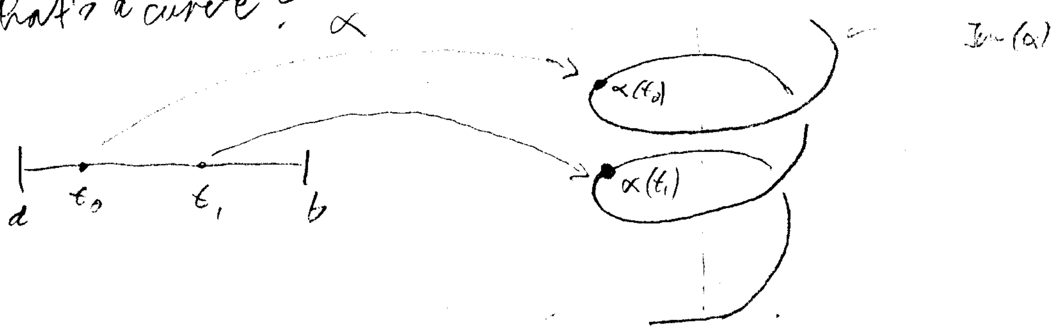
Local Curve Theory

Today's goal: What are curves, their length and arc length?

What's a curve?



"Think of a curve as path traced out by a moving particle in $\mathbb{R}^3$, within the time from a to b."

Def 1

A <u>regular</u> <u>curve</u> α is a C^1-function $\alpha : (a,b) \longrightarrow \mathbb{R}^3$, such that $\frac{d\alpha}{dt}(t) \neq 0 \quad \forall t \in (a,b)$.

$\frac{d\alpha}{dt}(t_0) \in \mathbb{R}^3$ is called <u>velocity vector</u> of α at $t = t_0$ and $\frac{d\alpha}{dt}$ <u>velocity vector field</u>.

Example 1

i) $\alpha : (0,1) \longrightarrow \mathbb{R}^3$, $\alpha(t) := (t, t^2, 1)$

$\Longrightarrow \frac{d\alpha}{dt}(t) = (1, 2t, 0) \neq 0 \quad \Longrightarrow \alpha$ regular curve

ii) $\beta : (-1,1) \longrightarrow \mathbb{R}^3$, $\beta(t) := (0, t^2, t^3)$

$\Longrightarrow \frac{d\beta}{dt}(t) = (0, 2t, 3t^2) \quad \Longrightarrow \frac{d\beta}{dt}(0) = 0 \quad \Longrightarrow$ not regular curve

iii) $\gamma : (0, \frac{1}{2}) \longrightarrow \mathbb{R}^3$, $\gamma(t) := (2t, 4t^2, 1)$ is regular curve and $\mathrm{Im}(\gamma) = \mathrm{Im}(\alpha)$.

"α & γ have same geometry but different parametrization."

Def 2

∴ A reparametrization of α: is a one-to-one and onto function $g : (c,d) \longrightarrow (a,b)$ such that g & g^{-1} are C^1.

$\beta := \alpha \circ g$ is called a reparametrized curve of α.

Remark: • $\beta(r) = (\alpha \circ g)(r)$ is regular, since $\frac{d\beta}{dr} = \underbrace{\frac{d\alpha}{dg}}_{\neq 0,\ \alpha \text{ regular}} \underbrace{\frac{dg}{dr}}_{\neq 0} \neq 0$

since $1 = \frac{d}{dr}(g^{-1} \circ g(r)) = \frac{dg}{dg}$

• $\mathrm{Im}(\beta) = \mathrm{Im}(\alpha)$ since g is onto

Def 3

$$\underbrace{T(t)}_{\in \mathbb{R}^3} := \frac{1}{\left|\frac{d\alpha}{dt}\right|} \frac{d\alpha}{dt}(t)$$ is called tangent vector to a regular cur

Rmk: tangent space of α at t_0 is line $\{\alpha(t_0) + T(t_0) \cdot c \mid c \in \mathbb{R}\}$

Prop. 1

Let T be tangent vector to α and S tangent vector to $\beta = \alpha \circ$

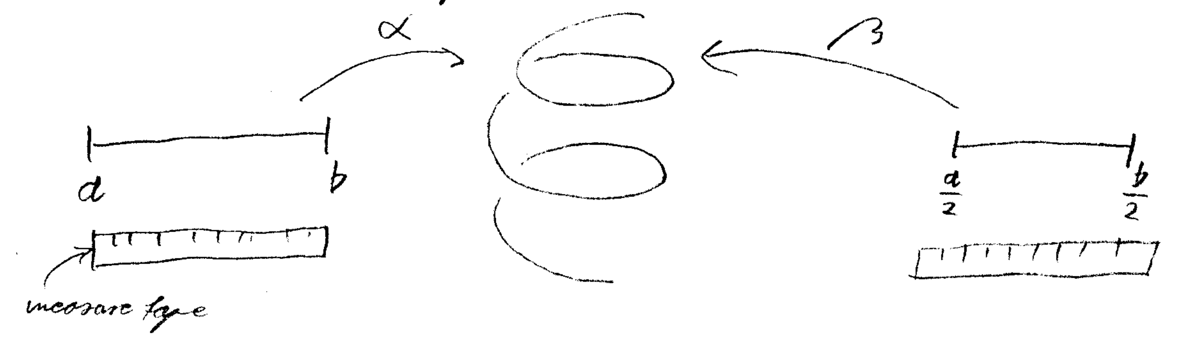
Then $T = S$ if $\frac{dg}{dr} > 0$ and $T = -S$ if $\frac{dg}{dr} < 0$!

proof:

$$S = \frac{1}{\left|\frac{d\beta}{dr}\right|} \underbrace{\frac{d\beta}{dr}}_{= \frac{d\alpha}{dg} \frac{dg}{dr}} = \overbrace{\frac{1}{\left|\frac{d\alpha}{dg}\right|} \frac{d\alpha}{dg}}^{= T} \frac{\overbrace{\frac{dg}{dr}}^{= \pm 1}}{\left|\frac{dg}{dr}\right|} = \pm T$$

What's length of a curve? "In real life one would use a measure tape."

Can we use meas. tape on interval?



Not invariant under reparametrization

Idea: length = $|b-a| \times |\text{velocity}|$

treat like time

Def 4

The length of $\alpha : [a,b] \longrightarrow \mathbb{R}^3$ is defined as $\ell(\alpha) := \int_a^b \left|\frac{d\alpha}{dt}\right| dt$.

Remark: Using closed interval $[a,b]$ is only possible if α is defined on $(a-\varepsilon, b+$

Prop. 2 Let $\beta = \alpha \circ g$ reparam. curve, then $\ell(\alpha) = \ell(\beta)$!

proof: If $\frac{dg}{dr} > 0$:

$$\ell(\beta) = \int_c^d \underbrace{\left|\frac{d\beta}{dr}\right|}_{=\left|\frac{d\alpha}{dg}\right| \cdot \frac{dg}{dr}} dr = \int_{g(c)=a}^{g(d)=b} \left|\frac{d\alpha}{dg}\right| dg = \ell(\alpha)$$

If $\frac{dg}{dr} < 0$: $\ell(\beta) = \int_c^d \left|\frac{d\alpha}{dg}\right| (-1) \frac{dg}{dr} dr = -\int_{g(c)=b}^{g(d)=a} \left|\frac{d\alpha}{dg}\right| dg = \ell(\alpha)$ ∎

Example 2:

Let $\alpha(t) = (at, bt, ct)$ a straight line, $t \in [0,1]$

$\Rightarrow \frac{d\alpha}{dt} = (a,b,c) \Rightarrow \left|\frac{d\alpha}{dt}\right| = \sqrt{a^2+b^2+c^2}$

$\Rightarrow \ell(\alpha, [0,1]) = \sqrt{a^2+b^2+c^2}$ = Euclidean length

In the following we introduce a "canonical" parametrization:

Thm 1

For a regular curve $\alpha : (a,b) \to \mathbb{R}^n$

define $h(t) := \int_a^t \left|\frac{d\alpha}{dt}\right| dt$.

Then h^{-1} is a reparametrization of α

with $h^{-1} : (0, \ell(\alpha)) \longrightarrow (a,b)$.

Moreover: $\alpha \in C^k(a,b) \Leftrightarrow h \,\&\, h^{-1} \in C^k$

proof:

Show: (i) h is one-to-one , (ii) h is onto , (iii) $\alpha \in C^k \Leftrightarrow h, h^{-1} \in C^k$

(i): $\frac{dh}{dt} = \left|\frac{d\alpha}{dt}\right| > 0 \Rightarrow h$ monotonly increasing $\Rightarrow h$ is 1-1 ✓

(ii): Let $\ell \in (0, \ell(\alpha))$. Set $f(t) := h(t) - \ell$,

$\Rightarrow f \in C^0([a,b])$ & $f(a) = -\ell < 0$ & $f(b) = \ell(\alpha) - \ell > 0$

Thus the Intermediate value Thm yield: $\exists t_0 \in (a,b)$ s.t. $f(t_0) = 0$

$\Rightarrow h(t_0) = \ell$ $\Rightarrow h$ is onto ✓

(iii): By def of h we see: $\alpha \in C^k \Leftrightarrow h \in C^k$

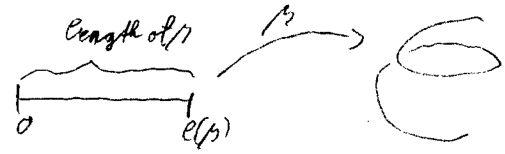
Moreover, $(\underset{\uparrow\, \in C^k}{h^{-1} \circ h})(t) = \underset{\nwarrow\, \in C^k}{t} \xRightarrow{\text{(expand difference quotients)}} h^{-1} \in C^k$ ✓

∎

Def 5

$s := h(t)$ is called <u>arc length</u> along α.

$\beta(s) := (\alpha \circ h^{-1})(s)$ is said to has been <u>parametrized by arc length</u>.



Example 3

Let $\alpha(t) = (r\cos t, r\sin t, 0)$, $r > 0$

$\Rightarrow \left|\frac{d\alpha}{dt}\right| = r \quad \Rightarrow \quad s = rt$ arc length

$\overset{t=\frac{s}{r}}{\Rightarrow} \beta(s) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right), 0\right)$ param. by arc length

$\boxed{\text{From now on assume } a = 0.}$ (WLOG)

Lemma 1

$\beta(s)$ parametrized by arc length $\Longleftrightarrow \left|\frac{d\beta}{ds}\right| = 1$

(β called <u>unit speed curve</u>)

proof

$\Rightarrow$) Let $\beta = \alpha \circ h^{-1}$ (as in Def 5)

$$\Rightarrow \left|\frac{d\beta}{ds}\right| \overset{t=h^{-1}(s)}{=} \left|\frac{d\alpha}{dt}\right| \cdot \left|\frac{dh^{-1}}{ds}\right| = \frac{\left|\frac{d\alpha}{dt}\right|}{\underbrace{\left|\frac{dh}{dt}\right|}_{=\left|\frac{d\alpha}{dt}\right|}} = 1 \quad ✓$$

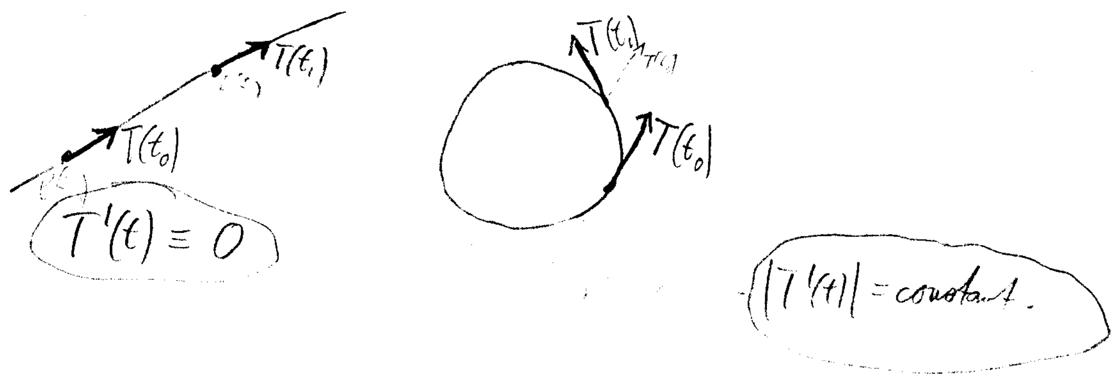
$\Leftarrow$)

$$h(s) = \int_{a=0}^{s} \overbrace{\left|\frac{d\beta}{ds}\right|}^{=1} ds = \int_0^s ds = s \quad ✓$$

- Next goal: curvature, torsion and Frenet-Serret apparatus
- From now on assume α is parametrized with arc length s. (WLOG)
- What's curvature (κ)?

Intuitively: straight line shall have $\kappa \equiv 0$

circle shall have $\kappa \equiv$ constant.



$T'(t) \equiv 0$

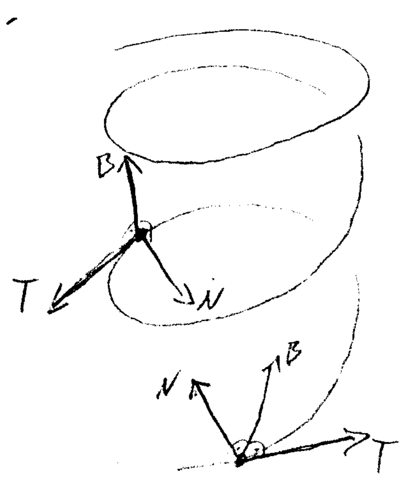
$|T'(t)| =$ constant.

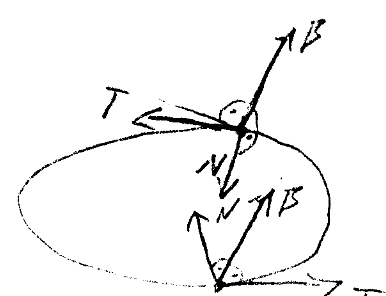
Def 6

The curvature of $\alpha(s)$ is $\kappa(s) := |T'(s)|$ (arc length!!)

Rmk: $\kappa \sim \left|\frac{d^2\alpha}{ds^2}\right| \sim |$acceleration of particle$|$

- Next introduce a basis of $\mathbb{R}^3$ $\{T(s), N(s), B(s)\}$ that "moves" with the curve.





"Think of a driver in a car, T is the direction he is driving in, N points to his left and B into the sky."

Def 7 Suppose $\kappa(s) \neq 0$. !! arclength !!

The principal normal vector field to $\alpha(s)$ is $N(s) := \dfrac{T'(s)}{\kappa(s)}$.

The binormal vector field to $\alpha(s)$ is $B(s) := T(s) \times N(s)$

The torsion of $\alpha(s)$ is $\tau(s) := -\langle B'(s), N(s) \rangle$

The Frenet-Serret apparatus of $\alpha(s)$ is $\{\kappa(s), \tau(s), T(s), N(s), B(s)\}$

Rmk:

- It's crucial that $\kappa(s) \neq 0$!
- The torsion τ measures how far α is from laying in a plane. (Think of a car going from one park deck to another.)

Lemma 2

Let $\kappa(s) \neq 0$. Then $\{T(s), N(s), B(s)\}$ is an orthonormal basis of $\mathbb{R}^3$.

proof:

- $\langle T, T' \rangle = 0$ (since: $1 = |T|^2 = \langle T, T \rangle \Rightarrow 0 = \frac{d}{ds}\langle T, T \rangle = 2\langle T, T' \rangle$)

$\Rightarrow T \perp N$

Moreover: $|N| = 1$

- $B \perp T$ & $B \perp N$ by definition ✓

$|B| = |T \times N| = |T| \cdot |N| = 1$ ✓

rest is trivial

Prop 3

Suppose $\kappa(s) \equiv 0 \quad \forall s \in [a,b]$, then $\alpha : [a,b] \to$ is a straight line.

proof:

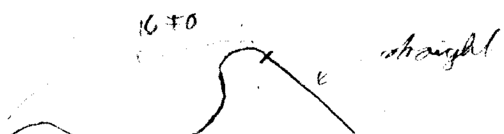
$$|T'| = \kappa \equiv 0 \iff T' = 0 \iff T \text{ constant}$$

$$\iff \frac{d\alpha}{ds} = \text{constant} \quad \text{since } T(s) = \frac{d\alpha}{ds} \text{ in arc length param}$$

$$\iff \alpha(s) \text{ straight line} \quad \blacksquare$$

Rmk

Conclude, we can either define the Frenet-Serret apparatus or we are only dealing with straight line segment.



We'll see below that all ($\kappa \neq 0$) curves are already determined by their curvature and torsion. ("Fundamental Thm of curves")
Next Thm is most important tool in local curve thy.:

Thm (Frenet-Serret)

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

proof:

Linear Algebra: Given orthonormal basis $(e_i)_{i=1,2,3}$, then $v = \sum_{i=1}^{3} \langle v, e_i \rangle e_i \quad \forall v \in \mathbb{R}^3$!

Now calculate components of T', N', B' in basis $\{T, N, B\}$.

(e.g.: to show $N' = -\kappa T + \tau B$ calculate $\langle N', T \rangle = \left[\langle (\frac{T'}{\kappa})', T \rangle \overset{T \perp T'}{=} \langle \frac{T''}{\kappa}, T \rangle = \right.$

$= \frac{1}{\kappa} \frac{d}{ds} \Big(\overbrace{\langle T', T \rangle}^{=0} - \overbrace{\langle T', T' \rangle}^{=\kappa^2} \Big) \Big| = -\kappa$ and $\langle N', B \rangle = \ldots = \tau$ ✓)

· We use next Frenet-Serret Thm to show that torsion indeed is "measuring how far α is away from being a plane curve".

Prop: Let $\kappa(s) \neq 0$.

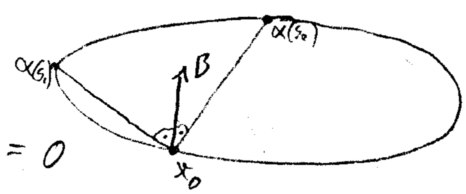
$\alpha(s)$ plane curve (i.e. $\mathrm{Im}(\alpha)$ lies in a plane) $\overset{!}{\Longleftrightarrow}$ $\tau(s) = 0 \;\forall s$

proof: Let $x_0 := \alpha(t_0)$ for some t_0.

$(\Leftarrow)$ Suppose $\tau \equiv 0$. $\Rightarrow B(s) =: B$ is constant (by F.S.: $B' = -\tau N$).

Show: $\langle B, \alpha(s) - x_0 \rangle = 0 \;\forall s$ ($\Rightarrow \alpha$ lies in plane perpendicular to B [spanned by T & N])

$$\frac{d}{ds}\langle B, \alpha(s) - x_0\rangle \overset{B'=0}{=} \langle B, \underbrace{\alpha'(s)}_{=T}\rangle = 0$$



$$\Rightarrow \langle B, \alpha(s) - x_0\rangle = \text{constant} = \langle B, \overbrace{\alpha(s_0)}^{=x_0} - x_0\rangle = 0$$

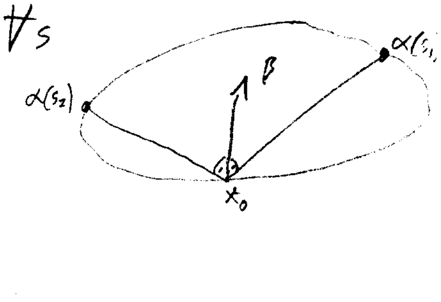
$\Rightarrow \alpha$ is plane curve with $\mathrm{Im}(\alpha)$ lying in plane perpendicular to B ✓

(<u>**Def**</u>: This is called the <u>osculating plane</u> of α.)

1^{st}

$(\Rightarrow)$ Suppose α is a plane curve

$\Rightarrow \exists V \in \mathbb{R}^3$ s.t. $\langle V, \alpha(s) - x_0\rangle = 0 \quad \forall s$



$\Rightarrow \frac{d}{ds}\langle V, \alpha(s) - x_0\rangle = 0 \;\forall s$

$\Rightarrow \langle V, \overbrace{T}^{=\alpha'}\rangle \equiv 0 \quad (*)$

$\overset{\frac{d}{ds}}{\Longrightarrow} \langle V, \underbrace{T'}_{=\kappa N \text{, F.S. Thm}}\rangle \equiv 0$

$\overset{(*)}{\Longrightarrow} V \perp (T(s) \,\&\, N(s)) \quad \forall s$

$\underset{\text{F.S. Thm}}{\Longrightarrow} V = cB, \; c \in \mathbb{R}\setminus\{0\} \quad \Rightarrow B \in \mathbb{R}^3$ constant

$\Longrightarrow \tau \equiv 0$ (F.S.: $B' = -\tau N$)

Observe that F.S. Thm yield an ODE $\frac{d}{ds}(u^i) = a^i_j u^j$ $\quad (*)$

with $(u^i) = \begin{pmatrix} T \\ N \\ B \end{pmatrix}$ and $(a^i_j) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$.

The theory of ODE's yield (Picard-Lindelöf-Thm.) that given $a^i_j(s)$ and initial values $u^i_0 = u^i(s_0)$ there exist a solution $u^i(s)$ to $(*)$ with $s \in (a,b) \ni s_0$ for some $a < b$. [idea: $u' = fu$ $\Rightarrow u = \exp(\int_{s_0}^{s} f \ldots$

This leads to the following:

<u>Thm</u> (Fundamental Theorem of Curves)

Any regular curve with $\kappa > 0$ is completely determined, up to position, by its curvature $\kappa(s)$ and its torsion $\tau(s)$.

More precisely, given to functions $\kappa(s) > 0$ & $\tau(s)$, some $x_0 \in \mathbb{R}^3$ and an orthonormal right-handed basis $\{T_0, N_0, B_0\}$ of $\mathbb{R}^3$.

Then $\exists \alpha : (a,b) \longrightarrow \mathbb{R}^3$ regular curve (with arc length parameterization) such that κ is its curvature and τ its torsion and $\exists s_0 \in (a,b)$ such that $\alpha(s_0) = x_0$ and $T(s_0) = T_0$, $N(s_0) = N_0$, $B(s_0) = B_0$.

<u>proof</u>:

As mentioned above we obtain solutions $T(s), N(s), B(s)$ to $(*)$.

We obtain $\alpha(s)$ by solving $\frac{d\alpha}{ds} = T$ $\quad \left(\Rightarrow \alpha = \int_{s_0}^{s} T(\tau)\, d\tau + x_0 \right)$.

The equ "$\frac{d\alpha}{ds} = T$" ensures that s is arc length parametrization.

The Frenet-Serret equ $(*)$ ensure that κ is curvature of α and τ its torsion. $\blacksquare$