Tensors

Tensors In General $\Leftrightarrow$ Summation Conventio (Summary)

- $\underset{\sim}{x} : \mathcal{M} \to \mathbb{R}^n$ a coordinate system on an n-dimensio manifold $\mathcal{M}$
- $T_p\mathcal{M} \equiv$ tangent space of $\mathcal{M}$ at P. All vectors can be represented in a coord. system
- $\left.\frac{\partial}{\partial x^i}\right|_p$, $i = 1, \dots, n$ the $\underset{\sim}{x}$-coordinate basis for $T_p\mathcal{M}$
- $X_p = a^i \frac{\partial}{\partial x^i}$ the $\underset{\sim}{x}$-coordinate name for a vector $X_p \in T_p\mathcal{M}$
- a^i the $\underset{\sim}{x}$-components of X_p
- Under change of coordinates $\underset{\sim}{x} \longmapsto \underset{\sim}{y}$ the components & bases transform by $\left(B \equiv B^i_\alpha = \frac{\partial x^i}{\partial y^\alpha}\right)$

$$\boxed{\frac{\partial}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i}, \quad \bar{a}^\alpha = \frac{\partial y^\alpha}{\partial x^i} a^i}$$

- $T_p^*M \equiv$ cotangent space of M at P. All covectors can be represented in a coord sys

- $dx^i|_p$, $i=1,\dots,n$ the $\underset{\sim}{x}$-coord basis for T_p^*M

- $\omega_p = a_i dx^i$ the $\underset{\sim}{x}$-coord name for a covector $\omega_p \in T_p^*M$

- a_i the $\underset{\sim}{x}$-components of ω_p

dot product in $\underset{\sim}{x}$-coords

- $\omega_p(X_p) = a_i dx^i(b^j \frac{\partial}{\partial x^j}) = a_i b^i \equiv \underset{\sim}{a} \cdot \underset{\sim}{b}$

- Under change of coordinates $\underset{\sim}{x} \mapsto \underset{\sim}{y}$ the components & basis transform by

$$\boxed{\bar{a}_\alpha = \frac{\partial x^i}{\partial y^\alpha} a_i \, , \quad dy^\alpha = \frac{\partial y^\alpha}{\partial x^i} dx^i}$$

- Covectors preserve the "coordinate dot product": $a_i b^i = \bar{a}_\alpha \bar{b}^\alpha$ & thereby keep track of hyperplanes — co-dimension 1 planes in $T_p M$.

- A coordinate dependent matrix $g_{ij}(p)$ tells how to compute the inner product on $T_p\mathcal{M}$:

$$\langle \underline{X}_p, \underline{Y}_p \rangle \equiv \text{"inner product between vectors up on } \mathcal{M}\text{"}$$

$$= g_{ij}\, a^i b^j = \text{"}\|\underline{X}_p\|\|\underline{Y}_p\| \cos\theta\text{" up on } \mathcal{M}$$

$$\underline{X}_p = a^i \frac{\partial}{\partial x^i}, \quad \underline{Y}_p = b^j \frac{\partial}{\partial x^j}$$

- In a different coordinate system $\underset{\sim}{y}$:

$$\langle \underline{X}_p, \underline{Y}_p \rangle = \bar{g}_{\alpha\beta}\, \bar{a}^\alpha \bar{b}^\beta$$

$$\underline{X}_p = \bar{a}^\alpha \frac{\partial}{\partial x^\alpha} \quad \underline{Y}_p = \bar{b}^\beta \frac{\partial}{\partial x^\beta}$$

where

$$\boxed{\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \iff \bar{g} = B^T g B}$$

(coordinate dependent)

A matrix A^i_j keeps track of linear transformations of T_pM:

$$T : T_pM \to T_pM, \quad \underline{X}_p \longmapsto \underline{Y}_p = T\underline{X}_p$$

$$\underline{X}_p = a^i \frac{\partial}{\partial x^i}, \quad \underline{Y}_p = b^j \frac{\partial}{\partial x^j}$$

$$\Downarrow$$

$$b^j = A^j_i \, a^i$$

- A^j_i is the $\underset{\sim}{x}$-coord representation of A
- Under change of coordinates, A^i_j transforms to $\bar{A}^\alpha_\beta$ where

$$\bar{b}^\alpha = \bar{A}^\alpha_\beta \, \bar{a}^\beta$$

($\underset{\sim}{y}$-components of $\underline{Y}_p$) ($\underset{\sim}{y}$-components of $\underline{X}_p$)

$$\boxed{\bar{A}^\alpha_\beta = A^i_j \frac{\partial x^j}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^i}}$$

$$\Leftrightarrow \quad \boxed{\bar{A} = B^{-1} A B}$$

4 Tensors:

- g_{ij} is a $\binom{0}{2}$-tensor

  A^i_j is a $\binom{1}{1}$-tensor

  R^i_{jkl} is a $\binom{1}{3}$-tensor (Riemann Curvature tensor)

Eg: $$\boxed{\bar{R}^{\alpha}_{\beta\gamma\delta} = R^i_{jkl} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma} \frac{\partial x^l}{\partial y^\delta} \frac{\partial y^\alpha}{\partial x^i}}$$

Extends to $\binom{m}{n}$-tensors

- Principle: up index transforms contravariantly

$$\bar{A}^{\cdots\alpha\cdots}_{\cdots\cdots} = A^{\cdots i\cdots}_{\cdots\cdots} \frac{\partial y^\alpha}{\partial x^i} \cdots$$

$B^{-1} = \frac{\partial y^\alpha}{\partial x^i}$

down index transforms covariantly:

$$\bar{A}^{\cdots\cdots}_{\cdots\beta\cdots} = A^{\cdots\cdots}_{\cdots j\cdots} \frac{\partial x^j}{\partial y^\beta}$$

$B = \frac{\partial x^i}{\partial y^\alpha}$

- We can use the metric to "raise & lower" indices —

$$A_{ij} \equiv g_{ik} A^{k}_{\;j}$$

A_{ij} transforms like a $\binom{0}{2}$-tensor

Eg: $\bar{A}^{\alpha}_{\;\beta} = \bar{g}_{\beta\sigma} \bar{A}^{\sigma}_{\;\beta}$ etc.

- Raising indices: let $g^{ij} = (g_{ij})^{-1}$

row col row col

Then $g^{ik} A_{kj} = A^{i}_{\;j}$

Same way —

- In general: $A_{\cdots i \cdots}^{\cdots\cdots} = g_{ik} A^{\cdots k \cdots}_{\cdots\cdots}$

$$A^{\cdots i \cdots}_{\cdots\cdots} = g^{ik} A^{\cdots\cdots}_{\cdots k \cdots}$$

<u>Theorem</u>: raising & lowering converts contravariant indicies to covariant and visa versa.

<u>Theorem</u>: contracting an up index with a down index is coordinate independent operation:

<u>Eg</u> A^i_j a $\binom{1}{1}$-tensor implies

$$A^i_i = \bar{A}^\alpha_\alpha$$ ⟸ "sum up-down indices from 1-n"

& $$A^{\cdots i \cdots}_{\cdots i \cdots} = \bar{A}^{\cdots \alpha \cdots}_{\cdots \alpha \cdots}$$

(HW)

$\boxed{\text{HW}}$ Prove that the <u>contraction</u> of 2 indices is indept of coords. I.e. show That if A^i_j is a 1-1 tensor, then

$$A^i_i = \bar{A}^\alpha_\alpha$$

Sum $i = 1, \dots, n$ ⇒ this is the trace(A) = sum of diagonal entries - in $\underset{\sim}{x}$-coordinate

trace of $\bar{A}$ in $\underset{\sim}{y}$-coords

• Finally: If A^i_j are the "components" what are the "coordinate basis elements"?

Basis: $dx^j \otimes \frac{\partial}{\partial x^i}$ basis

$$A = \underbrace{A^i_j}_{\text{component}} \underbrace{dx^j \otimes \frac{\partial}{\partial x^i}}_{\text{basis}}$$

$dx^j \otimes \frac{\partial}{\partial x^i}$ operates on pairs $(X_p, w_p) \in T_pM \times T^*_pM$

by

$$dx^j \otimes \frac{\partial}{\partial x^i}\left(a^i \frac{\partial}{\partial x^i}, b_j dx^j\right) = a^j b_i$$

Turns out: $$dy^\alpha \otimes \frac{\partial}{\partial y_\beta} = \underbrace{\frac{\partial y^\alpha}{\partial x^i}\left(dx^i \otimes \frac{\partial}{\partial x^j}\right)\frac{\partial x^j}{\partial y^\beta}}$$

i.e., by summation convention.

*This gives tensors an invariant interpretation as linear operator on $T_pM \times T^*_pM$ etc *