MATH 116 Solution Set # 3

§2.4 #4.1, 4.2, 4.6, 4.11 and E.1

#4.1 Assume $\bar{\alpha}(s)$ is a unit speed curve with $\kappa \neq 0$ and Frenet-Serret apparatus $\{\kappa, \tau, T, N, B\}$.

Then $-\langle T', B' \rangle = -\langle \kappa N, -\tau N \rangle$ by Theorem 4.2

$$\Rightarrow -\langle T', B' \rangle = \kappa\tau \langle N, N \rangle = \kappa\tau.$$

If $\kappa = 0$ then $|T'| = 0 \Rightarrow T' = \vec{0}$ since $\langle , \rangle$ is positive definite, hence $-\langle T', B' \rangle = 0 = \kappa\tau$.

4.2 Assume $\bar{\alpha}(s)$ is unit speed with $\kappa \neq 0$ and F-S: $\{\kappa, \tau, T, N, B\}$

Now, $[\alpha', \alpha'', \alpha'''] = \langle (\alpha' \times \alpha''), \alpha''' \rangle$ by definition.

But $\alpha' = T$, $\alpha'' = T' = \kappa N$ hence $\alpha''' = (\kappa N)' = \kappa' N + \kappa N'$

$$\Rightarrow [\alpha', \alpha'', \alpha'''] = \langle (T \times \kappa N), (\kappa' N + \kappa N') \rangle$$
$$= \langle \kappa B, \kappa' N + \kappa N' \rangle = \kappa\kappa' \langle B, N \rangle + \kappa^2 \langle B, N' \rangle$$
$$= \kappa^2 \langle B, N' \rangle \text{ since F-S is orthonormal.}$$

Now $N' = -\kappa T + \tau B$ by Thm 4.2, hence

$$\langle B, N' \rangle = -\kappa \langle B, T \rangle + \tau \langle B, B \rangle = \tau \text{ since F-S is orthonormal}$$

hence $[\alpha', \alpha'', \alpha'''] = \kappa^2 \tau$.

If $\kappa = 0$ then $\alpha'' = \bar{0}$ hence $[\alpha', \alpha'', \alpha'''] = [\alpha', \bar{0}, \alpha'''] = 0 = \kappa\tau$.

#4.6 Let $\bar{\alpha}(t) = (e^t, \cos t, 3t^2)$. Then $\bar{\alpha}'(t) = (e^t, -\sin t, 6t)$ and since $e^t > 0, \forall t$ $\bar{\alpha}'(t) \neq 0$ $\forall t$. So, $T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$, and the equation of the normal plane to $\bar{\alpha}(t)$ at $t = 1$ is

$$\langle \bar{x} - \bar{\alpha}(t), T(t) \rangle \Big|_{t=1} = 0 = \langle \bar{x} - \bar{\alpha}(t), \alpha'(t) \rangle \Big|_{t=1} \frac{1}{|\alpha'(1)|}$$

$$\Rightarrow (x - e)e - (y - \cos(1))\sin(1) + (z - 3)6 = 0 \cdot |\alpha'(1)|$$
$$\Rightarrow ex - \sin(1) y + 6z = 18 + e^2 + \cos(1)\sin(1).$$

#4.11 Let $\bar\alpha(s)$ be unit speed with $\kappa \neq 0$.

Let $\bar w(s) = \tau \bar T + \kappa \bar B$, Then $\bar w \times \bar T = (\tau \bar T + \kappa \bar B) \times \bar T$

or, $\bar w \times \bar T = \tau(\bar T \times \bar T) + \kappa(\bar B \times \bar T) = \tau(0) + \kappa(\bar N) = \bar T'$ by F-S.

And, $\bar w \times \bar N = (\tau \bar T \times \kappa \bar B) \times \bar N = \tau(\bar T \times \bar N) + \kappa(\bar B \times \bar N)$

$= \tau(\bar B) + \kappa(-\bar T) = -\kappa \bar T + \tau \bar B = \bar N'$ by F-S

and finally, $\bar w \times \bar B = (\tau \bar T \times \kappa \bar B) \times \bar B = \tau(\bar T \times \bar B) + \kappa(\bar B \times \bar B)$

$= \tau(-\bar N) + \kappa(0) = -\bar N \tau = \bar B'$ by F-S.

E.1 Assume $\bar\alpha(t)$ is a C^3 curve.

Since $\bar\alpha(t)$ is not unit speed, $\bar\alpha'(t) = |\bar\alpha'(t)| \bar T$

hence $\bar\alpha(t)'' = (|\bar\alpha'(t)| \bar T)' = |\bar\alpha'(t)|' \bar T + |\bar\alpha'(t)| \bar T'$

hence $\bar\alpha(t)' \times \bar\alpha(t)'' = |\bar\alpha'(t)| \bar T \times (|\bar\alpha'(t)|' \bar T + |\bar\alpha'(t)| \bar T')$

$= |\bar\alpha'(t)| |\bar\alpha'(t)|' (\bar T \times \bar T) + |\bar\alpha'(t)|^2 (\bar T \times \bar T') = |\bar\alpha'(t)|^2 (\bar T \times \bar T')$

Now, $\bar T' = \frac{d\bar T}{ds}\frac{ds}{dt} = \kappa \bar N |\bar\alpha'(t)|$ so, (Let $\bar\alpha(t) \equiv \bar\alpha$)

$\bar\alpha' \times \bar\alpha'' = |\bar\alpha'|^2 (\bar T \times \kappa \bar N |\bar\alpha'|) = \kappa |\bar\alpha'|^3 (\bar T \times \bar N) = \kappa |\bar\alpha'|^3 \bar B$.

Now, $\bar\alpha''' = (|\bar\alpha'|' \bar T + |\bar\alpha'| \bar T')' = |\bar\alpha'|'' \bar T + 2|\bar\alpha'|' \bar T' + |\bar\alpha'| \bar T''$,

$= |\bar\alpha'|'' \bar T + 2|\bar\alpha'|' (\kappa \bar N |\bar\alpha'|) + |\bar\alpha'| (\kappa' \bar N |\bar\alpha'| + \kappa \bar N' |\bar\alpha'| + \kappa \bar N |\bar\alpha'|')$

$\Rightarrow (\bar\alpha' \times \bar\alpha'') \cdot \bar\alpha''' = (\kappa |\bar\alpha'|^3 \bar B) \cdot (|\bar\alpha'|'' \bar T + 2|\bar\alpha'|^2 \kappa \bar N + |\bar\alpha'|^2 \kappa' \bar N + |\bar\alpha'|^2 \kappa \bar N' + |\bar\alpha'| |\bar\alpha'|' \kappa \bar N)$

$= \kappa^2 |\bar\alpha'|^5 (\bar B \cdot \bar N')$ since $\bar B \cdot \bar T = \bar B \cdot \bar N = 0$.

Now, $\bar N' = \frac{d\bar N}{ds}\frac{ds}{dt} = (-\kappa \bar T + \tau \bar B)|\bar\alpha'|$

$\Rightarrow (\bar\alpha' \times \bar\alpha'') \cdot \bar\alpha''' = \kappa^2 |\bar\alpha'|^6 (\bar B) \cdot (-\kappa \bar T + \tau \bar B) = \tau \kappa^2 |\bar\alpha'|^6$ since $\bar B \cdot \bar T = 0$

and $\bar B \cdot \bar B = 1$.

Hence with $\kappa \neq 0 \Rightarrow \tau = \dfrac{(\bar\alpha' \times \bar\alpha'') \cdot \bar\alpha'''}{\kappa^2 |\bar\alpha'|^6}$.