$\boxed{\text{Volume}}$

{also name of metric or 2x2 matrix!}

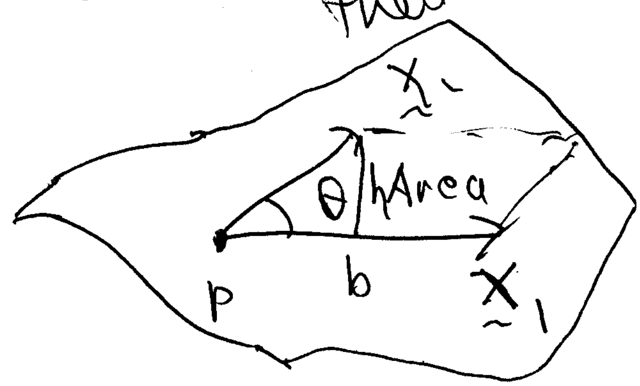
$\boxed{\Xi}$ $\det g_{ij} = |g_{ij}| = g$ is a function of p that depends on the coordinate system.

Lemma 1: $\sqrt{g} =$ (area spanned by $\parallel$-ogram $\underset{\sim}{X}_1, \underset{\sim}{X}_2$)

Pf. $g_{ij} = \langle \underset{\sim}{X}_i, \underset{\sim}{X}_j \rangle = \begin{bmatrix} \underset{\sim}{X}_1 \cdot \underset{\sim}{X}_1 & \underset{\sim}{X}_1 \cdot \underset{\sim}{X}_2 \\ \underset{\sim}{X}_1 \cdot \underset{\sim}{X}_2 & \underset{\sim}{X}_2 \cdot \underset{\sim}{X}_2 \end{bmatrix}$

$$g = |g_{ij}| = \|\underset{\sim}{X}_1\|^2 \|\underset{\sim}{X}_2\|^2 - (\underset{\sim}{X}_1 \cdot \underset{\sim}{X}_2)^2$$

$$\underset{\sim}{X}_1 \cdot \underset{\sim}{X}_2 = \|\underset{\sim}{X}_1\| \|\underset{\sim}{X}_2\| \cos\theta$$ angle betw them



$$g = \|\underset{\sim}{X}_1\|^2 \|\underset{\sim}{X}_2\|^2 - \|\underset{\sim}{X}_1\|^2 \|\underset{\sim}{X}_2\|^2 \cos^2\theta$$

$$= \|\underset{\sim}{X}_1\|^2 \|\underset{\sim}{X}_2\|^2 \sin^2\theta$$

$$= \text{Area}^2 \checkmark$$

Area $= bh = \|\underset{\sim}{X}_1\| \|\underset{\sim}{X}_2\| \sin$

• **Recall**: $\underline{X} \times \underline{Y} = \underbrace{\|\underline{X}\| \|\underline{Y}\| \sin\theta}_{\text{Area spanned by } \underline{X}, \underline{Y}} \; \vec{n}$

"length in $\mathbb{R}^3$"

unit normal by RH-rule

Conclude: $g = |\underset{\sim}{x}_1 \times \underset{\sim}{x}_2|^2$

• Note: By $g = \|\underset{\sim}{x}_1\|^2 \|\underset{\sim}{x}_2\|^2 \sin^2\theta > 0$

we know $g > 0$. $\Rightarrow \underset{2\times2}{B^T} \underset{2\times2}{g} \underset{2\times2}{B} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ $\delta_i = \pm 1$

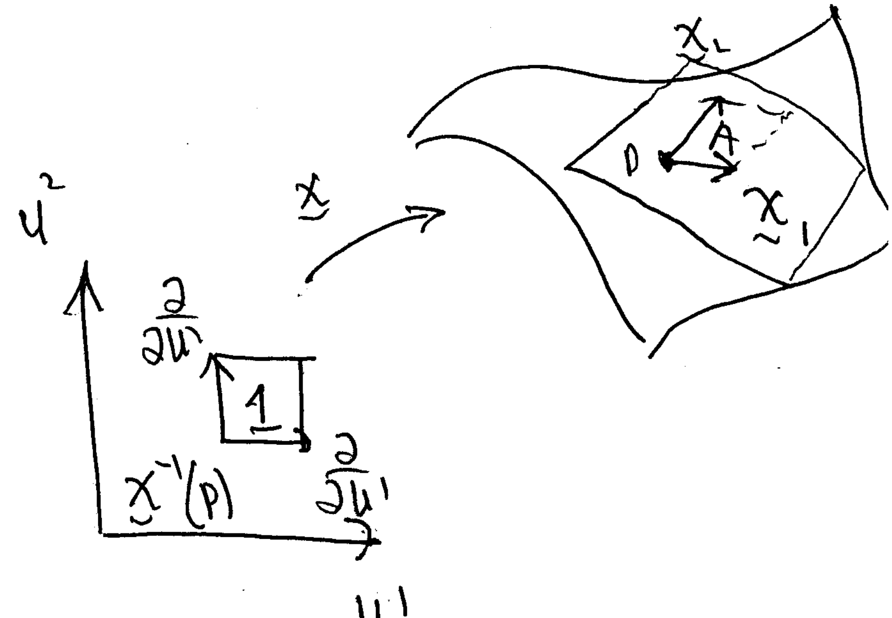
is pos definite as expected. Ie,

$|\underset{2\times2}{B^T} \underset{2\times2}{g} \underset{2\times2}{B}| = |\underset{2\times2}{B^T}| |\underset{2\times2}{g}| |\underset{2\times2}{B}| = |B|^2 g = \delta_1 \cdot \delta_2$

$\Rightarrow \delta_1 = \delta_2$. Since all vectors have pos length,

$\delta_1 = \delta_2 = 1$ ✓

<u>Conclude</u>: $|g|^{1/2}$ gives the amplification factor for the area in going from $\underset{\sim}{x}$-coords to M. That is:



So:
$$\frac{\text{Area up on } M \text{ @ } p}{\text{Area in } \underset{\sim}{x}\text{-coords @ } p} \approx \frac{\text{Area}[\underset{\sim}{x}_1, \underset{\sim}{x}_2]}{\text{Area}\left[\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right]}$$

$$= \frac{\|\underset{\sim}{x}_1 \times \underset{\sim}{x}_2\|}{\left\|\frac{\partial}{\partial u^1} \times \frac{\partial}{\partial u^2}\right\|} = \|\underset{\sim}{x}_1 \times \underset{\sim}{x}_2\| = (\det g)^{1/2}$$

integrate in $\underset{\sim}{x}$-coords to get surface area upon M

Suface area of image

<u>Defn</u>:
$$\boxed{\iint_U |g|^{1/2} du^1 du^2 = \iint_{\underset{\sim}{x}(U)} dS}$$

• <u>Conclude</u>: The condition that a coord system does not collapse areas to zero (regular coord systems do not) is the condition:

$$\underset{\sim}{X}_1 \times \underset{\sim}{X}_2 \neq 0 \iff \det g \neq 0$$

• Q: How does $g = \det g_{ij}$ transform to y-coords?

Ans: $\bar{g} = \det \bar{g}_{\alpha\beta}$, $g = \det g_{ij}$

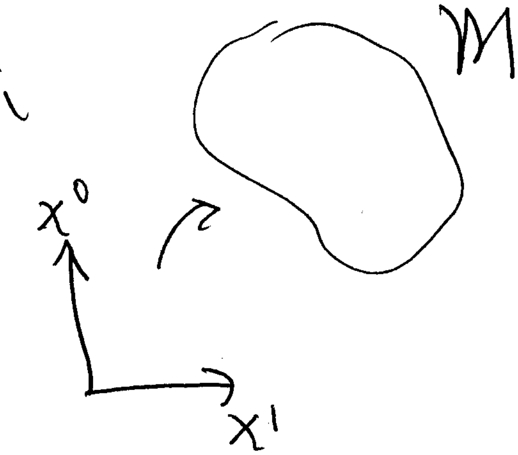
$$\bar{g} = \left|\bar{g}_{\alpha\beta}\right| = \left|\frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta}\right| = \left|\frac{\partial x^i}{\partial y^\alpha}\right|^2 g$$

$\left|\frac{\partial x^i}{\partial y^\alpha}\right|$ = "determinant of Jacobian deriv. of transformation at each p"

- In general Relativity: $M \equiv$ 4-d manifold

$\underset{\sim}{x}$-coordinates $\Rightarrow g_{ij}$ components of spacetime

$g_{ij}(p) \equiv$ gravitational metric tensor



It solves $G = 8\pi T$

$$G_{ij}[g] = 8\pi T_{ij}$$

$$\text{"} \partial^2_{ij}\, g_{ij}(x^0, \dots, x^3) = 8\pi\, \rho(x^0, \dots, x^3) \text{"}$$

↑ second derivative that make curvature ↑ energy density

- Point mass (like sun) in empty space –

Soln: $ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{1 - \frac{2GM}{r}}\, dr^2 + r^2 d\Omega^2$

$d\Omega^2 = d\varphi^2 + \sin^2\varphi\, d\theta^2$ metric on sphere!

(Schwarzschild 1915) $G = \frac{G_{Newton}}{c^2}$

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \frac{1}{1-\frac{2GM}{r}}dr^2 + r^2 d\Omega^2$$

- $r = 2GM$ = Schwarzschild radius of mass M

$r_{earth} = 2GM_{earth} \cong 1$ cm

$r_{sun} = 2GM_{sun} \cong 3$ km

$r_{Milky Way} = 2GM_{Milky Way} \cong 10^{12} \times 3$ km $\cong \frac{1}{3}$ light yr

$c = 3 \times 10^8 \frac{M}{s}$

$yr = 60 \times 60 \times 24 \times 365 \approx$

$\approx 3 \times 10^7$ seconds

$1 lyr \approx 3 \times 10^5 \times 3 \times 10^7$ km $\approx 1 \times 10^{13}$ km

- In GR, an observer traversing path $\gamma(\tau)$ will experience proper time change (age) of

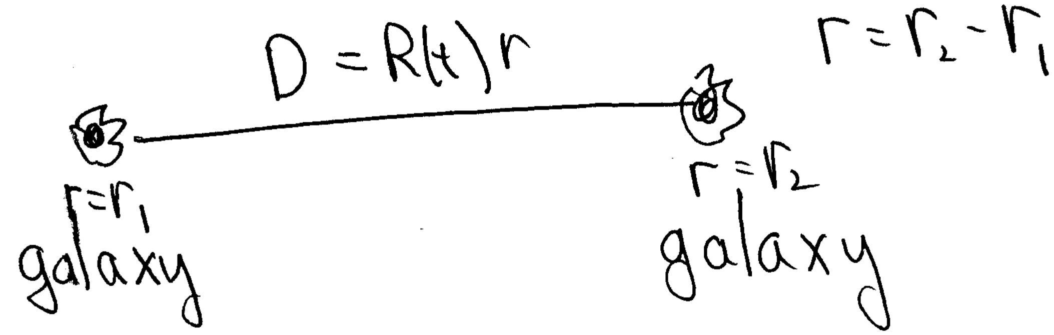
$$\int_a^b ds = \int_{\tau_a}^{\tau_b} \sqrt{|g_{ij}\dot{\gamma}^i(\tau)\dot{\gamma}^j(\tau)|}\, d\tau = \underbrace{\int_{\tau_a}^{\tau_b} \|\dot{\gamma}(\tau)\| d\tau}$$

take $|\cdot|$ since timelike vectors have neg. length squared $\|\dot{\gamma}\|^2 < 0$.

- In Cosmology: $ds^2 = -dt^2 + R(t)^2 \{\underbrace{dr^2 + r^2 d\Omega^2}_{\mathbb{R}^3 \text{ in spherical coor}}\}$

$R(t)$ increases in time $\approx$ expansion of universe

galaxies traverse paths $r = \text{const}$, radial distance at fixed time is $D = R(t) r$



$$\dot D = \dot R(t) r = \frac{\dot R}{R} R r = \underbrace{\frac{\dot R}{R}} D$$

galaxies move at $r = \text{const}$

$H = \frac{\dot R}{R}$ Hubbles constant

Hubbles Law (1929) $\dot D = H D$

"galaxies are receding at rate $\propto$ distance"

The volume tensor η:

- Consider two vectors X, Y

$$X = a^i \frac{\partial}{\partial u^i} = a^i \underset{\sim}{x}_i \qquad Y = b^j \frac{\partial}{\partial u^j} = b^j \underset{\sim}{x}_j$$

name in $\underset{\sim}{x}$-coords

as a vector in $\mathbb{R}^3$

- We can compute the area spanned in $\mathbb{R}^3$ by cross product: $X \times Y = \underbrace{A\,\vec{n}}_{\text{direction given by RH-r}}$

- To orient the sign of A, let $\vec{n}$ be a continuous unit normal to M.

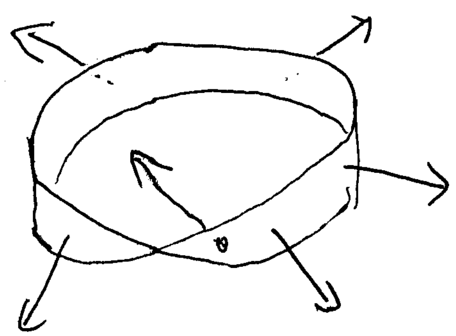
Eg, in each coord patch we could pick $\vec{n} = \dfrac{\underset{\sim}{x}_1 \times \underset{\sim}{x}_2}{\|\underset{\sim}{x}_1 \times \underset{\sim}{x}_2\|}$, but $\vec{n}$ could

switch direction from patch to patch.

- **Defn**: M is <u>orientable</u> if $\exists$ a continuous choice of unit normal to M.

Ex: Sphere is orientable (choose <u>outer</u> normal)

Ex: Mobius band is <u>not</u> orientable



We can only orient the sign of A in formula $X \times Y = A\vec{n}$ <u>when</u> $\exists$ a normal $\vec{n}$ that orients M. So: Assume M orientable, & $\vec{n}$ is unit normal which varies continuously on M. $\vec{n}$ is called an <u>orientation</u> for M.

Thm: The Mobius strip cannot be covered by one regular coord chart.

Pf. If so, then $\underset{\sim}{x}_1 \times \underset{\sim}{x}_2$ is a continuous nonzero normal to M

Defn: If M is orientable, then a choice of continuous normal $\vec{n}$ on M is called an orientation.

Given an orientation $\vec{n}$:

- **Defn**: We say a coordinate chart is positively oriented if $\dfrac{\underset{\sim}{x}_1 \times \underset{\sim}{x}_2}{\|\underset{\sim}{x}_1 \times \underset{\sim}{x}_2\|} = \vec{n}$, and negatively oriented if $\dfrac{\underset{\sim}{x}_1 \times \underset{\sim}{x}_2}{\|\underset{\sim}{x}_1 \times \underset{\sim}{x}_2\|} = -\vec{n}$.

- **Defn**: define the signed area associated with the $\|$-opiped spanned by $\underline{X}, \underline{Y} \in T_p M$ as A where

$$\underline{X} \times \underline{Y} = A\vec{n}.$$

$A > 0$ if $\underline{X} \times \underline{Y}$ in direction $\vec{n}$,
$A < 0$ if $\underline{X} \times \underline{Y}$ in direction $-\vec{n}$

- Now assume $\underset{\sim}{x}$ is pos. oriented coord system, & $\underline{X} = a^i \dfrac{\partial}{\partial u^i} = a^i \underset{\sim}{x}_i$, $\underline{Y} = b^j \dfrac{\partial}{\partial u^j} = b^j \underset{\sim}{x}_j$

$\underset{\sim}{x}$-coord name

in $\mathbb{R}^3$

Then $\quad \underset{\sim}{X} \times \underset{\sim}{Y} = (a^i \underset{\sim}{x}_i) \times (b^j \underset{\sim}{x}_j) = a^i b^j \, \underset{\sim}{x}_i \times \underset{\sim}{x}_j$

Cross product is bi-linear like $\binom{0}{2}$ tensors!

$(-1)^{\mathcal{O}} = +1$ if $\underset{\sim}{x}$ pos oriented

$(-1)^{\mathcal{O}} = -1$ if $\underset{\sim}{x}$ neg oriented

$$\underset{\sim}{x}_1 \times \underset{\sim}{x}_1 = 0, \quad \underset{\sim}{x}_1 \times \underset{\sim}{x}_2 = (-1)^{\mathcal{O}} \sqrt{g}\, \vec{n}, \quad \underset{\sim}{x}_2 \times \underset{\sim}{x}_1 = -\sqrt{g}\,\vec{n}, \quad \underset{\sim}{x}_2 \times \underset{\sim}{x}_2 = 0$$

So

$$\underset{\text{row}}{\underset{\sim}{x}_i} \times \underset{\text{col}}{\underset{\sim}{x}_j} = (-1)^{\mathcal{O}} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\varepsilon_{ij}} \sqrt{g}\, \vec{n}$$

and we have

$$\underset{\sim}{X} \times \underset{\sim}{Y} = \underbrace{(-1)^{\mathcal{O}} a^i b^j \, \varepsilon_{ij} \sqrt{g}}_{\text{signed area spanned by } \underset{\sim}{X} \,\&\, \underset{\sim}{Y}.} \, \vec{n}$$

unit normal given by orientation

<u>Claim</u>: $\eta \equiv (-1)^{\mathcal{O}} \varepsilon_{ij} \sqrt{g}$ is a $\binom{0}{2}$ - tensor!

- <u>Conclude</u>: we can calculate the signed area spanned by $\underline{X}$ & $\underline{Y}$ in $\underset{\sim}{x}$-coords by

$$\boxed{\text{Area} = (-1)^0 \sqrt{g}\, \varepsilon_{ij}\, a^i b^j}$$

$\varepsilon_{ij} \equiv$ Levi-Civita completely antisymmetric Tensor $= \begin{cases} +1 & \pi(i,j) > 0 \\ -1 & \pi(i,j) < 0 \\ 0 & \pi(i,j) = 0 \end{cases}$

$\pi(1,2) = +1$ (no transposition required to bring order increasing)

$\pi(2,1) = -1$ (odd # transpositions bring it to increasing order)

$\pi(1,1) = \pi(2,2) = 0$ (not a permutation of $(1,2)$)

- <u>Note</u>: $\varepsilon_{ij} = -\varepsilon_{ji} \Leftrightarrow$ <u>antisymmetric</u>

- The summary: on the one hand we have

$$\underline{X}\times\underline{Y} = (-1)^\sigma|\text{Area}|\,\vec{n} \quad \text{where } \vec{n} \equiv \text{orientation normal} \quad (\underline{X},\underline{Y}\in T_pM)$$

But also:

$$\underline{X}\times\underline{Y} = a^i \underset{\sim}{X}_i \times b^j \underset{\sim}{X}_j = a^i b^j\, \underset{\sim}{X}_i \times \underset{\sim}{X}_j =$$

$$= (-1)^\sigma \sqrt{g}\, a^i b^j \varepsilon_{ij}\, \vec{n}$$

Thus: signed area of $[\underline{X},\underline{Y}] = (-1)^\sigma|\text{Area}| = (-1)^\sigma\sqrt{g}\, a^i b^j \varepsilon_{ij}$

<u>Conclude</u>: $\eta_{ij} = (-1)^\sigma\sqrt{g}\,\varepsilon_{ij}$ computes signed area b

$$\eta(\underline{X},\underline{Y}) = \eta_{ij}a^i b^j = (-1)^\sigma|\text{Area}|$$

<u>Theorem</u>: η_{ij} transform like a $\binom{0}{2}$-tensor

That is: $\eta \equiv \underbrace{\eta_{ij}\, du^i \otimes du^j}_{\underset{\sim}{x}\text{-coord rep of } \eta}$ operates on $(\underline{X},\underline{Y})$ to output $(-1)^\sigma|\text{Area}|$

I.e.,

$$\eta(\underline{X},\underline{Y}) = \eta_{ij}\, du^i \otimes du^j(\underline{X},\underline{Y}) = \eta_{ij}\, du^i(\underline{X})\, du^j(\underline{Y})$$

$$= \eta_{ij}a^i b^j = (-1)^\sigma|\text{Area}|$$

<u>Proof</u>: We see how ε_{ij} transforms:

Compute: $\varepsilon_{ij} \dfrac{\partial u^i}{\partial v^\alpha} \dfrac{\partial v^j}{\partial v^\beta}$ to see how far off it is from a tensor –

$$\varepsilon_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta} = \underset{2\times2}{J^T} \, \underset{2\times2}{\varepsilon} \, \underset{2\times2}{J} \quad \text{for } J = \frac{\partial u^i}{\partial v^\alpha} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$J^T \varepsilon J = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} = \det J \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\bar{\varepsilon}_{\alpha\beta}}$$

or

$$\boxed{\bar{\varepsilon}_{\alpha\beta} = \varepsilon_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta} \frac{1}{\det J}}.$$

But $\bar{g} = (\det J)^2 g$ $\cancel{\Rightarrow}$

$\cancel{\sqrt{\bar{g}}\, \bar{\varepsilon}_{\alpha\beta} \quad \sqrt{g} \dfrac{}{\det J} \varepsilon_{ij} \dfrac{\partial u^i}{\partial v^\alpha} \dfrac{\partial u^j}{\partial v^\alpha}}$

~~Claim: $\det J > 0$ iff $o(x) = o(y) \Leftrightarrow (-1)^{o(x)} = (-1)^{o(y)} \dfrac{\det J}{|\det J|}$~~

• We next show $\det J > 0$ iff $(-1)^{o(x)} = (-1)^{o(y)}$

Said differently: $(-1)^{o(x)} = (-1)^{o(y)} \dfrac{\det J}{|\det J|}$

± 1 ± 1 ± 1

Pf. We have: $\sqrt{\bar{g}} = \sqrt{g} \det J$, $\sqrt{g} = \sqrt{\bar{g}} \det J^{-1} = \dfrac{\sqrt{\bar{g}}}{\det J}$

$$\underset{\sim}{x}_i \times \underset{\sim}{x}_j = (-1)^{o(x)} \sqrt{g}\, \varepsilon_{ij} \vec{n}$$

$$\underset{\sim}{y}_\alpha \times \underset{\sim}{y}_\beta = (-1)^{o(y)} \sqrt{\bar{g}}\, \bar{\varepsilon}_{\alpha\beta} \vec{n}$$

$$\varepsilon_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta} = \bar{\varepsilon}_{\alpha\beta} \det J$$

$$\underset{\sim}{y}_\alpha = \underset{\sim}{x}_i \frac{\partial u^i}{\partial v^\alpha}$$

Thus:

$$(-1)^{o(y)}\sqrt{g}\,\varepsilon_{\alpha\beta}\,\vec{n} = \underset{\sim}{y}_\alpha \times \underset{\sim}{y}_\beta = \left(\frac{\partial u^i}{\partial v^\alpha}\underset{\sim}{x}_i\right)\times\left(\frac{\partial y^j}{\partial v^\beta}\underset{\sim}{x}_j\right)$$

$$= \frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^j}{\partial v^\beta}\,\underset{\sim}{x}_i \times \underset{\sim}{x}_j$$

$$= \underbrace{\frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^j}{\partial v^\beta}\,\varepsilon_{ij}}_{\varepsilon_{\alpha\beta}\det J}\,\underbrace{\sqrt{g}}_{\frac{\sqrt{\bar{g}}}{|\det J|}}\,(-1)^{o(x)}\,\vec{n}$$

$$\therefore\ (-1)^{o(y)}\sqrt{\bar{g}}\,\varepsilon_{\alpha\beta} = (-1)^{o(x)}\sqrt{\bar{g}}\,\frac{\det J}{|\det J|}\,\varepsilon_{\alpha\beta}.$$

$$\therefore\ \boxed{(-1)^{o(y)} = (-1)^{o(x)}\,\frac{\det J}{|\det J|}}$$

✓

- We now show $\eta_{ij} = (-1)^{o} \sqrt{g}\, \varepsilon_{ij}$ transforms like a $\binom{0}{2}$-tensor:

$$\eta_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta} = (-1)^{o(x)} \sqrt{g}\, \varepsilon_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta}$$

$$= (-1)^{o(x)} \frac{\sqrt{g}}{|\det J|} \det J \; \varepsilon_{\alpha\beta}$$

$$= (-1)^{o(y)} \sqrt{g}\, \varepsilon_{\alpha\beta} = \eta_{\alpha\beta} \quad \checkmark$$

<u>Defn</u>: $\eta \equiv \eta_{ij}\, du^i \otimes du^j$ is the volume form on an oriented manifold M,

$$\eta_{ij} = (-1)^{o} \sqrt{g}\, \varepsilon_{ij}$$

so $\boxed{\eta_{ij} = -\eta_{ji} \quad \text{antisymmetric}}$

$\eta \equiv$ antisymmetric tensor.

• Note: In GR, where spacetime is a 4-D manifold $\mathcal{M}$ of "events", we don't have a space that $\mathcal{M}$ "lies in" like $\mathbb{R}^3$, all we have is the coord systems. I.e., no cross product to define "volume". In this case,

$$\eta_{ijkl} = (-1)^{s} \sqrt{g}\, \varepsilon_{ijkl}$$

is the volume form: A differential form is a tensor that is "completely antisymmetric": $\varepsilon_{\pi(ijkl)} = (-1)^{\pi(ijkl)} \varepsilon_{ijkl}$

• Orientation = consistent choice of pos/neg oriented coord system

• $\eta(X, Y, Z, W) =$ (signed) 4-vol of ‖-piped spanned by X, Y, Z, W

☒ Volume as a <u>differential</u> <u>form</u>:

- We have: $\eta_{ij} = (-1)^0 \sqrt{g}\, \varepsilon_{ij}$ are the component of a $\binom{0}{2}$-tensor

Tensors as operators: $\eta = \eta_{ij}\, du^i \otimes du^j$

"η operates on two vectors $\underline{X}_p$, $\underline{Y}_p$ to compute the signed volume of the ∥-opiped spanned by $\underline{X}_p$ & $\underline{Y}_p$" ; I.e.,

$$\eta(\underline{X}_p, \underline{Y}_p) = \eta_{ij}\, du^i \otimes du^j (\underline{X}_p, \underline{Y}_p) \qquad \left\langle \begin{array}{l} \underline{X}_p = a^i \dfrac{\partial}{\partial u^i} \\ \underline{Y}_p = b^j \dfrac{\partial}{\partial u^j} \end{array} \right.$$

$$\equiv \eta_{ij}\, du^i(\underline{X}_p)\, du^j(\underline{Y}_p) \quad \text{(bilinear tensor!)}$$

$$= \eta_{ij}\, a^i b^j \quad (du^i \text{ picks out } i\text{-comp of } \underline{X}_p)$$

<u>Conclude</u>:

$$\eta(\underline{X}_p, \underline{Y}_p) = \eta_{ij}\, a^i b^j$$

$$= (-1)^0 \sqrt{g}\, \underbrace{\varepsilon_{ij}\, a^i b^j}_{(a^1, a^2)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} b^1 \\ b^2 \end{pmatrix} = (a^1, a^2)\begin{pmatrix} b^2 \\ -b^1 \end{pmatrix}}$$

$$= a^1 b^2 - a^2 b^1$$

$$= \det \begin{vmatrix} a^1 & b^1 \\ a^2 & b^2 \end{vmatrix} = \left(du^1 \otimes du^2 - du^2 \otimes du^1\right)(\underline{X}_p, \underline{Y}_p)$$

<u>Defn</u>: $du^1 \wedge du^2 \equiv du^1 \otimes du^2 - du^2 \otimes du^1$

acts on two vectors $\underline{X}_p, \underline{Y}_p$ to compute

$$du^1 \wedge du^2 (\underline{X}_p, \underline{Y}_p) = \det \begin{vmatrix} a^1 & b^1 \\ a^2 & b^2 \end{vmatrix}$$

$du^1 \wedge du^2$ is called a <u>differential form</u>

$\wedge$ is called the <u>wedge product</u>

• The volume tensor η as a differential form:

$$\eta(\mathbb{X}_p, \mathbb{Y}_p) = (-1)^0 \sqrt{g}\, du^1 \wedge du^2 (\mathbb{X}_p, \mathbb{Y}_p)$$

$$du^1 \wedge du^2 = du^1 \otimes du^2 - du^2 \otimes du^1$$

$$du^1 \wedge du^2 (\mathbb{X}_p, \mathbb{Y}_p) = \varepsilon_{ij} a^i b^j = \det \begin{vmatrix} a^1 & b^1 \\ a^2 & b^2 \end{vmatrix}$$

$$= a^1 b^2 - a^2 b^1$$

Then $du^2 \wedge du^1 = -du^1 \wedge du^2$

so $du^1 \wedge du^2 = \frac{1}{2} \varepsilon_{ij}\, du^i \wedge du^j$

We have the equivalent statements:

$$\boxed{\eta = (-1)^0 \sqrt{g}\, \varepsilon_{ij}\, du^i \otimes du^j = (-1)^0 \sqrt{g}\, du^1 \wedge du^2 = (-1)^0 \sqrt{g}\, \frac{1}{2} \varepsilon_{ij}\, du^i \wedge du^j}$$

* In texts you will see the volume form written in one of these 3 ways *

⊡ The abstract point of view is to integrate a differential form to get the volume:

$$\int_{\mathcal{U}} \eta = \int_{\mathcal{U}} \underbrace{\eta\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right)} du^1 du^2 = \int_{M} dS$$

computes the signed volume# of the ∥-opiped spanned by $\frac{\partial}{\partial u^1} \equiv \underset{\sim}{x}_1$ & $\frac{\partial}{\partial u^2} \equiv \underset{\sim}{x}_2$ up on M
$\equiv$ amplification factor for volume

$$\eta\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right) = (-1)^0 \sqrt{g}\, du^1 \wedge du^2 \left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right)$$

$$= (-1)^0 \sqrt{g} \cdot 1 \quad \checkmark$$

(In 2-d, "volume" means "area"!)