Review of Linear Algebra -

Linear Algebra - The study of vectors, vector spaces, transformations of vectors

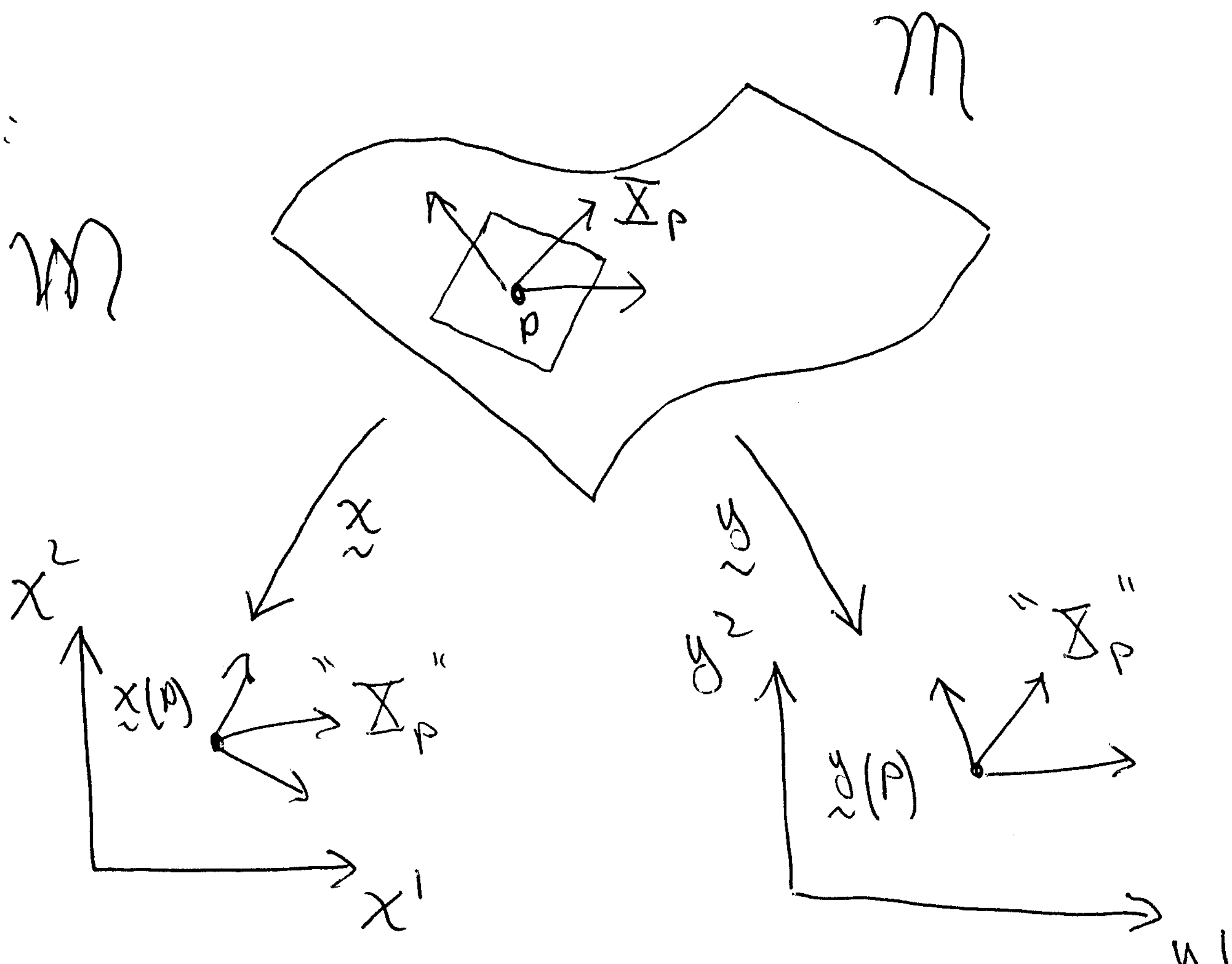
Q: Why is this central to differential geometry?

- Big Picture: We wish to study surfaces by studying them in coordinate systems placed on the surface:

Picture:

Surface M

$M \subseteq \mathbb{R}^3$



Eg: Surface $z = f(x^1, x^2)$ defines $\mathcal{M}$

$$P = (x^1, x^2, f(x^1, x^2))$$

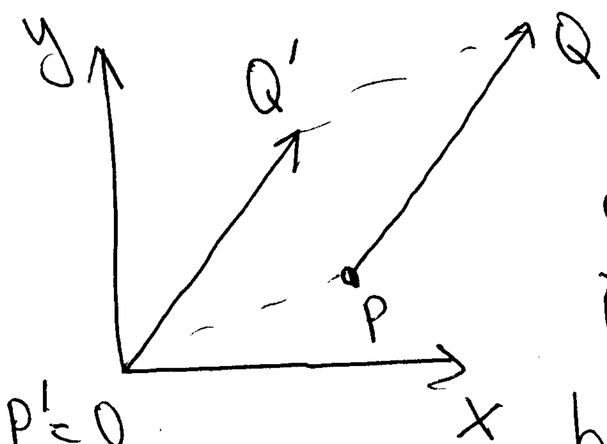
$$\underset{\sim}{x}(P) = (x^1, x^2), \quad P = \underset{\sim}{x}^{-1}(x^1, x^2)$$

- The collection of vectors tangent to the surface at $P \in \mathcal{M}$ is called the "tangent space of $\mathcal{M}$ at P" or $T_P\mathcal{M}$
- We denote the vectors tangent to $\mathcal{M}$ at P as $\underline{X}_P \in T_P\mathcal{M}$.
- Each $\underline{X}_P$ is named by a corresponding vector at $\underset{\sim}{x}(P)$ in coordinate system $\underset{\sim}{x}$

This raises some very interesting issues:

1. In 2D we identify vectors in $\mathbb{R}^2$ with points in $\mathbb{R}^2$ — we can ∥-translate vectors from point to point — can't do that on <u>curved</u> surfaces

Eg:



In (flat) $\mathbb{R}^2$ we can identify direction $\overrightarrow{PQ}$ with point Q' by ∥-translating the vector to the origin. On manifolds, the vectors are <u>pinned</u> to fixed points

X_p pinned to point p

2 When the surface is <u>curved</u>, the vectors don't lie in the surface - they "stick out" into a larger space.

Q: what if M is the 3-d space we live in? There is no "bigger space" for them to "stick out" into.

<u>Then</u>: we have to find a way to define the vector directions in M by defining them in the coordinate systems <u>alone</u>

3. If $\underset{\sim}{x}(P) = \underset{\sim}{x}$ & $\underset{\sim}{y}(P) = \underset{\sim}{y}$

Then $$\underset{\sim}{x} = \underbrace{\underset{\sim}{x} \circ \underset{\sim}{y}^{-1}}_{\varphi}(\underset{\sim}{y}) \iff \underset{\sim}{x} = \varphi(\underset{\sim}{y})$$

So φ gives the mapping between coord. systems. Turns out:

The matrix $$\left.\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}}\right|_p = \begin{bmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} \end{bmatrix}_p = \left(\frac{\partial x^i}{\partial y^j}\right) =:$$

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tells how the vectors that name $\mathbb{X}_p$ in $\underset{\sim}{x}$-coords get <u>transformed</u> to vectors that name $\mathbb{X}_p$ in $\underset{\sim}{y}$-coords.

$$\mathbb{X}_p = a^i \frac{\partial}{\partial x^i} = b^j \frac{\partial}{\partial y^j}$$

Two names for same vector on surface

components in $\underset{\sim}{x}$-coords

$\bar{e}_i$ in x-coords

Comp's in $\underset{\sim}{y}$-coords

basis vectors e_j in y-coords

Turns out:

$$a^i = \frac{\partial x^i}{\partial y^j} b^j \iff \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = J \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$(2\times 1) \quad (2\times 2)(2\times 1)$$

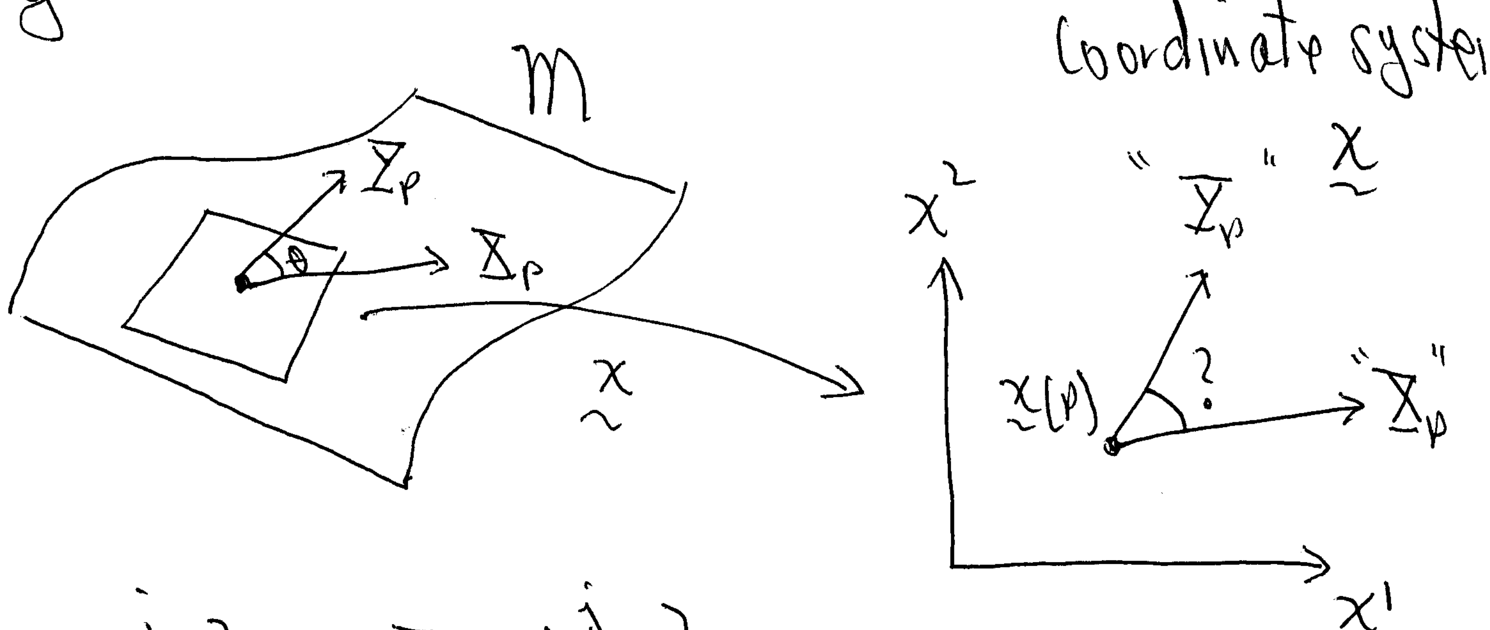
$$\underset{\substack{\text{basis vectors} \\ \text{in } \underset{\sim}{x}}}{\frac{\partial}{\partial x^i}} = \frac{\partial y^j}{\partial x^i} \underset{\substack{\text{basis vectors} \\ \text{in } \underset{\sim}{y}}}{\frac{\partial}{\partial y^j}} \iff (\vec{e}^{\,x}_1, \vec{e}^{\,x}_2) = (\vec{e}^{\,y}_1, \vec{e}^{\,y}_2)\, J^{-1}$$

$$(1\times 2) \quad (1\times 2)\ (2\times 2)$$

Conclude: Linear Transformations take vectors at P in one coordinate system to vectors at P in another coord system.

(4) Central Problem: How, in your coordinate system, do you compute the lengths of vectors and angles between vectors on M at p?

<u>Ans</u>: Each coordinate system has a <u>matrix</u> $g_{ij}(p)$ that tells how to compute lengths & angles on M at p



$$\underline{X}_p = a^i \frac{\partial}{\partial x^i} \qquad \underline{Y}_p = b^j \frac{\partial}{\partial x^j}$$

$$\underline{X}_p \cdot \underline{Y}_p = \langle \underline{X}_p, \underline{Y}_p \rangle = g_{ij}\, a^i b^j = a^T g a$$

dot product in $\mathbb{R}^3$

$$\boxed{\langle \underline{X}_p, \underline{Y}_p \rangle = (a^1, a^2) \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

- Once you know the inner product, you know angles & lengths —

$$\bar{X}_p \cdot \bar{Y}_p = \|\bar{X}_p\| \|\bar{Y}_p\| \cos\theta$$

lengths: $\|\bar{X}_p\|^2 = \bar{X}_p \cdot \bar{X}_p$

angles: $\cos\theta = \dfrac{\bar{X}_p \cdot \bar{Y}_p}{\|\bar{X}_p\| \|\bar{Y}_p\|}$

Thus the <u>metric</u> tells you <u>everything</u>

In particular, <u>curvature</u> is encoded in g_{ij}

- <u>Problem</u>: in our 3 space, we <u>only</u> have the angles and lengths in our <u>coordinate systems</u> since the vectors are only defined in the coord systems!!

"No bigger space in which to put the vectors $\bar{X}_p$ on M"

5 1 – 4 goes on in vector space $T_p\mathcal{M}$

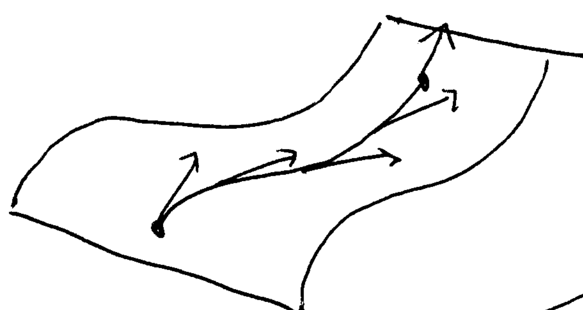
vector space at fixed point p.

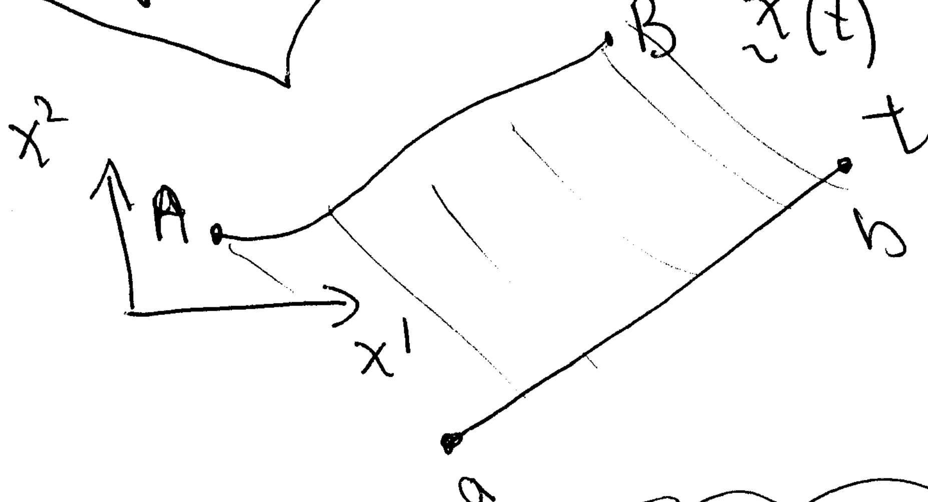
Q1: How to compute lengths of curves that go from point to point?

Q2: How to find curves of shortest length!
(geodesics)

Q3: How to compute curvature

Ans (Q1)





Length = $\int_a^b \left\| \frac{d\underset{\sim}{x}}{dt} \right\| dt$

"tangent vectors in $\underset{\sim}{x}$-coords! "

Ans (Q2): Curves of shortest ~~distance~~ length (geodesics) solve 2nd order ODE:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^i \dot{x}^j = 0$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma} \{ -g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j} \}$$

depend only on 1st derivative of metric components!

Ans (Q3): Curvature given by RCT

$$R^i_{jkl} = \text{"depends on } g_{ij} \text{ \& derivatives up to 2nd order"}$$