1,1)\to(-\infty,\infty)$ be given by $f(t)=\tan(\frac{\pi t}{2})$.
Let y_1 and y_2 be in $(-\infty,\infty)$. If $y_1=y_2$ then $\tan(\frac{\pi t_1}{2})=\tan(\frac{\pi}{2}t_2)$
hence $\frac{\pi t_1}{2}=\frac{\pi}{2}t_2+n\pi$ since tan is periodic where $n\in\mathbb{Z}$.
hence $t_1=t_2+2n$. But if $n\neq 0$, then for $t_2\in(-1,1)$,
$t_1\notin(-1,1)$ hence $n=0$ and $t_1=t_2$. Also, since
$\lim\limits_{t\to 1}\tan(\frac{\pi t}{2})=\infty$ and $\lim\limits_{t\to -1}\tan(\frac{\pi t}{2})=-\infty$ it is clear that
F is 1 to 1 and onto. Now, if $f(t)=\tan(\frac{\pi}{2}t)$
then $f'(t)=\frac{\pi}{2}\sec^2(\frac{\pi t}{2})$, $f''(t)=\frac{\pi^2}{2}\sec(\frac{\pi t}{2})\sec(\frac{\pi t}{2})\tan(\frac{\pi}{2}t)$.
etc. hence $f\in C^k$ for any $k\geq 1$.
Similarly, $f^{-1}(t)=g(r)=\frac{2}{\pi}\arctan(r)$ and $g'(r)=\frac{2}{\pi}\frac{1}{1+r^2}$
$g''(r)=-\frac{4r}{\pi}(1+r^2)^{-2}$, etc hence $g(r)\in C^k$ for any $k\geq 1$.
Therefore $f(t)$ is a reparameterization

#1.7 Let $\bar{\alpha}(t)$ be a regular curve, and that there is a point
$\bar{a}\in\mathbb{R}^3$ such that $\alpha(t)-\bar{a}$ and $T(t)$ are orthogonal for all t.
Consider $\frac{d}{dt}\langle\bar{\alpha}(t)-\bar{a},\bar{\alpha}(t)-\bar{a}\rangle=\langle\frac{d\bar{\alpha}}{dt},\bar{\alpha}(t)-\bar{a}\rangle+\langle\bar{\alpha}(t)-a,\frac{d\bar{\alpha}}{dt}\rangle$
$=2\langle\frac{d\bar{\alpha}}{dt},\bar{\alpha}(t)-\bar{a}\rangle=2\langle|\frac{d\bar{\alpha}}{dt}|T(t),\bar{\alpha}(t)-\bar{a}\rangle=2|\frac{d\bar{\alpha}}{dt}|\langle T(t),\bar{\alpha}(t)-\bar{a}\rangle$.
Now $T(t)$ is $\perp$ to $\alpha(t)-\bar{a}$ hence $\langle T(t),\bar{\alpha}(t)-\bar{a}\rangle=0$
$\Rightarrow\frac{d}{dt}\langle\bar{\alpha}(t)-\bar{a},\bar{\alpha}(t)-\bar{a}\rangle=0\Rightarrow\langle\bar{\alpha}(t)-\bar{a},\bar{\alpha}(t)-\bar{a}\rangle=$ const.
But this is the equation of a sphere hence $\bar{\alpha}(t)$
must lie on a sphere centered at $\bar{a}$.

§2.2

#2.1 Let $\bar{\alpha}(t) = (r\cos t, r\sin t, ht)$. Find s for $0 \le t \le 10$.

$$s = \int_0^{10} \left|\frac{d}{dt}\bar{\alpha}(t)\right| dt = \int_0^{10} |(-r\sin t, r\cos t, h)|\, dt = \int_0^{10} (r^2+h^2)^{1/2}\, dt$$

$$= (r^2+h^2)^{1/2}\int_0^{10} dt = 10\,(r^2+h^2)^{1/2} \quad (\text{remember } |\ | = \sqrt{\langle\ ,\ \rangle})$$

#2.2 Let $\alpha(t) = (2\cosh 3t, -2\sinh 3t, 6t)$, find s for $0 \le t \le 5$

$$\frac{d\alpha}{dt} = (6\sinh 3t, -6\cosh 3t, 6) \Rightarrow \left|\frac{d\alpha}{dt}\right| = 6(\sinh^2 3t + \cosh^2 3t + 1)^{1/2}$$

$$= 6(\cosh^2 3t + \cosh^2 3t)^{1/2}$$

$$= 6\sqrt{2}\cosh 3t.$$

So, $s = \int_0^5 6\sqrt{2}\cosh 3t\, dt$

$$= 6\sqrt{2}\,\tfrac{1}{3}\sinh 3t\Big|_0^5 = 2\sqrt{2}(\sinh 15).$$

#2.3 Reparametrize $\bar{\alpha}(t) = (r\cos t, r\sin t, ht)$ by arclength.

Since $0 \in$ the domain of $\bar{\alpha}(t)$, set $t_0 = 0$.

So, $s = s(t) = \int_0^t \left|\frac{d\bar{\alpha}}{dt}\right| dt = \int_0^t \sqrt{r^2+h^2}\, dt = t\sqrt{r^2+h^2}$. $\Rightarrow t = \frac{s}{\sqrt{r^2+h^2}}$.

hence, $\bar{\alpha}(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+h^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+h^2}}\right), \frac{hs}{\sqrt{r^2+h^2}}\right)$.

#2.8 Let $\bar{\alpha}(t)$ be regular with $\left|\frac{d\alpha}{dt}\right| = a$, a const > 0.

Then $s = \int_{t_0}^t \left|\frac{d\bar{\alpha}}{dt}\right| dt = \int_{t_0}^t a\, dt = a(t - t_0)$.

$\Rightarrow t = \frac{s}{a} + t_0$ where t_0 is an arbitrary pt in the domain of $\bar{\alpha}(t)$. (clearly a constant).

§2.3

#3.1 $\bar{\alpha}(s) = \left(\frac{5}{13}\cos s, \frac{8}{13} - \sin s, -\frac{12}{13}\cos s\right)$

hence $\bar{\alpha}'(s) = \left(-\frac{5}{13}\sin s, -\cos s, \frac{12}{13}\sin s\right) \Rightarrow |\bar{\alpha}'(s)| = \left(\frac{25}{169}\sin^2(s) + \cos^2(s) + \frac{144}{169}\sin^2 s\right)^{1/2} = 1$ ✓

#3.1 cont

Since $\bar{\alpha}(s)$ is unit speed,

$T(s) = \frac{d\alpha}{ds} = \left(\frac{-5}{13}\sin s, -\cos(s), \frac{12}{13}\sin s\right) \Rightarrow T'(s) = \left(\frac{-5}{13}\cos(s), \sin(s), \frac{12}{13}\cos s\right)$

So $\kappa(s) = \left|\frac{dT}{ds}\right| = \left(\frac{25}{169}\cos^2(s) + \sin^2(s) + \frac{144}{169}\cos^2(s)\right) = (\cos^2 s + \sin^2(s))^{1/2} = \underline{1}$

Hence, $N(s) = \frac{T'(s)}{\kappa(s)} = \left(\frac{-5}{13}\cos(s), \sin(s), \frac{12}{13}\cos(s)\right)$

Now, $B(s) = T(s) \times N(s) = \left(\frac{-12}{13}\cos^2(s) - \frac{12}{13}\sin^2(s), \frac{-60}{13}\sin(s)\cos(s) + \frac{60}{13}\sin(s)\cos(s),\right.$

$\left.-\frac{5}{13}\sin^2(s) - \frac{5}{13}\cos^2(s)\right) = \left(\frac{-12}{13}, 0, -5/13\right)$

hence $B'(s) = 0 \Rightarrow \tau(s) = -\langle B', N\rangle = 0$.

So, the Frenet-Serret apparatus of $\alpha(t)$ is:

$\left\{\kappa = 1, \tau = 0, T = \begin{pmatrix} \frac{-5}{13}\sin(s) \\ -\cos(s) \\ \frac{12}{13}\sin(s) \end{pmatrix}, N = \begin{pmatrix} \frac{-5}{13}\cos(s) \\ \sin(s) \\ \frac{12}{13}\cos(s) \end{pmatrix}, B = \begin{pmatrix} -12/13 \\ 0 \\ -5/13 \end{pmatrix}\right\}$

#3.7 Let $\alpha(s) = (x(s), y(s), 0)$ be a unit speed curve.

First, since $\bar{\alpha}(s)$ is unit, $T(s) = (x', y', 0)$ and $T' = (x'', y'', 0)$.

hence $B = T \times N = T \times \frac{T'}{\kappa} = \frac{1}{\kappa}(T \times T') = \frac{1}{\kappa}(0, 0, x'y'' - y'x'')$.

But $\bar{\alpha}(s)$ is unit $\Rightarrow |B| = 1 \Rightarrow 1 = \left[\frac{1}{\kappa^2}(0 + 0 + (x'y'' - y'x'')^2)\right]^{1/2}$

$\Rightarrow 1 = \sqrt{\frac{1}{\kappa^2}(x'y'' - y'x'')^2} \Rightarrow \kappa = |x'y'' - y'x''|$.

E.1 → E.4 from sheet.

E.1 Let $y^1 = x^1 + 2x^2$, $y^2 = -x^1 + x^2$

then $\frac{\partial y^1}{\partial x^1} = 1$, $\frac{\partial y^1}{\partial x^2} = 2$, $\frac{\partial y^2}{\partial x^1} = -1$, $\frac{\partial y^2}{\partial x^2} = 1 \Rightarrow (A^\alpha_i) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \equiv A$

Now, $\det A = 3 \neq 0$ hence A is invertable.

E.3 (Note)

Since $g_{ij}a^i a^j = \bar g_{\alpha\beta}\bar a^\alpha \bar a^\beta$ and $a^i = A^i_\alpha \bar a^\alpha$ and $a^j = A^j_\beta \bar a^\beta$

$$\Rightarrow g_{ij}(A^i_\alpha)\bar a^\alpha (A^j_\beta)\bar a^\beta = g_{ij}A^i_\alpha A^j_\beta \bar a^\alpha \bar a^\beta = \bar g_{\alpha\beta}\bar a^\alpha\bar a^\beta$$

hence, we see that $\underline{\bar g_{\alpha\beta} = g_{ij}A^i_\alpha A^j_\beta}$.

Now, if $A \equiv (A^\alpha_i)$ from #E.1, then $A^i_\alpha = A^{-1}$

hence $\bar G = (A^{-1})^{Tr} G (A^{-1})$ in matrix form.

E.2 If $G = \begin{bmatrix} a & b\\ b & c\end{bmatrix}$ and $A = \begin{bmatrix}1&2\\-1&1\end{bmatrix} \Rightarrow A^{-1} = \frac13\begin{bmatrix}1&-2\\1&1\end{bmatrix}$

$$\Rightarrow \bar G = \frac13\begin{bmatrix}1&1\\-2&1\end{bmatrix}\begin{bmatrix}a&b\\b&c\end{bmatrix}\frac13\begin{bmatrix}1&-2\\1&1\end{bmatrix} = \frac19\begin{bmatrix}1&1\\-2&1\end{bmatrix}\begin{bmatrix}a+b & -2a+b\\ b+c & -2b+c\end{bmatrix}$$

$$= \frac19 \begin{bmatrix} a+2b+c & -2a-b+c\\ -2a-b+c & 4a-4b+c\end{bmatrix}.$$

E.4 Let $a_1 = 1$ and $a_2 = 2$. Then $\bar a_\alpha = A^i{}_\alpha a_i$

$$\Rightarrow \bar a_\alpha = \frac13\begin{bmatrix}1&1\\-2&1\end{bmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \frac13\begin{pmatrix}3\\0\end{pmatrix} = \underline{\begin{pmatrix}1\\0\end{pmatrix}}.$$

Whoops!

#3.4 Let $\alpha(s) = (\sqrt{1+s^2}, 2s, \ln(s+\sqrt{1+s^2}))/\sqrt5$

Then $\frac{d\alpha}{ds} = \left(\frac{s}{\sqrt{1+s^2}}, 2, \left(\frac{1}{s+\sqrt{1+s^2}}\right)\left(1+\frac{s}{\sqrt{1+s^2}}\right)\right)/\sqrt5 = \left(\frac{s}{\sqrt{1+s^2}}, 2, \left(\frac{1}{s+\sqrt{1+s^2}}\right)\left(\frac{s+\sqrt{1+s^2}}{\sqrt{1+s^2}}\right)\right)/\sqrt5$

and $\left|\frac{d\alpha}{ds}\right| = \left\{\left(\frac{s^2}{1+s^2}+4+\frac{1}{1+s^2}\right)/5\right\}^{1/2} = \left\{\left(\frac{s^2+1}{s^2+1}+4\right)/5\right\}^{1/2} = \underline{1}$

So, since $\alpha(s)$ is of unit speed,

$$T = \frac{d\alpha}{ds} = \boxed{\left(\frac{s}{\sqrt{1+s^2}}, 2, \frac{1}{\sqrt{1+s^2}}\right)/\sqrt5 \quad (III.)}$$

and $T' = \frac{dT}{ds} = \left(\frac{\sqrt{1+s^2} - s^2(\sqrt{1+s^2})^{-1/2}}{1+s^2}, 0, \frac{-s}{(1+s^2)^{3/2}}\right)/\sqrt5 = \left(\frac{1}{(1+s^2)^{3/2}}, 0, \frac{-s}{(1+s^2)^{3/2}}\right)/\sqrt5$

hence $\kappa(s) = \left|\frac{dT}{ds}\right| = \left(\frac{1}{(1+s^2)^3}+0+\frac{s^2}{(1+s^2)^3}\right)^{1/2}/\sqrt5 = \boxed{\left(\frac{1}{1+s^2}\right)/\sqrt5 \quad (I)}$

#3.4 cont

So, $N(s) = \dfrac{T'(s)}{\kappa(s)} = \left(\dfrac{1}{\sqrt{5}(1+s^2)^{3/2}}, 0, \dfrac{-s}{\sqrt{5}(1+s^2)^{3/2}}\right) \cdot \sqrt{5}(1+s^2)$

$$= \boxed{\left(\frac{1}{\sqrt{1+s^2}}, 0, \frac{-s}{\sqrt{1+s^2}}\right) \quad (\mathrm{IV})}$$

And, $B(s) = T(s) \times N(s)$

$$= \left(\frac{-2s}{\sqrt{1+s^2}}, \frac{1}{1+s^2} + \frac{s^2}{1+s^2}, \frac{-2}{\sqrt{1+s^2}}\right) / \sqrt{5}$$

$$= \boxed{\left(\frac{-2s}{\sqrt{1+s^2}}, 1, \frac{-2}{\sqrt{1+s^2}}\right) / \sqrt{5} \quad (\mathrm{V})}$$

and finally, $\tau(s) = -\langle B'(s), N(s) \rangle$, where

$$B'(s) = \left(\frac{-2}{\sqrt{1+s^2}} + \frac{2s^2}{(1+s^2)^{3/2}}, 0, \frac{2s}{(1+s^2)^{3/2}}\right) / \sqrt{5}$$

$$= \left(\frac{-2}{(1+s^2)^{3/2}}, 0, \frac{2s}{(1+s^2)^{3/2}}\right) / \sqrt{5}$$

So, $\tau(s) = -\left(\dfrac{-2}{(1+s^2)^2} + 0 + \dfrac{-2s^2}{(1+s^2)^2}\right) / \sqrt{5} = \dfrac{2(1+s^2)}{\sqrt{5}(1+s^2)^2}$

$$= \boxed{\frac{2}{\sqrt{5}(1+s^2)} \quad \mathrm{II}}$$

So, the Frenet-Serrat apparatus is given by

$\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}\}$ above