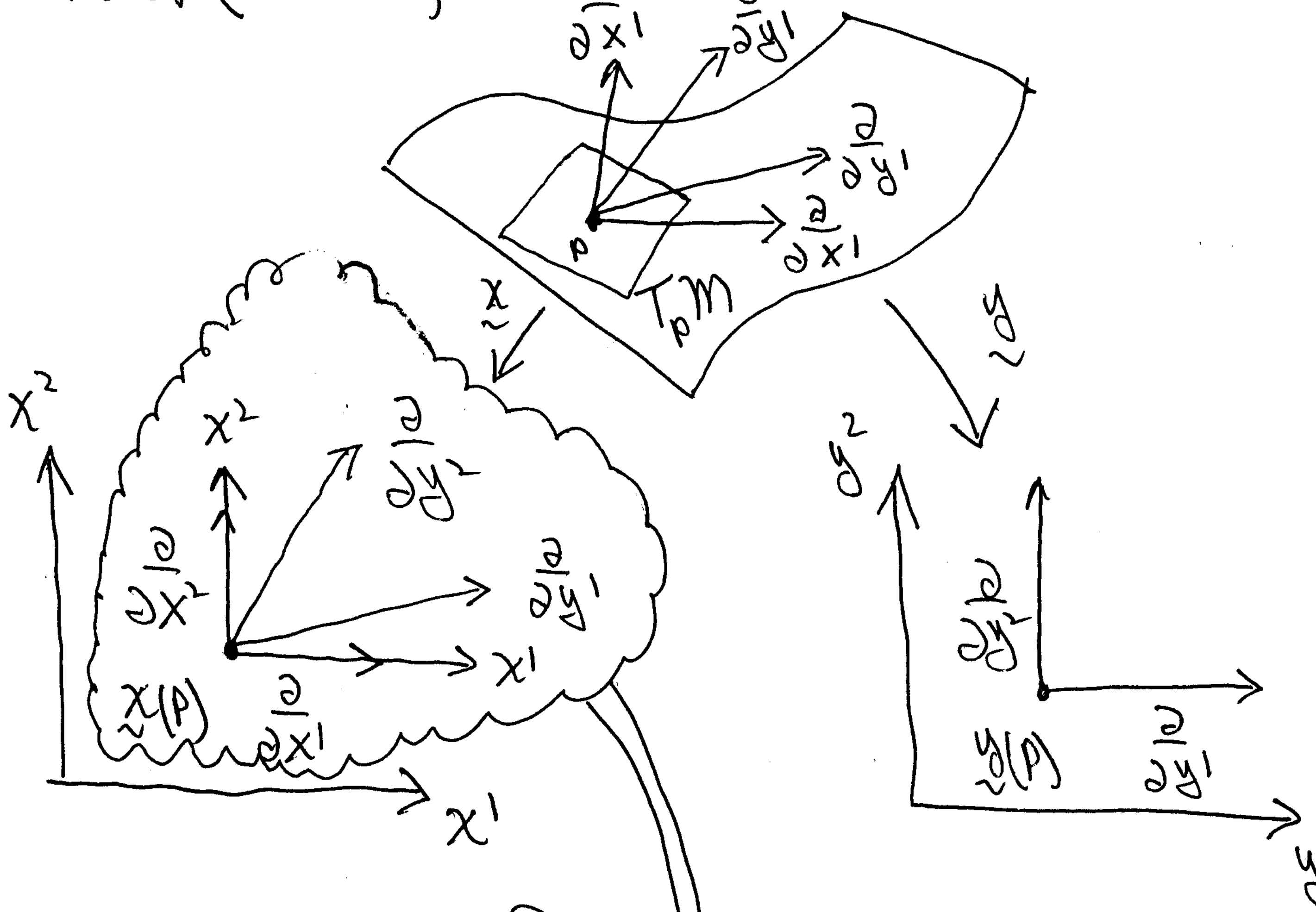
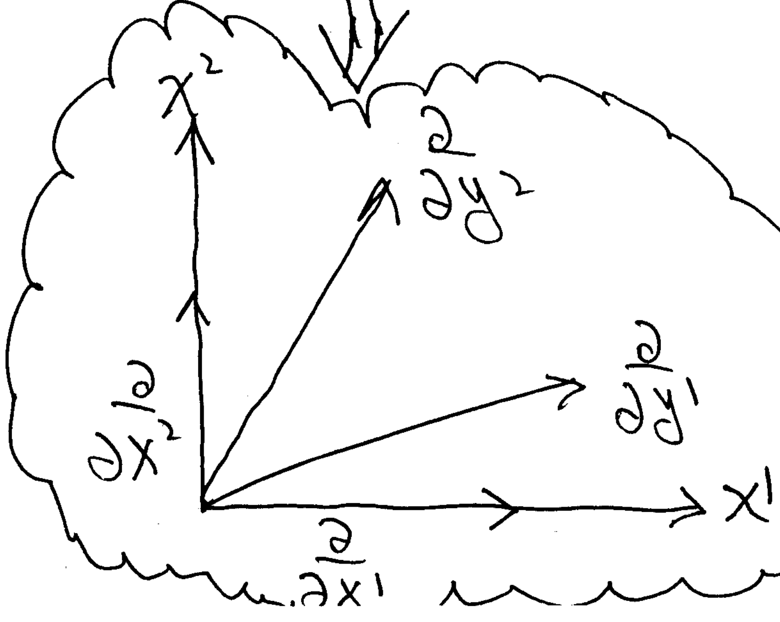
The linear algebra of the tangent space $T_p M$:

Manifold (Surface)



Tangent Space in $\underset{\sim}{x}$-coords is $\mathbb{R}^2$



- We identify vectors up on $T_p M$ with their representations in each coord. syst.
- Thus: $\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}$ have representations in the $\underset{\sim}{x}$-coordinate system
- Vector Space $\mathbb{R}^2$ is the collection of vectors in $\underset{\sim}{x}$-coordinates based at $\underset{\sim}{x}(P)$
- $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right) \equiv (\vec{e}_1, \vec{e}_2)$ are the <u>coordinate basis vectors</u>
- (x^1, x^2) now denote the <u>components</u> of a vector relative to the coord. basis

$\boxed{\text{Q1:}}$ Given a vector $\vec{V} = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2}$, what are the components of $\vec{V}$ wrt new basis $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right)$? I.e., find $\bar{a}^1, \bar{a}^2$ s.t.

$$a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} = \vec{V} = \bar{a}^1 \frac{\partial}{\partial y^1} + \bar{a}^2 \frac{\partial}{\partial y^2}$$

Summation notation: $a^i \frac{\partial}{\partial x^i} = \vec{V} = b^\alpha \frac{\partial}{\partial y^\alpha}$

[use i, j, k as summation indices in $\underset{\sim}{x}$-coords
use $\alpha, \beta, \gamma, \sigma$ as summation indices in $\underset{\sim}{y}$-coords

Sum repeated up-down indices from $1 \to 2$

To answer this, we need to know $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right)$ in terms of $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$:

Eg assume we know:

$$\frac{\partial}{\partial y_1} = B^1_1 \frac{\partial}{\partial x^1} + B^2_1 \frac{\partial}{\partial x^2} \, ; \, \frac{\partial}{\partial y_2} = B^1_2 \frac{\partial}{\partial x^1} + B^2_2 \frac{\partial}{\partial x^2}$$

Summation notation: $\frac{\partial}{\partial y_1} = B^i_1 \frac{\partial}{\partial x^i}$; $\frac{\partial}{\partial y_2} = B^i_2 \frac{\partial}{\partial x^i}$

Or altogether: $\frac{\partial}{\partial y_\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$ $\quad \alpha = 1, 2$

Matrix Notation: $\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right) = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right) \begin{bmatrix} B^1_1 & B^1_2 \\ B^2_1 & B^2_2 \end{bmatrix}$

[Convention: down indices form row vectors & called co-variant
Up indices form column vectors & are called contravariant]

$\underset{2\times 2}{B} = B^i_\alpha$ (i ← row, α ← colm)

Note: from summation notation, no difference betw $\mathbb{R}^2$ & $\mathbb{R}^n$ ≡ sum from $1 \cdots n$.

(It turns out: $B^i_\alpha = \frac{\partial x^i}{\partial y_\alpha}$!)

- From B^i_α we can see how a^i transform to $\bar{a}^\alpha$:

I.e.,

$$\vec{V} = \bar{a}^\alpha \frac{\partial}{\partial y^\alpha} = \bar{a}^\alpha \left(B^i_\alpha \frac{\partial}{\partial x^i} \right) = \underbrace{\left(\bar{a}^\alpha B^i_\alpha \right)}_{} \frac{\partial}{\partial x^i}$$

$$a^i = \bar{a}^\alpha B^i_\alpha = B^i_\alpha \bar{a}^\alpha$$

order doesn't matter in summation convention

- **Conclude:**

$$a^i = B^i_\alpha \bar{a}^\alpha \iff \underset{\sim}{a} = B \underset{\sim}{\bar{a}}$$

row (i), colm (α)

$2 \times 1 = (2\times 2)(2\times 1)$

Summation Notation — Matrix Notation

Thus: $B^{-1} \underset{\sim}{a} = \bar{a} \iff B^\alpha_i a^i = \bar{a}^\alpha$

Define: $B^\alpha_i = \left(B^i_\alpha \right)^{-1} \underset{\text{correct}}{\Longrightarrow} \dfrac{\partial y^\alpha}{\partial x^i} = \left(\dfrac{\partial x^i}{\partial y^\alpha} \right)^{-1}$

$$B^\alpha_i = \frac{\partial y^\alpha}{\partial x^i}$$

$\boxed{\text{Q2:}}$ How does the matrix g that computes inner products from components in one basis transform to the matrix wrt another basis

Ex: Standard inner product:

$$\vec{V} = a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} \quad \vec{W} = b^1 \frac{\partial}{\partial x^1} + b^2 \frac{\partial}{\partial x^2}$$

$$\vec{V} \cdot \vec{W} = a^1 b^1 + a^2 b^2 \equiv \langle \vec{V}, \vec{W} \rangle$$

geometric: $\vec{V} \cdot \vec{W} = \|\vec{V}\| \|\vec{W}\| \cos\theta$ ← computes length b/c's in $\underset{\sim}{x}$-coords!

$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$ is an ON basis $\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \delta_{ij}$

Q: what matrix $g \equiv g_{ij}$ computes $\vec{V} \cdot \vec{W}$ from their components wrt $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$?

<u>Ans</u>: $\vec{v} \cdot \vec{w} = [a^1, a^2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$

Matrix: $\underset{\sim}{a}^T g \, \underset{\sim}{b}$

$$\underbrace{g}_{\text{matrix}} \equiv \underbrace{g_{ij}}_{\text{comp's}} = \delta_{ij}$$

$\underset{\sim}{a} = \begin{bmatrix} a^1 \\ a^2 \end{bmatrix}$ vertical since components up

Summation: $\vec{v} \cdot \vec{w} = a^i g_{ij} b^j \equiv g_{ij} a^i b^j$

sum repeated up-down indices from $1 \to 2$

order doesn't matter!

<u>Conclude</u>: $g_{ij} = \delta_{ij}$ is the matrix that tells how to compute $\vec{v} \cdot \vec{w}$ in $\underset{\sim}{x}$-coords (i.e. wrt standard $\underset{\sim}{x}$-basis $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$)

Q: what matrix $\bar{g}_{\alpha\beta}$ computes the inner product $\langle \vec{v}, \vec{w} \rangle$ from components wrt $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right)$ (ie., in $\underset{\sim}{y}$-coordinates...)

Soln: Inner product determined on a basis (by linearity!)

$$\left\langle a^1\frac{\partial}{\partial y^1} + a^2\frac{\partial}{\partial y^2}, b^1\frac{\partial}{\partial y^1} + b^2\frac{\partial}{\partial y^2} \right\rangle = a^1b^1\left\langle \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^1}\right\rangle + a^1b^2\left\langle \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right\rangle$$

$$+ a^2b^1\left\langle \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^1}\right\rangle + a^2b^2\left\langle \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^2}\right\rangle$$

$$= a^\alpha b^\beta \left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right\rangle \Rightarrow \bar{g}_{\alpha\beta} = \left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right\rangle$$

Compute: $\bar{g}_{\alpha\beta} = \left\langle \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right\rangle = \left\langle B^i_\alpha \frac{\partial}{\partial x^i}, B^j_\beta \frac{\partial}{\partial x^j}\right\rangle$

$$= B^i_\alpha B^j_\beta \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle}_{g_{ij}} = B^i_\alpha\, g_{ij}\, B^j_\beta$$

B^i_α : i ← row, α ← colm

g_{ij} : i ← row, j ← colm

$$\Downarrow$$

$\boxed{\bar{g} = B^T g B}$ matrix

$\boxed{\bar{g}_{\alpha\beta} = B^i_\alpha\, g_{ij}\, B^j_\beta}$ summation

<u>Conclude</u>: We never assumed $g_{ij} = \delta_{ij}$, so in general: if

$$\langle \vec{v}, \vec{w} \rangle = g_{ij}\, a^i b^j$$

represents an inner product wrt $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$

then

$$\langle \vec{v}, \vec{w} \rangle = \bar{g}_{\alpha\beta}\, \bar{a}^\alpha \bar{a}^\beta$$

represents it in terms of $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\right)$

where:

$$\boxed{\begin{aligned} \frac{\partial}{\partial y^\alpha} &= B^i_\alpha \frac{\partial}{\partial x^i} \\ \bar{g}_{\alpha\beta} &= B^i_\alpha\, g_{ij}\, B^j_\beta \end{aligned}}$$

<u>Note</u>: usual notation leave off bars & let ~~e~~ α, β tell you you are in y-coords: $\bar{g}_{\alpha\beta} \equiv g_{\alpha\beta}$; $\bar{a}^\alpha \equiv a^\alpha$

- **Defn**: We say g_{ij} symmetric if $g_{ij} = g_{ji}$ for $i, j \in \{1, \dots, n\}$. $(n=2)$

In matrix notation: $\underset{2\times 2}{g} = (g_{ij})$

row ↑ ↑ colm

$\Rightarrow g_{ij} = g_{ji}$ iff $g = g^T$

Theorem: If g_{ij} is symmetric then so is $\bar{g}_{\alpha\beta}$

Proof: $\bar{g} = B^T g B$ (matrix)

$$\bar{g}^T = (B^T g B)^T = B^T g^T B = B^T g B = \bar{g}$$

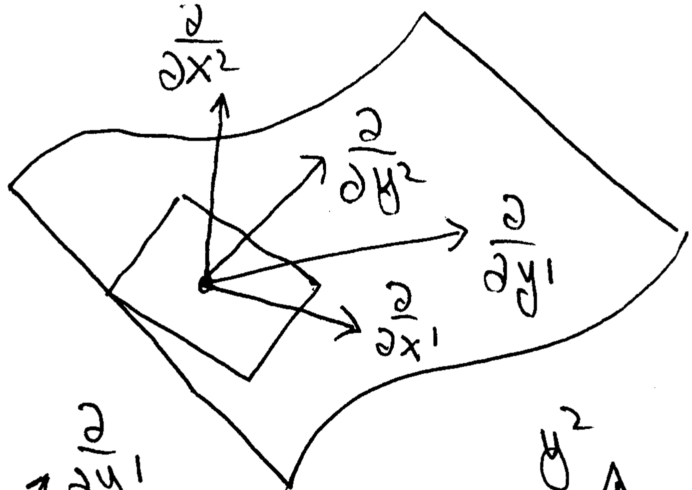
$\Rightarrow \bar{g}$ is symmetric ✓

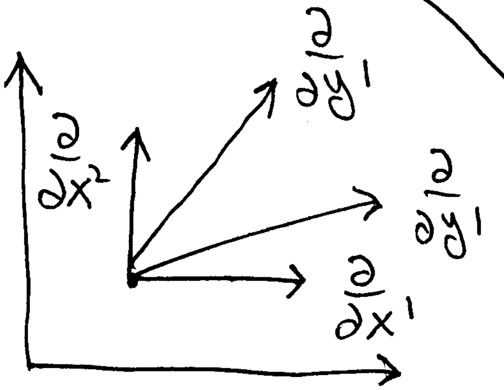
<u>Conclude</u>: The standard inner product is symmetric in <u>every</u> <u>basis</u>.

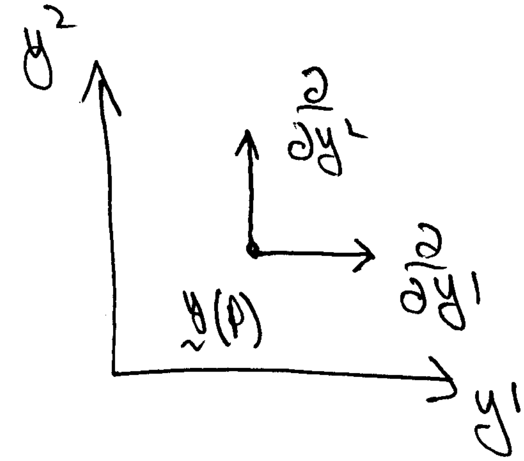
Ie., $g_{ij} = \delta_{ij}$ is symmetric ✓

In differential geometry we <u>always</u> restrict to symmetric inner products.

Conclude:







- A vector X_p has a representation in every coordinate system
- If $\frac{\partial}{\partial y^\alpha}$ are represented in $\underset{\sim}{x}$-coords as

$$\frac{\partial}{\partial y^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$$

then: $\vec{V} \equiv X_p = \underbrace{a^i}_{\underset{\sim}{x}\text{-comp's}} \frac{\partial}{\partial x^i} = \underbrace{\bar{a}^\alpha}_{\underset{\sim}{y}\text{-comp's}} \frac{\partial}{\partial y^\alpha}$; $\vec{W} = Y_p = \underbrace{b^i}_{\underset{\sim}{x}\text{-comps}} \frac{\partial}{\partial x^i} = \underbrace{\bar{b}^\alpha}_{\underset{\sim}{y}\text{-comp's}} \frac{\partial}{\partial y^\alpha}$,

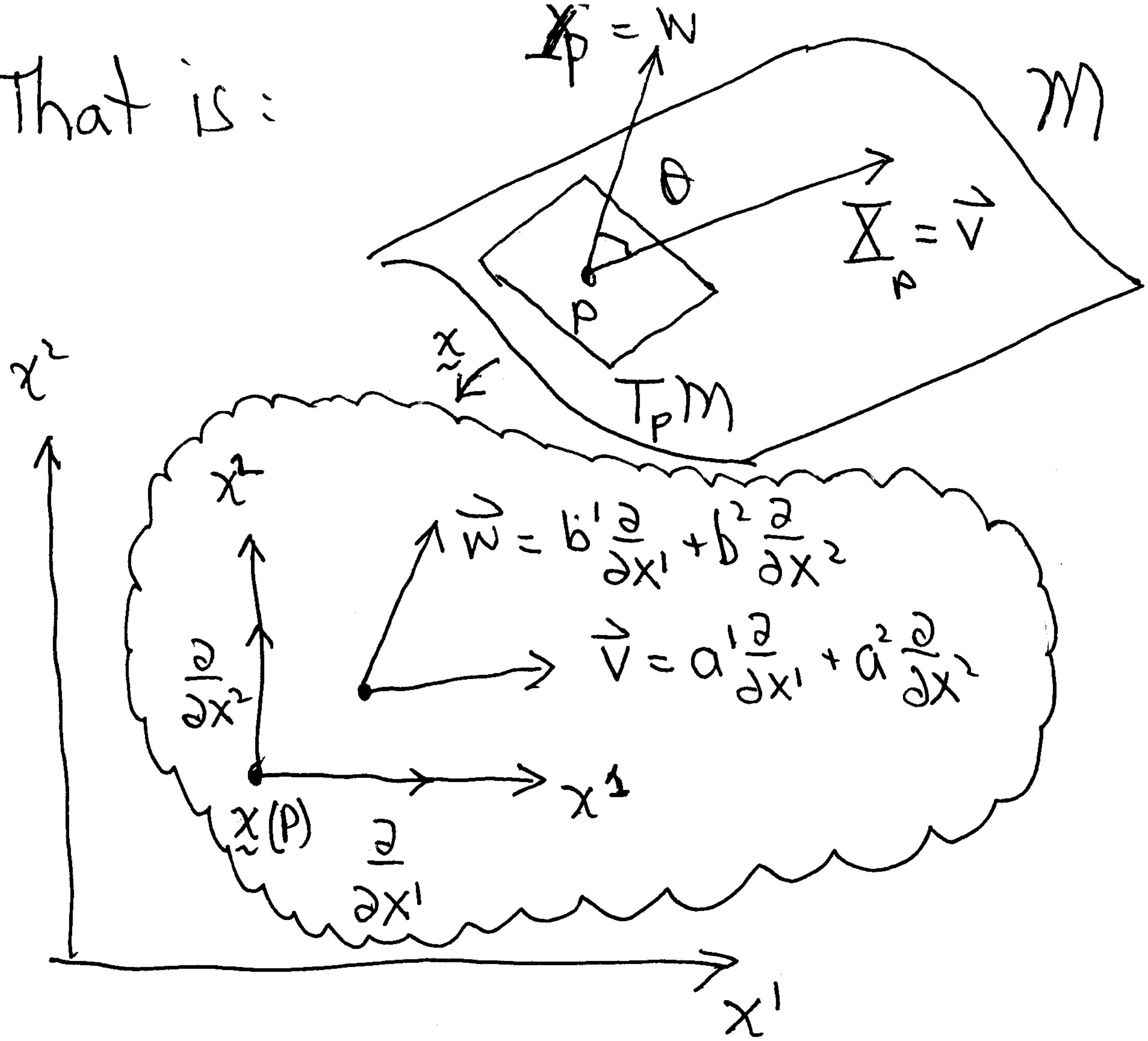
$$a^i = B^i_\alpha \bar{a}^\alpha \quad , \quad b^j = B^j_\beta \bar{a}^\beta$$

$$\langle X_p , Y_p \rangle = g_{ij} a^i b^j = \bar{g}_{\alpha\beta} \bar{a}^\alpha \bar{b}^\beta \; ; \; \bar{g}_{\alpha\beta} = B^i_\alpha g_{ij} B^j_\beta$$

Back To The Big Picture: We have that if g_{ij} is how the inner product in $\underset{\sim}{x}$-coordinates must be modified in order that $\langle \vec{v}, \vec{w} \rangle = g_{ij} a^i b^j$ gives length & angles up at P on M, then in $\underset{\sim}{y}$-coords it must be $\bar{g}_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta$ $\left(B^i_\alpha = \frac{\partial x^i}{\partial y^\alpha} \right)$

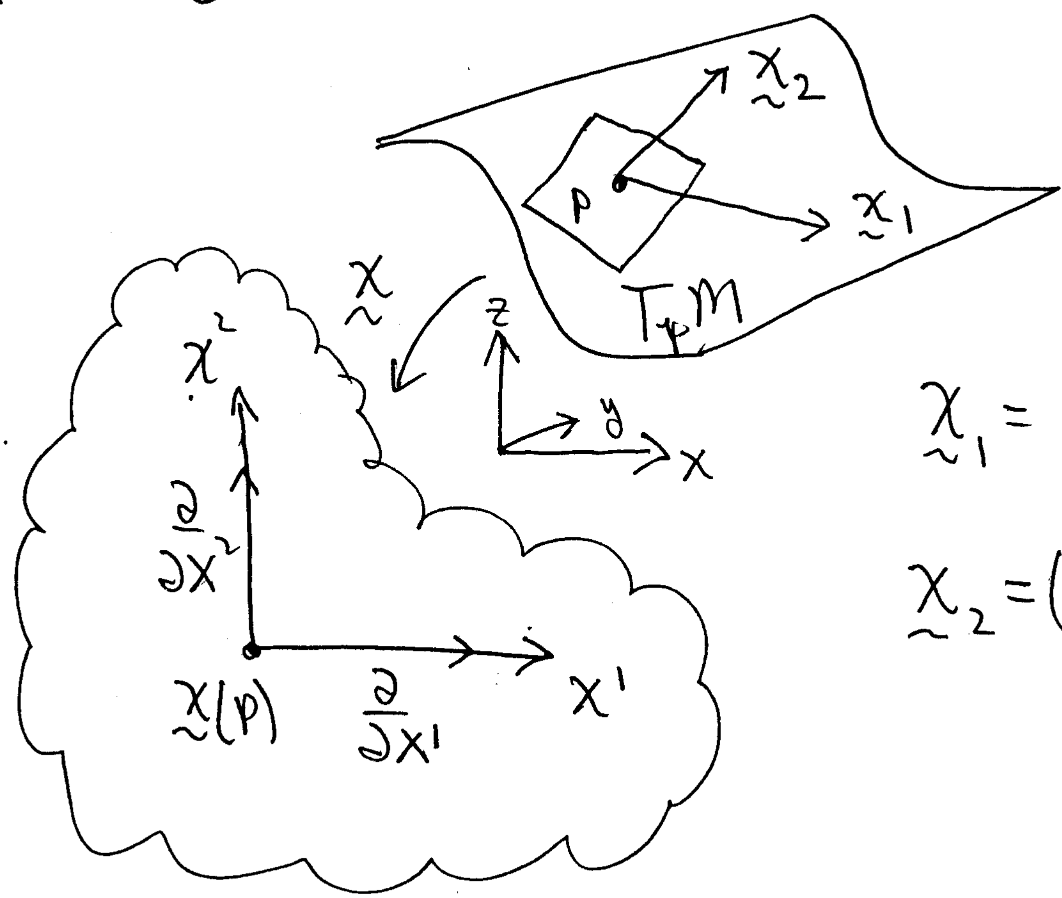
$\Rightarrow$ Begs Question: what must g_{ij} in $\underset{\sim}{x}$-coords be to do this?

That is:



In the tangent space at $\underset{\sim}{x}(p)$ as represented in $\underset{\sim}{x}$-coords, $\vec{v}$ & $\vec{w}$ are vectors in $\mathbb{R}^2$ with (x^1, x^2)-components $(a^1, a^2)^t$ & $(b^1, b^2)^t$. The usual inner product gives us $\underset{\sim}{x}$-coord lengths & angles, so how do we modify this to get what we want: lengths & angles for the "real" $\vec{v}$ & $\vec{w}$ they name up on surface M?

• To get g_{ij}: Let $\underset{\sim}{x}_1$ & $\underset{\sim}{x}_2$ denote the vectors up in T_pM named by coordinate vectors $\frac{\partial}{\partial x^1}$ & $\frac{\partial}{\partial x^2}$ down in the $\underset{\sim}{x}$-coord system.



$M \subseteq \mathbb{R}^3$

$\underset{\sim}{x}_1 = \overrightarrow{(\alpha^1, \alpha^2, \alpha^3)} \in \mathbb{R}^3$

$\underset{\sim}{x}_2 = \overrightarrow{(\beta^1, \beta^2, \beta^3)} \in \mathbb{R}^3$

$\vec{V} = \underbrace{a^i \frac{\partial}{\partial x^i}}_{\text{name in } \underset{\sim}{x}\text{-coords}} = \underbrace{a^i \underset{\sim}{x}_i}_{\text{up on surface}}$

$\vec{w} = \underbrace{b^i \frac{\partial}{\partial x^i}}_{\text{in } \underset{\sim}{x}\text{-coords}} = \underbrace{b^i \underset{\sim}{x}_i}_{\text{up on surface}}$

Thus:

$$\langle \vec{v}, \vec{w} \rangle_p = \langle a^i \underset{\sim}{x}_i, b^j \underset{\sim}{x}_j \rangle_p = a^i b^j \underbrace{\langle \underset{\sim}{x}_i, \underset{\sim}{x}_j \rangle_p}$$

The real inner product we want up on surface (dot prod in $\mathbb{R}^3$)

this must be g_{ij} !

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{x}_1 \cdot \underset{\sim}{x}_1 & \underset{\sim}{x}_1 \cdot \underset{\sim}{x}_2 \\ \underset{\sim}{x}_2 \cdot \underset{\sim}{x}_1 & \underset{\sim}{x}_2 \cdot x_2 \end{bmatrix}$$

computed by the dot product in $\mathbb{R}^3$ where M Lives !

<u>Note</u>: $g_{ij} = g_{ji}$

because

$$\underset{\sim}{x}_1 \cdot \underset{\sim}{x}_2 = \underset{\sim}{x}_2 \cdot \underset{\sim}{x}_1$$

<u>Conclude</u>: once we know g_{ij} in one coordinate system $\underset{\sim}{x}$,

$$g_{ij} = \underset{\sim}{x}_i \cdot \underset{\sim}{x}_j$$

then we know it in every other coord system by

$$\bar{g}_{\alpha\beta} = g_{ij} \, B^i_\alpha \, B^j_\beta$$

$$B^i_\alpha = \frac{\partial x^i}{\partial y^\alpha} \qquad B^j_\beta = \frac{\partial x^j}{\partial y^\alpha}$$

(we still need to confirm these formulas for B^i_α!)

☐ Invariant Defn of inner product. (Pg 2)

<u>Defn</u>: an inner product on a vector space V is a function $\langle\, ,\, \rangle : V \times V \to \mathbb{R}$ st

(a) $\langle u, v\rangle = \langle v, u\rangle$ symmetric

(b) $\langle u, rv + sw\rangle = r\langle u,v\rangle + s\langle u,w\rangle$ (multilinearity)

(c) $\langle u, u\rangle > 0,\ u \neq 0$ pos definite (Euclidean)

(c') Specify signature (Lorentzian case)

<u>Note</u>: If $\frac{\partial}{\partial x^i}$ is a basis for V, then

(a) & (b) $\Rightarrow$ that if $u = a^i \frac{\partial}{\partial x^i}$ $v = b^j \frac{\partial}{\partial x^j}$

then

$$\langle u,v\rangle = \langle a^i \frac{\partial}{\partial x^i}, b^j \frac{\partial}{\partial x^j}\rangle \underset{\text{linearity}}{\uparrow}{=} a^i b^j \underbrace{\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle}_{g_{ij}}$$

$\Leftrightarrow$ $\langle u,v\rangle = g_{ij} a^i b^j$ some $g_{ij} = g_{ji}$

⊡ <u>Preview</u>: In General Relativity (GR)

- $\mathcal{M} \subseteq \mathbb{R}^4$ denotes <u>spacetime</u>
- spacetime ≡ the "manifold of events" that can be named by coordinate systems $\underset{\sim}{x} = (\underbrace{x^0}_{\text{time}}, \underbrace{x^1, x^2, x^3}_{\text{space}})$
- Time $x^0 = ct$ gives x^0, dimensions of length like x^1, x^2, x^3
- The <u>gravitational</u> <u>field</u> is the metric g_{ij}, given in every coordinate system as a 4x4 symmetric matrix

$$g_{ij} = g_{ji}$$

- The matrix g_{ij} depends on P; in general it changes from point to point because spacetime is <u>curved</u>
- At each P you can find a coord system in which (Locally Inertial Coordinates)

$$g_{ij} = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

In every other coord system $\underset{\sim}{y}$ we have:

$$\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\alpha}$$

sum i,j $0 \ldots 3$

- g_{ij} tells how to compute (infinitessmal) changes in time & space for observers passing thru the point P, computed just like in special relativity, with erros due to the fact that g_{ij} cannot be made diagonal near P when ∃ spacetime curvature.

in a whole open nbdd of P...

- $g_{ij}(\underset{\sim}{x}(P))$ <u>is</u> the gravitational field

[Everything observable by gravity can be computed from $g_{ij}(\underset{\sim}{x}(P))$
* Not every metric can be a gravitational field *]

- 1915 (Einstein) The gravitational fied Solves the Einstein Eqvations: $G = 8\pi T$

* This is the constraint that it be a gravitational field *

- $$G = 8\pi T$$

curvature in metric G
(Einstein Curvature Tensor)

energy density & flux
(Stress energy tensor)

In $\underset{\sim}{x}$ -coords:

$$\text{"}\ \partial^2_{\underset{\sim}{x}}\, g_{ij} = 8\pi\, T_{ij}(\rho, u, p)\ \text{"}$$

energy density   velocity   pressure

<u>Conclude</u>: $G = 8\pi T$ is the <u>equation</u> for the gravitational field ...

A 2nd order (involves 2nd partials $\frac{\partial g_{ij}}{\partial x^i \partial x^j}$ etc.)
PDE for g_{ij} (10 functions!)

② The remarkable thing about curvature —

- g_{ij} at a point p transforms from coord system to coordinate system like a $\binom{0}{2}$-tensor:

$$g_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$$

- In general, derivatives of g_{ij}: like

$$\frac{\partial}{\partial x^k} g_{ij}(\underset{\sim}{x}) \quad , \frac{\partial^2}{\partial x^k \partial x^\ell} g_{ij}(\underset{\sim}{x})$$

The matrix g_{ij} depends on $\underset{\sim}{x} = (x^1, x^2)$ where T_pM is pinned

are determined by <u>nearby</u> values of g_{ij} — i.e., values of $g_{ij}(\underset{\sim}{x}(\hat{p}))$ for $\hat{p}$ near p — thus they <u>do not transform</u> like a $\binom{0}{2}$ tensor —

• <u>Remarkable fact</u>: The curvature tensor is the special combination of 2nd derivatives of g_{ij} that <u>DO</u> transform like a tensor. Thus $G = 8\pi T$ is a tensor equation:

$$G = 8\pi T \Leftrightarrow \partial^2_{x^i x^j} g_{ij}(\underset{\sim}{x}) = T(\underset{\sim}{x})$$

$\Downarrow$ Curvature is a <u>tensor</u>

$$G_{ij} = 8\pi T_{ij}$$

Transforms like g_{ij} <u>itself</u> = a $\binom{0}{2}$-tensor

I.e.
$$G_{\alpha\beta} = G_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\alpha}$$

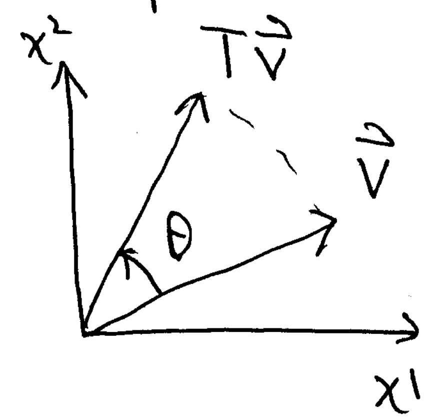
$$T_{\alpha\beta} = T_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$$

!!!

- <u>Linear Transformations</u> of $\mathbb{R}^2$:

$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \vec{v} \longmapsto T\vec{v} = \vec{w}$

"T maps vectors in plane to vectors in plane"



<u>Ex</u>: Rotation through angle θ:

$$T\left(\frac{\partial}{\partial x^1}\right) = \cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial x^2}$$

$$T\left(\frac{\partial}{\partial x^2}\right) = -\sin\theta \frac{\partial}{\partial x^1} + \cos\theta \frac{\partial}{\partial x^2}$$

Linearity $\Rightarrow$ T is determined on a basis:

$$T\cdot\left(a^1\frac{\partial}{\partial x^1} + a^2\frac{\partial}{\partial x^2}\right) = a^1 T\left(\frac{\partial}{\partial x^1}\right) + a^2 T\left(\frac{\partial}{\partial x^2}\right)$$

$$= a^1\left(\cos\theta\frac{\partial}{\partial x^1} + \sin\theta\frac{\partial}{\partial x^2}\right) + a^2\left(-\sin\theta\frac{\partial}{\partial x^1} + \cos\theta\frac{\partial}{\partial x^2}\right)$$

$$= \left(a^1\cos\theta - a^2\sin\theta\right)\frac{\partial}{\partial x^1} + \left(a^1\sin\theta + a^2\cos\theta\right)\frac{\partial}{\partial x^2}$$

$$= \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}a^1 \\ a^2\end{bmatrix}$$

<u>Conclude</u>: Rotation thru θ takes vector $\vec{v}$ with components $\begin{pmatrix} a^1 \\ a^2 \end{pmatrix}$ wrt $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$ to vector $T\vec{v} = \vec{w}$ with components

$$\begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \end{bmatrix}$$

We say: $\underset{2\times 2}{A} \equiv A^{j}_{\;i} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$j \leftarrow$ row

$i \leftarrow$ colm

represents T in the basis $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$

<u>Notation</u>: "down $\equiv$ covariant $\equiv$ row vector"

"up $\equiv$ contravariant $\equiv$ colm vector"

Matrix: $\boxed{\underset{\sim}{b} = A\,\underset{\sim}{a}}$ Summation: $\boxed{b^{j} = A^{j}_{\;i}\, a^{i}}$

$\boxed{\text{Q3:}}$ General Question: If A represents T wrt one basis (say $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$), and another basis is given (say $\frac{\partial}{\partial y^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$), then what matrix represents T wrt $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2})$?

[use i, j, k, l as indices on $\underset{\sim}{x}$-basis $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$
use $\alpha, \beta, \gamma, \delta$ as indices on $\underset{\sim}{y}$-basis $\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}$]

Theorem: $\underset{2\times 1}{\underset{\sim}{b}} = \underset{(2\times 2)}{A}\underset{(2\times 1)}{\underset{\sim}{a}}$ represents T in $\underset{\sim}{x}$-basis, then in $\underset{\sim}{y}$-basis A is represented by $\bar{A}$ with

$$\bar{A} = B^{-1} A B \qquad \text{(matrix notation)}$$

$$(\bar{A})^\alpha_\beta = (B^{-1}AB)^\alpha_\beta \qquad \text{(indices)}$$

$$\bar{A}^\alpha_\beta = B^\alpha_j A^j_i B^i_\beta \qquad \text{(summation notation)},$$

$$B^\alpha_j = (B^i_\beta)^{-1} = (B^{-1})^\alpha_j$$

$\alpha \leftarrow$ row, $j \leftarrow$ column

<u>Proof</u>: $\vec{V} = a^i \frac{\partial}{\partial x^i}$, $\vec{W} = b^j \frac{\partial}{\partial x^j}$ (in $\underset{\sim}{x}$-coords)

$$= \bar{a}^\alpha \frac{\partial}{\partial y^\alpha}, \qquad = \bar{b}^\beta \frac{\partial}{\partial y^\beta} \quad \text{(in } \underset{\sim}{y}\text{-coords)}$$

where $\bar{a}^\alpha = B^\alpha_i a^i$ & $\bar{b}^\beta = B^\beta_j b^j$.

We want $\underset{2\times 2}{\bar{A}}$ s.t. $\underset{2\times 1}{\underset{\sim}{b}} = \underset{(2\times 2)}{A}\,\underset{(2\times 1)}{\underset{\sim}{a}}$ iff $\underset{(2\times 1)}{\bar{\underset{\sim}{b}}} = \underset{(2\times 2)}{\bar{A}}\,\underset{(2\times 1)}{\bar{\underset{\sim}{a}}}$

But

$$\bar{b}^\beta = B^\beta_j b^j \qquad (b^j = A^j_i a^i) \Rightarrow$$

$$= B^\beta_j A^j_i a^i \qquad (a^i = B^i_\alpha \bar{a}^\alpha) \Rightarrow$$

$$= \underbrace{B^\beta_j A^j_i B^i_\alpha}_{\underset{(2\times 2)(2\times 2)(2\times 2)}{B^{-1}AB}} \underbrace{\bar{a}^\alpha}_{\underset{(2\times 1)}{\bar{\underset{\sim}{a}}}}$$

✓

<u>Note</u>: A^i_j is a 2x2 matrix that represents a <u>transformation</u>

$i \leftarrow$ row, $j \leftarrow$ colm

g_{ij} is a 2x2 matrix that represents a <u>metric</u>

$i \leftarrow$ row, $j \leftarrow$ colm

But they <u>don't</u> transform the same way so they are <u>not</u> the same thing!!

$$\boxed{\begin{aligned} \bar{A}^\alpha_\beta &= A^i_j B^j_\beta B^\alpha_i \iff \bar{A} = B^{-1} A B \\ \bar{g}_{\alpha\beta} &= g_{ij} B^i_\alpha B^j_\beta \iff \bar{g} = B^T A B \end{aligned}}$$

<u>Covectors</u>: To complete the basic objects of differential geometry we need co-vectors

$$\boxed{\omega = a_i \, dx^i}$$

- dx^i is the <u>covector</u> "dual" to $\frac{\partial}{\partial x^i}$:

dx^i operates on vectors to compute the i-th component of the vector in $\underset{\sim}{x}$-coords!

$$X \in T_p M \qquad X = a^i \frac{\partial}{\partial x^i} = \bar{a}^\alpha \frac{\partial}{\partial y^\alpha}$$

$dx^i(X) = a^i \Rightarrow dx^i$ acts linearly on vectors

<u>Note</u>: a^i depends on all the vectors in the $\underset{\sim}{x}$-basis, not just $\frac{\partial}{\partial x^i}$

$b_i dx^i(X) = b_i a^i \equiv$ "coord dot product of $\underset{\sim}{b}$ with $\underset{\sim}{a}$ in $\underset{\sim}{x}$-coords"

i.e. $= (b_1 dx^1 + \cdots + b_n dx^n)(X) = b_1 dx^1(X) + \cdots + b_n dx^n(X) = b_i a^i$ ✓

- Q: If $b_i\, dx^i(\underline{X}) = b_i a^i$, what 1-form computes this in $\underline{y}$-coordinates?

<u>Defn</u>: we say $b_i\, dx^i = \bar{b}_\alpha dy^\alpha$ if

$$b_i\, dx^i(\underline{X}) = \bar{b}_\alpha\, dy^\alpha(\underline{X})$$

$$\underline{X} = a^i \frac{\partial}{\partial x^i} = \bar{a}^\alpha \frac{\partial}{\partial y^\alpha} = \underline{X}$$

$$b_i\, dx^i\left(a^i \frac{\partial}{\partial x^i}\right) = \bar{b}_\alpha\, dy^\alpha\left(\bar{a}^\alpha \frac{\partial}{\partial y^\alpha}\right)$$

$$b_i a^i = \bar{b}_\alpha \bar{a}^\alpha$$

But $\bar{a}^\alpha = B^\alpha_{\ i}\, a^i$ so

$$b_i a^i = \bar{b}_\alpha B^\alpha_{\ i}\, a^i$$

$$\boxed{b_i = \bar{b}_\alpha B^\alpha_{\ i}}$$

"b_i transforms like $\frac{\partial}{\partial x^i}$ a <u>down</u> index"

- b_i are the components of the covector dx^i. Q: how does dx^i transform?

$$dx^i(\underline{X}) = \bar{b}_\alpha \, dy^\alpha(\underline{X})$$

$b_j = \begin{cases} 1 & j=i \\ 0 & \text{ow} \end{cases}$

$\bar{b}_\alpha = B^i_\alpha$ ← fixed i since $b_j = \begin{cases} 1 & j=i \\ 0 & \text{ow} \end{cases}$

$$\boxed{dx^i = B^i_\alpha \, dy^\alpha}$$

Conclude: dx^i transforms contravariantly (like a^i)

b_i transforms covariantly (like $\frac{\partial}{\partial x^i}$)

$\omega = b_i \, dx^i$ called a covector

$\underline{X} = a^i \frac{\partial}{\partial x^i}$ called a vector

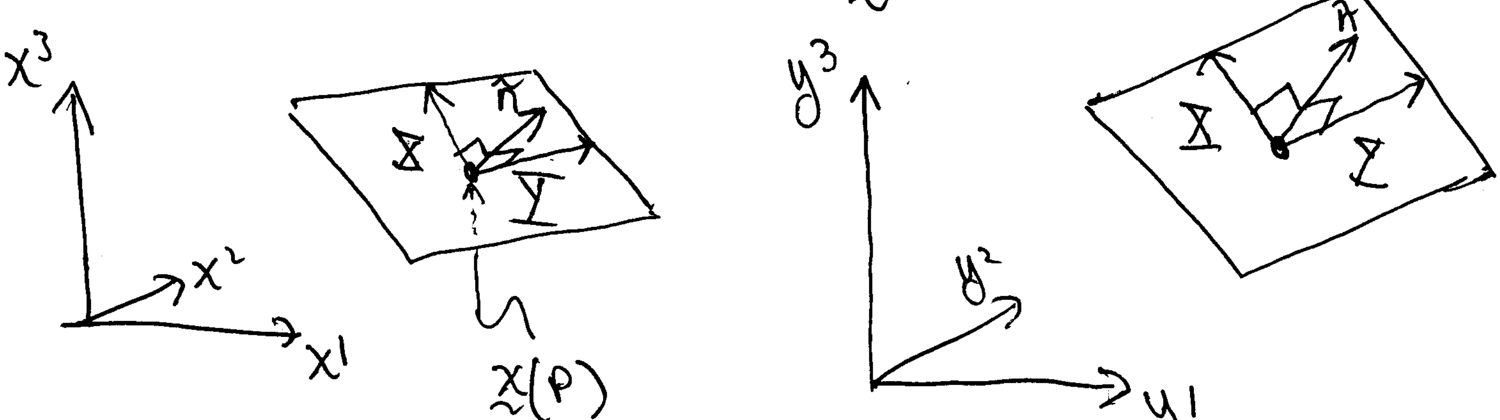
These are the fundamental objects of diff geom.

- Vectors keep track of directions on M,

Covector keep track of normals to hypersurfaces

[A hypersurface is a Surface (Manifold) of dimension $n-1 \equiv$ co-dimension $=1$

To identify a hypersurface in $\underset{\sim}{x}$-coordinates you locate the normal $\underset{\sim}{n}$



Really: for $\underset{\sim}{n}$ to remain $\perp$ to surface in all coord systems, it has to transform like a covector

$$n = n_i \, dx^i$$

Then if $\underline{X}$ is tangent to the hypersurface at P,

$$n(\underline{X}) = 0 \iff n_i \, dx^i\left(a^i \frac{\partial}{\partial x^i}\right) = 0 = n_\alpha \, dy^\alpha\left(\bar{a}^\alpha \frac{\partial}{\partial y^\alpha}\right)$$

$$n_i a^i = 0 \quad \left\{ \underset{\sim}{n} \cdot \underset{\sim}{a} = 0 \text{ in every coord!} \right\} \quad n_\alpha \bar{a}^\alpha = 0$$

• Another way to say it: each coord system $\underset{\sim}{x}$ has its own coordinate inner product (at $\underset{\sim}{x}(p)$), the <u>dot product</u>

$$\left(a^i \frac{\partial}{\partial x^i}\right) \underset{\text{dot}}{\cdot} \left(b^j \frac{\partial}{\partial x^j}\right) = \sum_{i=1}^{n} a_i b_i$$

The fact that the summation convention fails tells us that the coordinate dot product in $\underset{\sim}{y}$-coords at p is different from $\underset{\sim}{x}$-dot product.

Q: if $\underset{\sim}{n}$ has a "dot product" with tangent vectors in $\underset{\sim}{x}$-coords, how must it transform so you get the <u>same</u> dot product in $\underset{\sim}{y}$-coords?

Ans: $\underset{\sim}{n} \equiv n_i$, $\bar{n}_\alpha = B^i_\alpha n_i$ implies

$$\underbrace{n_i a^i}_{\underset{\sim}{x}\text{-dot prod}} = \underbrace{\bar{n}_\alpha \cdot \bar{a}^\alpha}_{\underset{\sim}{y}\text{-dot prod}} \Rightarrow \text{dot prod preserved!}$$

✓