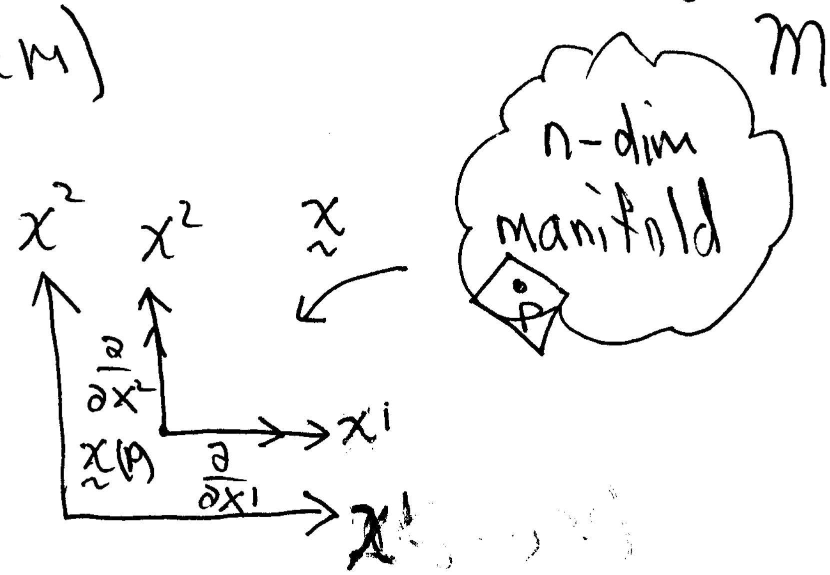
Symmetric Tensors & The inner product

IV

- Recall: g_{ij} is a symmetric matrix that represents an inner product wrt a given basis (coord system)

$$\overline{X}_p = a^i \frac{\partial}{\partial x^i}$$

$$\overline{Y}_p = b^j \frac{\partial}{\partial x^j}$$



$$\langle \overline{X}_p, \overline{Y}_p \rangle = g_{ij} a^i b^j$$

$n \times n$ symmetric matrix that represents $\langle , \rangle$ in basis $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$

Alway assume g symmetric: $g_{ij} = g_{ji}$

We now show:

"Law of inertia"

Theorem (8) (Sylvester, 1852) For any metric g given in $\underset{\sim}{x}$-coords by g_{ij} st $|g| = \det|g_{ij}|$ & $g_{ij} = g_{ji}$ (symmetric), there exists an ON basis $\frac{\partial}{\partial y_\alpha}$ such that in $\underset{\sim}{y}$-coords (at P!)

$$g_{\alpha\beta} = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix}$$

where $\delta_k = \pm 1$.

Defn: n_- = "negative index" = # of neg 1's

n_+ = "positive index" = # of pos 1's

$$n_- + n_+ = n \quad (\text{because } \det g \neq 0)$$

[(n_-, n_+, σ) = inertia of the matrix (σ = # of zeros)]

Signature of $g = n_- - n_+$.

Thm: Every ON. basis has same signature

Pf. We use the symmetric matrix theorem:

- <u>Fundamental</u> <u>Thm</u> <u>of</u> <u>Symmetric</u> <u>Matrices</u>:

A symmetric $n\times n$ matrix $\underset{n\times n}{A}$, $A^T = A$

has <u>real</u> eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$

(repeated by multiplicity) & an O.N. basis of

eigenvectors $\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_n$ so that

$$A \underset{\sim}{r}_k = \lambda_k \underset{\sim}{r}_k$$

Matrix notation: $\underset{\sim}{r} = \begin{bmatrix} r^1 \\ \vdots \\ r^n \end{bmatrix} \in \mathbb{R}^n$

$$\underset{\sim}{r}_k \cdot \underset{\sim}{r}_\ell = \delta_{k\ell}$$

dot prod

Pf (See Strang Linear Alg)

☒ What does the Symmetric Matrix Thm tell us about symmetric tensors? Like A^i_j, g_{ij}?
Here are the two subtleties:

(1) "Symmetry" is only a coordinate independent property of $\binom{0}{2}$ tensors g_{ij}, not $\binom{1}{1}$-tensors A^i_j

(2) Eigenvalues and Eigenvectors are coordinate independent for $\binom{1}{1}$-tensors A^i_j but not $\binom{0}{2}$-tensors g_{ij}

Check (1): $g_{ij} = g_{ji} \Rightarrow \bar{g}_{\alpha\beta} = B^i_\alpha \, g_{ij} \, B^j_\beta = \bar{g}_{\beta\alpha}$

$$\bar{g}^T = (B^T g B)^T = \bar{g}$$

So either way, $\bar{g}_{\alpha\beta} = \bar{g}_{\beta\alpha} \Rightarrow$ symmetry is coord indept for $\binom{0}{2}$-tensors

On the other hand:

$$A^i_{\ j} = A^j_{\ i} \Rightarrow \bar{A}^\alpha_{\ \beta} = B^\alpha_{\ i} A^i_{\ j} B^j_{\ \alpha}$$

$$\bar{A}^T = (B^{-1}AB)^T = B^T A B^{-T} \neq \bar{A}$$

not always!

(2) If $\underset{2\times 2}{A}$ is a $\binom{1}{1}$-tensor A^i_j and

$$A\underset{\sim}{r} = \lambda \underset{\sim}{r} \quad \text{for } \underset{\sim}{r} = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix}, \text{ this says}$$

$$\boxed{A^i_j a^j = \lambda a^i} \qquad \underset{\sim}{x}\text{-coordinates}$$

makes sense because $\binom{1}{1}$-tensors represent Linear Transformations that take vectors to vectors, so $A\underset{\sim}{r} = \lambda \underset{\sim}{r} \Leftrightarrow A^i_j a^j = \lambda a^i$

vector vector vector comp's up vector comp's up

In $\underset{\sim}{y}$-coordinates: $\bar{a}^\alpha = B^\alpha_k a^k$

$$\bar{A}^\alpha_\beta = B^\alpha_i A^i_j \bar{B}^j_\beta$$

so

$$\bar{A}^\alpha_\beta \bar{a}^\beta = B^\alpha_i A^i_j \underbrace{\bar{B}^j_\beta B^\beta_k}_{j=k} a^k = \underbrace{B^\alpha_i A^i_k a^k = B^\alpha_i \lambda a^i}_{\text{(coord indep!)}} = \lambda \bar{a}^\alpha$$

- On the other hand, $\underset{\sim}{g}\,\underset{\sim}{r} = \lambda \underset{\sim}{r}$ (matrix) doesn't transform right ⇒ not "covariant" ≡ coordinate indept. Ie.

$$g_{ij}\, a^j = \lambda a^i \Rightarrow \bar{g}_{\alpha\beta}\, \bar{a}^\beta \neq \lambda \bar{a}^\alpha$$

g_{ij} ↑ down $\qquad$ a^i ↑ up

Eg: $$\bar{g}_{\alpha\beta}\,\bar{a}^\beta = B^i_\alpha\, g_{ij}\, \underbrace{B^j_\beta B^\beta_k}_{j=k} a^k = B^i_\alpha\, g_{ij}\, a^j = \lambda B^T \underset{\sim}{a}$$

$$\lambda \bar{a}^\alpha = \lambda B^\alpha_j\, a^j = \lambda B^{-1} \underset{\sim}{a} \neq \lambda B^T \underset{\sim}{a}$$

↑ in general $B^{-1} \neq B^T$

◈ To prove Theorem (g):

<u>Assume</u>: g_{ij} given in $\underset{\sim}{x}$-coords wrt $\frac{\partial}{\partial x^1} \cdots \frac{\partial}{\partial x^n}$

To give it e-vectors & e-vals, define

$$G^i_{\ j} \equiv g_{ij}$$

(Same symmetric matrix in $\underset{\sim}{x}$-coords, but transforms differently to other coords)

Assume symmetry: $g_{ij} = g_{ji}$ so $G^i_{\ j} = G^j_{\ i}$

(In other coords, $G^\alpha_{\ \beta}$ <u>not</u> symmetric!)

- Symm matrix thm $\Rightarrow$ $G^i_{\ j}$ has real evals λ_i & o.n. basis of eigenvectors $\underset{\sim}{\vec{r}}_k = r^j_{\ k} \frac{\partial}{\partial x^j}$

Now represent $G^i_{\ j}$ in coord. system in which $\frac{\partial}{\partial y^\alpha} \equiv \underbrace{r^i_{\ \alpha}}_{\text{components of } \vec{r}_\alpha \text{ wrt } \frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^i} = \underset{\sim}{\vec{r}}_\alpha$ the α-th e-vector

• <u>Conclude</u>: $r^i_\alpha \equiv B^i_\alpha$ is the matrix that transforms from $\underset{\sim}{x}$ to $\underset{\sim}{y}$-coords at Point P. Thus in terms of $\underset{\sim}{y}$-basis $\frac{\partial}{\partial y^\alpha}$ we have G represented as

$$\bar{G}^\alpha_\beta = B^\alpha_i \, G^i_j \, B^j_\beta \qquad \text{(summation)}$$

$$\Leftrightarrow \quad \bar{G} = B^{-1} G B \qquad \text{(matrix)}$$

where $B = \begin{bmatrix} | & & | \\ \underset{\sim}{r}_1 & \cdots & \underset{\sim}{r}_n \\ | & & | \end{bmatrix}$ Colm's $\underset{\sim}{r}_i$ are O.N. basis wrt dot prod.

B orthogonal matrix $\Rightarrow B^{-1} = B^T$

Ie. $B^T B = \begin{bmatrix} - \underset{\sim}{r}_1 - \\ \vdots \\ - \underset{\sim}{r}_n - \end{bmatrix} \begin{bmatrix} | & & | \\ \underset{\sim}{r}_1 & \cdots & \underset{\sim}{r}_n \\ | & & | \end{bmatrix} = \underset{\sim}{r}_i \cdot \underset{\sim}{r}_j = \delta_{ij}$

But g_{ij} transforms over to $\frac{\partial}{\partial y_\alpha}$ basis as

$$\bar{g}_{\alpha\beta} = B^i_\alpha \, g_{ij} \, B^j_\beta$$

$$\bar{g} = B^T g B$$

So since $B^T = B^{-1}$ & $g_{ij} = G^i_j$ in $\underset{\sim}{x}$-coords they also agree in $\underset{\sim}{y}$-coordinates!

• Now: $\bar{G} = B^T G B$, $B = \begin{bmatrix} | & & | \\ \underset{\sim}{r}_1 & \cdots & \underset{\sim}{r}_n \\ | & & | \end{bmatrix}$

$$GB = \begin{bmatrix} | & & | \\ G\underset{\sim}{r}_1 & \cdots & G\underset{\sim}{r}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\underset{\sim}{r}_1 & \cdots & \lambda_n\underset{\sim}{r}_n \\ | & & | \end{bmatrix}$$

So

$$\bar{G} = B^T G B = \begin{bmatrix} - & \underset{\sim}{r}_1 & - \\ & \vdots & \\ - & \underset{\sim}{r}_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \lambda_1\underset{\sim}{r}_1 & \cdots & \lambda_n\underset{\sim}{r}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$$

& $\lambda_i \neq 0$ because $|\bar{G}| = |B^T||G||B| \underset{|B^T| = |B^{-1}| = 1/|B|}{\uparrow =} |\Lambda| \neq 0$.

<u>Conclude</u>: in coordinates $\underline{y}$ where

$$\frac{\partial}{\partial y^\alpha} = r^i_\alpha \frac{\partial}{\partial x^i}$$

we have G & g represented as

$$\bar{G}^\alpha_\beta = \bar{g}_{\alpha\beta} = \Lambda_{\alpha\beta} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}_{\alpha\beta}.$$

Finally let $Q = \begin{bmatrix} (\sqrt{|\lambda_1|})^{-1} & & \\ & \ddots & \\ & & (\sqrt{|\lambda_n|})^{-1} \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{\sqrt{|\lambda_1|}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{|\lambda_n|}} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{|\lambda_1|}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{|\lambda_n|}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\lambda_1}{|\lambda_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n}{|\lambda_n|} \end{bmatrix} = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix} = \Delta$$

where $\delta_k = \pm 1 = \text{sign}(\lambda_k)$. $\boxed{\cancel{\text{as claimed in Thm}}\ (g)}$.

Thus

$$Q^T B^T g B Q = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix}$$

& letting $\tilde{B} = BQ$ we get

$$\tilde{B}^T g \tilde{B} = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix}$$

for some sequence $\delta_1 \cdots \delta_n$ of ± 1's.

Thus $\tilde{B} = \tilde{B}^i_\alpha \Rightarrow \frac{\partial}{\partial \tilde{y}^\alpha} = \tilde{B}^i_\alpha \frac{\partial}{\partial x^i}$ is

an O.N. basis for metric g_{ij} !

[Note: this says at each point P $\exists$ a basis $\frac{\partial}{\partial y^i}$ in which g is $\mathrm{diag}(\delta_1, \ldots, \delta_n)$. But this basis will <u>not</u> in general be the basis for a coordinate system simultaneously at each pt in an open nbd of P, in fact, only possible when curvature $= 0$]

<u>Thm</u> (Sylvester) g has the same signature in every ON basis.

<u>Proof</u>: if not, then $\exists$ B such that

$$\bar{B}^T \Delta \bar{B} = \bar{\Delta}$$

and $\bar{\Delta} = \begin{bmatrix} \bar{\delta}_1 & & 0 \\ 0 & & \bar{\delta}_n \end{bmatrix}$ has a different # of ± 1's. Now consider all vectors st $g_{ij} a^i a^j < 0$. This is not a sub space, <u>but</u> there is a dimension d^+ of the largest subspace such that $g_{ij} a^i a^j < 0$ and d^- st $g_{ij} a^i b^j > 0$. We know from $g = \Delta$ that $d^- \geq n^-$ $d^+ \geq n^+$, and from $g = \bar{\Delta}$ that $d^- \geq \bar{n}^-$ $d^+ \geq \bar{n}^+$ $\Rightarrow n^- = \bar{n}^-$ & $n^+ = \bar{n}^+$ ✓

<u>Cor</u>: An inner product $\langle\, ,\, \rangle$ is positive definite, i.e.,

$$\langle u, u\rangle > 0 \quad \forall\, u \neq 0$$

iff $\delta_1 = \delta_2 = \cdots = \delta_n = 1$.

Pf. $\langle u, u\rangle$ can be calculated in any coord system, & in ON. basis the theorem is immediate:

$$[u^1, \ldots, u^n] \begin{bmatrix} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix}$$

$$= -(u^1)^2 \cdots - (u^p)^2 + (u^{p+1})^2 + \cdots + (u^n)^2 > 0$$

$\forall\, u \neq 0$ iff $p = 0$ ✓

☒ Symmetry with respect to an inner product —

- Note that $\binom{1}{1}$-tensors A^i_j are correct for eigenvalues & eigenvectors: that is, $(\lambda, \underset{\sim}{a})$ is an eigenpair if

$$A^i_j a^j = \lambda a^i$$

Moreover, under change of coordinates $\frac{\partial}{\partial y^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i}$ we have

$$\bar{A}^\alpha_\beta \bar{a}^\beta = \lambda \bar{a}^\alpha$$

where

$$\bar{A}^\alpha_\beta = B^\alpha_i A^i_j B^j_\beta \qquad \bar{a}^\alpha = a^i B^\alpha_i$$

$$\bar{A} = B^{-1} A B \qquad \underset{\sim}{\bar{a}} = B^{-1} \underset{\sim}{a}$$

so $\bar{A} \underset{\sim}{\bar{a}} = B^{-1} A B B^{-1} \underset{\sim}{a} = B^{-1} A \underset{\sim}{a} = \lambda B^{-1} \underset{\sim}{a} = \lambda \underset{\sim}{\bar{a}}$ ✓

<u>Conclude</u>: (i) tensors transform correctly so that eigenvalues & eigenvectors are numbers & vectors defined <u>independent of coordinates</u>

- On the other hand, symmetry of A^i_j is a coordinate dependent notion — we can only apply the symmetric matrix theorem in coordinates where $A^i_j = A^j_i$; in other coordinates, where symmetry does not hold, the thm is true (A^α_β still has real evals even when $A^\alpha_\beta \neq A^\beta_\alpha$) but we can't apply it.

- Resolution of the conundrum – Symmetry is really a property of inner products

<u>Defn</u>: A is <u>symmetric</u> wrt <u>inner product</u> $\langle\, ,\rangle$ if $\langle Au, v\rangle = \langle u, Av\rangle$ $\forall$ $u, v \in V$.

<u>Thm</u> (Symmetric matrix Thm) if A is symmetric wrt a <u>positive</u> <u>definite</u> inner product, then A has real eigenvalues & a basis of e-vectors orthogonal wrt the inner product $\langle\, ,\rangle$.

<u>Ex</u>: $\langle\, ,\rangle \equiv$ dot product $\Rightarrow \langle u, v\rangle = u^T \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} v$

Symmetry: $\langle Au, v\rangle \underset{\substack{\uparrow \\ \forall u,v}}{=} \langle u, Av\rangle \Leftrightarrow \delta_{ij} A^i_k u^k v^j = \delta_{ij} A^j_k v^k u^i$ $\forall u, v$

$\Leftrightarrow A^i_j = A^j_i$

① Eigenvectors from different eigenspaces are orthogonal

Pf. $\langle A\underset{\sim}{a}_1, \underset{\sim}{a}_2 \rangle = \langle \underset{\sim}{a}_1, A\underset{\sim}{a}_2 \rangle$

$$\lambda_1 \langle \underset{\sim}{a}_1, \underset{\sim}{a}_2 \rangle = \lambda_2 \langle \underset{\sim}{a}_1, \underset{\sim}{a}_2 \rangle$$

$\lambda_1 \neq \lambda_2 \Rightarrow \langle \underset{\sim}{a}_1, \underset{\sim}{a}_2 \rangle = 0$

② Eigenvalues of A are <u>real</u>

Pf. define complex inner product

$$\langle \underset{\sim}{c}_1, \underset{\sim}{c}_2 \rangle = g_{ij}\, \bar{c}_1^{\,i}\, c_2^{\,j}$$

$\underset{\sim}{c}_1 = (c_1^1, \dots, c_1^n)$

$\underset{\sim}{c}_2 = (c_2^1, \dots, c_2^n)$

matrix that defines the inner product

$\underset{\sim}{c}_i = \underset{\sim}{a}_i + i\underset{\sim}{b}_i$

complex real real

It follows that $\langle , \rangle$ extends to a bilinear form on $\mathbb{C}^n$, &

$$\langle \lambda \underset{\sim}{c}_1, \underset{\sim}{c}_2 \rangle = \bar{\lambda} \langle \underset{\sim}{c}_1, \underset{\sim}{c}_2 \rangle$$

$$\langle \underset{\sim}{c}_1, \lambda \underset{\sim}{c}_2 \rangle = \lambda \langle \underset{\sim}{c}_1, \underset{\sim}{c}_2 \rangle$$

<u>Thus</u>: if $A \underset{\sim}{c} = \lambda \underset{\sim}{c}$ λ an e-value, then

$$\langle A \underset{\sim}{c}, \underset{\sim}{c} \rangle = \langle \underset{\sim}{c}, A \underset{\sim}{c} \rangle$$

$$\Downarrow$$

$$\bar{\lambda} \langle \underset{\sim}{c}, \underset{\sim}{c} \rangle = \lambda \langle \underset{\sim}{c}, \underset{\sim}{c} \rangle$$

<u>Conclude</u>: if $\langle \underset{\sim}{c}, \underset{\sim}{c} \rangle \neq 0$, then λ must be real.

• **Cor ①**: If $\langle\, ,\, \rangle$ is positive definite,

$$\langle \underset{\sim}{c}, \underset{\sim}{c} \rangle > 0 \text{ for } \underset{\sim}{c} \neq 0,$$

then A has real eigenvalues

• **Cor ②**: In general, only lightlike vectors ($\langle \underset{\sim}{c}, \underset{\sim}{c} \rangle = 0$) can be eigendirection with complex eigenvalues.

~~**Ex**: Say $g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ "Lorentzian metric"~~

~~$A^i{}_j$ symmetric if $\langle A\underset{\sim}{a}, \underset{\sim}{b} \rangle = \langle \underset{\sim}{a}, A\underset{\sim}{b} \rangle$~~

~~$$g_{jk} A^k{}_i a^i b^j = g_{ik} a^i A^k{}_j b^j$$~~

~~$$g_{jk} A^k{}_i a^i b^j = g_{ik} A^k{}_j a^i b^j$$~~

~~$\boxed{(gA)^T = gA}$~~ $\Leftrightarrow$ $\boxed{g_{jk} A^k{}_i = g_{ik} A^k{}_j}$

Note: $\langle A\underset{\sim}{a}, \underset{\sim}{b}\rangle = \langle \underset{\sim}{a}, A\underset{\sim}{b}\rangle \quad (*)$

for every $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^n$ iff

$$A_{ij} = A_{ji} \quad \text{for } A_{ij} = g_{ih} A^h_{\ j}$$

I.e.: $(*) \Rightarrow \quad g_{ij} A^i_{\ k} a^k b^j = g_{ij} a^i A^j_{\ k} b^k$

$$\Leftrightarrow \quad \underbrace{g_{ki}}_{ik} A^k_{\ j} a^j b^i = \underbrace{g_{ik} A^k_{\ j}}_{A_{ij}} a^i b^j$$

$$\Leftrightarrow \quad A_{ij} a^j b^i = A_{ij} a^i b^j$$

for $\underline{\text{all}}$ $\underset{\sim}{a}, \underset{\sim}{b}$

$$\Leftrightarrow \quad A_{ij} = A_{ji}$$

where $\bar{A}_{ij} \equiv A_{ij} = g_{ih} A^h_{\ j} \Leftrightarrow \bar{A} = gA$ ✓

$\underset{\bar{A}}{\equiv}$

<u>Claim</u>: $A_{ij} = g_{ik} A^k_j$ transforms like a $\binom{0}{2}$-tensor, so symmetry of A_{ij} is coordinate independent statement!

<u>Corollary</u>: If there exists a positive symmetric matrix $\underset{n\times n}{g}$ such that $\underset{n\times n}{gA}$ is a symmetric matrix ($(gA)^T = gA$), then A has real eigenvalues.