(20) Gauss Curv, & 2nd Fundamental Form — VII

- Curvature has something to do with 2nd derivative : $\therefore$ define

$$\underset{\sim}{x}_{ij} = \frac{\partial}{\partial u^i} \underset{\sim}{x}_j (u^1, u^2)$$

Since $\underset{\sim}{x}_j = \frac{\partial}{\partial u^j} \underset{\sim}{x}(u^1, u^2)$ we have

$$\underset{\sim}{x}_{ij} = \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} \underset{\sim}{x}(u^1, u^2)$$

$$= \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} \left(x(u^1,u^2), y(u^1,u^2), z(u^1,u^2) \right)$$

is a vector in $\mathbb{R}^3$.

- Since derivative of vectors tangent to M have normal components, (like curves!), $\underset{\sim}{x}_{ij}$ has a component in direction $\vec{n}$. (Assume $\vec{n}$ is an orientation on M)

• Since $\vec{n}, \underset{\sim}{x}_1, \underset{\sim}{x}_2$ is a basis for $\mathbb{R}^3$ at each $P \in M \subseteq \mathbb{R}^3$, and since $\underset{\sim}{x}_{ij} \in \mathbb{R}^3$, we can write $\underset{\sim}{x}_{ij}$ as a linear comb of $\{\vec{n}, \underset{\sim}{x}_1, \underset{\sim}{x}_2\}$

Define L_{ij} & Γ^k_{ij} at P by

$$\boxed{\underset{\sim}{x}_{ij} = L_{ij}\vec{n} + \Gamma^k_{ij}\,\underset{\sim}{x}_k}$$

L & Γ describe all 2nd deriv vector $\underset{\sim}{x}_{ij}$ @ $P \in M$

<u>Defn</u>: $L_{ij} \equiv$ 2nd Fundamental form @ P

$\Gamma^k_{ij} \equiv$ Christofel symbols $\equiv$ Connection Coefficients

$\underset{\sim}{x}_{ij} = \underset{\sim}{x}_{ji} \Rightarrow L_{ij} = L_{ji}, \ \Gamma^k_{ij} = \Gamma^k_{ji}$.

Both are Symmetric in every coord system.

• <u>Note</u>: $\Gamma^k_{ij}\,\underset{\sim}{x}_k$ are the 2nd deriv's in the surface $\Rightarrow$ the part that can by expressed by $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ in $\underset{\sim}{x}$-coords.

<u>Theorem</u> ① L_{ij} transforms like a $\binom{0}{2}$-tensor (like g_{ij} & η_{ij}) $\Rightarrow$ L_{ij} is a symmetric $\binom{0}{2}$-tensor.

<u>Theorem</u> ② $\Gamma^k_{ij} = \frac{1}{2} g^{k\sigma}\{g_{ij,k} + g_{ki,j} + g_{jk,i}\}$ is <u>not</u> a tensor, but is <u>intrinsic</u> in that they can be measured in the coord syst via knowing the g's.

<u>Theorem</u> 3 : The geodesics on M are solutions of the ODE's :

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0$$

<u>These</u> are curves of shortest (critical) length (locally not globally), & have property that their normal ~~derivatives~~ vectors $\vec{N}$ have no component tangent to M.

The second Fundamental form —

- Recall: g_{ij}, η_{ij} are $\binom{0}{2}$ - tensors I.e.,

$$\eta_{ij} \frac{\partial u^i}{\partial v^\alpha} \frac{\partial u^j}{\partial v^\beta} = \bar{\eta}_{\alpha\beta}$$

- As $\binom{0}{2}$ tensors they operate on pairs of vectors

$$\eta(X, Y) = \eta_{ij}\, du^i \otimes du^j (X, Y) = \eta_{ij}\, a^i b^j$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\eta(X, Y) = \bar{\eta}_{\alpha\beta}\, dv^\alpha \otimes dv^\beta (X, Y) = \bar{\eta}_{\alpha\beta}\, \bar{a}^\alpha \bar{b}^\beta$$

- $$\underset{\sim}{x}_{ij} = \frac{\partial}{\partial u^i} \underset{\sim}{x}_j(u^1, u^2) = \frac{\partial}{\partial u^i} \frac{\partial}{\partial u^j} \underset{\sim}{x}(u^1, u^2)$$

$$\underset{\sim}{y}_{\alpha\beta} = \frac{\partial}{\partial v^\alpha} \underset{\sim}{y}_\alpha (v^1, v^2) = \frac{\partial}{\partial v^\alpha} \frac{\partial}{\partial v^\beta} \underset{\sim}{y}(v^\alpha, v^\beta)$$

$\underset{\sim}{x}_i$ are <u>vectors</u>: $\underset{\sim}{y}_\alpha = \underset{\sim}{x}_i \dfrac{\partial u^i}{\partial v^\alpha}$

$\underset{\sim}{x}_{ij}$ are <u>not</u> tensors.

- Let $\vec{n}(p)$ be a cont choice of unit normal so $(\vec{n}, \underset{\sim}{x}_1, \underset{\sim}{x}_2)$ is a basis @ P. (orientation)

Thus we can write $\underset{\sim}{x}_{ij}$, $\underset{\sim}{y}_{ij}$ as linear combs of $(\vec{n}, \underset{\sim}{x}_1, \underset{\sim}{x}_2)$ @ each p:

$$\underset{\sim}{x}_{ij} = L_{ij}\vec{n} + \Gamma^k_{ij}\,\underset{\sim}{x}_k$$

$$\underset{\sim}{y}_{\alpha\beta} = L_{\alpha\beta}\vec{n} + \overline{\Gamma}^{\sigma}_{\alpha\beta}\,\underset{\sim}{y}_\sigma$$

__Claim:__ L_{ij} transforms like a $\binom{0}{2}$-tensor:

i.e., $$\overline{L}_{\alpha\beta} = L_{ij}\frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^j}{\partial v^\beta}$$

Pf. First note: $L_{ij} = \underset{\sim}{x}_{ij}\cdot\vec{n}$

$$\overline{L}_{\alpha\beta} = \underset{\sim}{y}_{\alpha\beta}\cdot\vec{n}$$

Then:

$$\bar{L}_{\alpha\beta} = y_{\alpha\beta} \cdot \vec{n} = \left(\frac{\partial}{\partial v^\alpha} \underset{\sim}{y}_\beta\right) \cdot \vec{n}$$ ③

But $\frac{\partial}{\partial v^\alpha} = \frac{\partial u^i}{\partial v^\alpha}\frac{\partial}{\partial u^i}$, $\underset{\sim}{y}_\beta = \frac{\partial u^j}{\partial v^\beta}\underset{\sim}{x}_j$ $\Rightarrow$

$$\bar{L}_{\alpha\beta} = \frac{\partial u^i}{\partial v^\alpha}\frac{\partial}{\partial u^i}\left(\frac{\partial u^j}{\partial v^\beta}\underset{\sim}{x}_j\right) \cdot \vec{n}$$

$$= \left\{\frac{\partial u^i}{\partial v^\alpha}\frac{\partial}{\partial u^i}\left(\frac{\partial u^j}{\partial v^\beta}\right)\underset{\sim}{x}_j + \frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^j}{\partial v^\beta}\underset{\sim}{x}_{ij}\right\} \cdot \vec{n}$$

$=0$!

$$= \frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^j}{\partial v^\beta}\underbrace{\underset{\sim}{x}_{ij} \cdot \vec{n}}_{L_{ij}}$$

$$= \frac{\partial u^i}{\partial v^\alpha}\frac{\partial u^j}{\partial v^\beta} L_{ij}$$ as claimed $\boxed{G}$

We have: L_{ij} is a tensor

$L_{ij} = L_{ji} \Rightarrow$ symmetric.

Now recall: $L^i_j = g^{i\sigma} L_{\sigma j}$ is the $\binom{1}{1}$-tensor obtained by raising the index with the metric.

Said Differently: L^i_j is a transformation matrix symmetric wrt the inner product g_{ij}:

$$\langle L\mathbf{X}, \mathbf{Y}\rangle \overset{?}{=} \langle \mathbf{X}, L\mathbf{Y}\rangle$$

$$\left\langle L^i_k a^k \frac{\partial}{\partial u^i}, b^j \frac{\partial}{\partial u^j}\right\rangle = \left\langle a^i \frac{\partial}{\partial u^i}, L^j_\ell b^\ell \frac{\partial}{\partial u^j}\right\rangle$$

$$\underline{g_{ij} L^i_k} a^k b^j = g_{ij} a^i L^j_\ell b^\ell$$

$$L_{jk} a^k b^j = L_{i\ell} a^i b^\ell$$

Yes because $L_{ij} = L_{ji}$

- Symmetric Matrix Thm: if L^i_j is symmetric wrt a pos def. inner prod g, then L has real e-vals & o.n. basis of e-vectors. That is: $\exists$ N_1 & N_2

$$N_1 = a^i \underset{\sim}{x}_i \ , \ N_2 = b^j \underset{\sim}{x}_j \ , \ \langle N_1, N_2 \rangle = 0$$

$$\boxed{\begin{array}{l} L N_1 = K_1 N_1 \\ L^i_j a^j = K_1 a^i \end{array}} \quad \boxed{\begin{array}{l} L N_2 = K_2 N_2 \\ L^i_j b^j = K_2 b^i \end{array}} \quad K_1 \leqslant K_2$$

- Now recall: comp's L^i_j of a $\binom{1}{1}$-tensor form a transformation matrix $\underset{2\times 2}{L}$ which transforms by $\bar{L} = B^{-1} L B$

Evals satisfy: $|L - \lambda I| = 0 \Leftrightarrow \lambda^2 - (\mathrm{tr} L)\lambda + \det L = 0$

$$|\bar{L} - \lambda I| = |B^{-1} L B - \lambda B^{-1} I B| = |B^{-1}(L - \lambda I) B| = |L - \lambda B| = 0$$

$$\Leftrightarrow \lambda^2 - (\mathrm{tr}\bar{L})\lambda + \det L$$

Since both poly's are the same,

$$\operatorname{tr}\bar{L} = \operatorname{tr}L$$
$$\det\bar{L} = \det L$$

Conclude: the trace & determinant of a $\binom{1}{1}$-tensor are coord indept numbers!

- Another Way — In basis N_1, N_2 we know

$$\bar{L} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix},$$

so $$\operatorname{trace}\bar{L} = \kappa_1 + \kappa_2 = \operatorname{trace}(L)$$
$$\det\bar{L} = \kappa_1\kappa_2 = \det L$$

Defn: $\frac{1}{2}(\kappa_1+\kappa_2)$ = Mean Curvature (Extrinsic)

$K \equiv \kappa_1\kappa_2$ = Gaussian Curvature (Intrinsic)

- **Thm**: A surface that has $K_1+K_2=0$ everywhere $\equiv$ zero mean curvature minimizes the surface area with given boundary $\approx$ minimal surface $\equiv$ soap bubble. [Topic for another class]

(K_1+K_2 does not just depend on g but also on how M is embedded in $\mathbb{R}^3$ $\Rightarrow$ not intrinsic)

- Interpretation of L & K

- Since $L \equiv L^i_j \, du^j \otimes \frac{\partial}{\partial u^i}$ is a $\binom{1}{1}$-tensor it operates on vectors to give vectors:

Eg $X = a^i \frac{\partial}{\partial u^i} \Rightarrow L(X) = \left(L^i_j \, du^j \otimes \frac{\partial}{\partial u^i}\right)(X)$

$= L^i_j \, du^j(X) \frac{\partial}{\partial u^i}$

$= L^i_j a^j \frac{\partial}{\partial u^i} = Y$

Or in matrix form based on $\underset{\sim}{x}$-comp's

$$\boxed{\begin{aligned} &La = b \\ &L^i_j a^j = b^i \end{aligned}}$$

<u>Theorem</u>: $L(\underline{X})$ = minus the rate at which $\vec{n}$ changes in the $\underline{X}$ direction on M (The way the normal vector changes keeps track of curvature.) $L\underline{X} = -\underline{X}(\vec{n})$: $L: T_pM \to T_pM$

- $\vec{n} = \underset{\sim}{n}(u^1, u^2)$ is a coord system

(really: $\vec{n} = \vec{n}(P) = \vec{n}(\underset{\sim}{x}(u^1,u^2)) \equiv \underset{\sim}{n}(u^1,u^2) \in S^2$)

S^2 = unit sphere in $\mathbb{R}^3$, $\vec{n} \in S^2$.

- Thus $\underset{\sim}{n}_1 = \frac{\partial}{\partial u^1}\underset{\sim}{n}(u^1,u^2)$ & $\underset{\sim}{n}_2 = \frac{\partial}{\partial u^2}\underset{\sim}{n}(u^1,u^2)$ both make sense

- Since $\underset{\sim}{n}_i = \frac{\partial}{\partial u^i}\underset{\sim}{n}(u^1,u^2)$ holds ($u^{\hat{i}}$ fixed), $\|\underset{\sim}{n}\| \equiv 1$

the other one!

we must have $\underset{\sim}{n}_i \perp \vec{n}$ (if $\|\gamma(t)\| = \text{const}$, then $\dot{\gamma}(t) \cdot \gamma(t) = 0$)

Thus $\underset{\sim}{n}_i$ lie in span of $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \Rightarrow$

$\exists\, L^i_j$ such that $\boxed{\underset{\sim}{n}_j = L^i_j\, \underset{\sim}{x}_i}$

- Now we can show $\bar{L}_{ij} = -L_{ij}$

P.f. $\langle \underset{\sim}{n}, \underset{\sim}{x}_i \rangle = 0$ (Key starting identity!)

$$\Rightarrow \; 0 = \frac{\partial}{\partial u_j} \langle \underset{\sim}{n}, \underset{\sim}{x}_i \rangle = \langle \frac{\partial}{\partial u_j} \underset{\sim}{n}, \underset{\sim}{x}_i \rangle + \langle \underset{\sim}{n}, \frac{\partial}{\partial u_j} \underset{\sim}{x}_i \rangle$$

$$= \langle \underset{\sim}{n}_j, \underset{\sim}{x}_i \rangle + \langle \underset{\sim}{n}, \underset{\sim}{x}_{ji} \rangle$$

$$= \langle \bar{L}^k_j \underset{\sim}{x}_k, \underset{\sim}{x}_i \rangle + L_{ji}$$

$$= \bar{L}^k_j \underbrace{\langle \underset{\sim}{x}_k, x_i \rangle}_{g_{ki}} + L_{ji}$$

$$=$$

$$= \bar{L}_{ij} + L_{ji}$$

$$\boxed{\bar{L}_{ij} = -L_{ij}} \; ! \quad \Rightarrow \quad \boxed{\underset{\sim}{n}_i = -L^k_i \underset{\sim}{x}_k}$$

• Conclude: $\underset{\sim}{n}_j = \frac{\partial}{\partial u^j}\underset{\sim}{n} = -L^i_j\,\underset{\sim}{x}_i$

If $X = a^i\frac{\partial}{\partial u^i}$, then

$$\left(a^j\frac{\partial}{\partial u^j}\right)\underset{\sim}{n} = -\left(L^i_j a^j\right)\underset{\sim}{x}_i$$

$$X(\underset{\sim}{n}) = -L(X) \qquad \left(\underset{\sim}{x}_i \equiv \frac{\partial}{\partial u^i}\ !!\right)$$

$$\boxed{L(X) = -X(\underset{\sim}{n})}$$

← Weingarten map L

output $L(X)$

rate at which $\underset{\sim}{n}$ changes in X direction

In words: LX has geometrical meaning as the rate at which $\vec{n}$ changes in X direction — ($\|\vec{n}\| = 1$, so $X(\vec{n}) \in T_p\mathcal{M}$ ✓)

- <u>Corollary</u>: Assume $\|\underline{X}\| = 1$. Then ("curvature") $\|L\underline{X}\| = \|\underline{X}(\vec{n})\|$ is maximized & minimized in directions N_1 & N_2.

<u>Proof</u>: $\underline{X} = \alpha_1 N_1 + \alpha_2 N_2$

$$1 = \|\underline{X}\| = \|\alpha_1 N_1\|^2 + \|\alpha_2 N_2\|^2 = \alpha_1^2 + \alpha_2^2 \quad (N_i \text{ ON}$$

Thus

$$\|L\underline{X}\|^2 = \|L(\alpha_1 N_1 + \alpha_2 N_2)\|^2 = \|\alpha_1 K_1 N_1\|^2 + \|\alpha_2 K_2 N_2\|$$

$$= \alpha_1^2 K_1^2 + \alpha_2^2 K_2^2$$

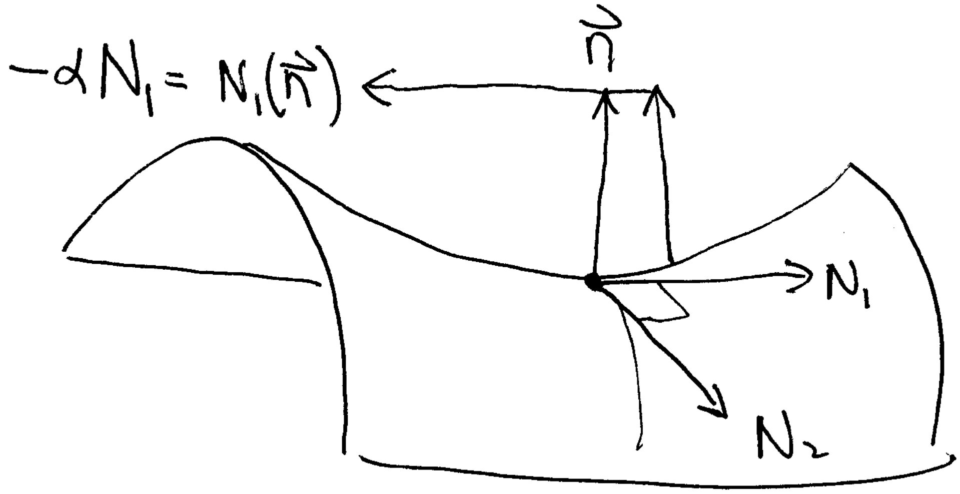
But $|K_1| \leq |K_2| \Rightarrow$

$$K_1^2 \leq \alpha_1^2 K_1^2 + \alpha_1^2 K_2^2 \leq \alpha_1^2 K_1^2 + \alpha_2^2 K_2^2 \leq \alpha_1^2 K_2^2 + \alpha_2^2 K_2^2 \leq K_2^2$$

$\Rightarrow \|L\underline{X}\| = \|\underline{X}(\vec{n})\|$ maximized/minimized at values $|K_2|$ & $|K_1|$. ✓

Picture: @ each pt $\exists$ two O.N. directions N_1 & N_2 in which $L_i N_i = K_i N_i$.

Eg:



$$N_1(\vec{n}) = \lim_{\varepsilon \to 0} \frac{\vec{n}(P+\varepsilon N_1) - \vec{n}(P)}{\varepsilon} = -\alpha N_1$$

$\alpha, \beta \geq 0$

$$N_2(\vec{n}) = \lim_{\varepsilon \to 0} \frac{\vec{n}(P+\varepsilon N_2) - \vec{n}(P)}{\varepsilon} = +\beta N_2$$

$-\underline{X}(\vec{n}) = L\underline{X}$, L has e-vals K_1 & K_2 $\Rightarrow$

$-N_1(\vec{n}) = LN_1 = K_1 N_1 \Rightarrow \alpha = K_1$

$-N_2(\vec{n}) = LN_2 = K_2 N_2 \Rightarrow \beta = -K_2$

Idea: convex up (rel to $\vec{n}$) = $K_i > 0$
convex dow (rel to $\vec{n}$) = $K_i < 0$

- Result: $K = k_1 k_2 < 0 \Rightarrow$ saddle shape
  convex up in direction N_i
  convex down in direction $N_{\bar{i}}$
  (signs of k_1 & k_2 change with normal, but <u>not</u> $k_1 k_2$)

$K = k_1 \cdot k_2 > 0 \Rightarrow$ convexity the same in all directions $\approx$ sphere.

⑧ Geometrical Interpretation of Gauss Curvature $\underline{K}$

We have:

$$\underset{\sim}{x}(u^1,u^2) = \left(x(u^1,u^2),\, y(u^1,u^2),\, z(u^1,u^2)\right)$$

$$\underset{\sim}{x}_1 = \frac{\partial \underset{\sim}{x}}{\partial u^1}, \quad \underset{\sim}{x}_2 = \frac{\partial \underset{\sim}{x}}{\partial u^2}, \quad g_{ij} = \langle \underset{\sim}{x}_i, \underset{\sim}{x}_j \rangle$$

$$\underset{\sim}{n}(u^1,u^2) = \vec{n}\left(\underset{\sim}{x}(u^1,u^2)\right)$$

$$\underset{\sim}{n}_1 = -L^k_1\, \underset{\sim}{x}_k, \quad \underset{\sim}{n}_2 = -L^\ell_2\, \underset{\sim}{x}_\ell$$

so
$$\hat{g}_{ij} = \langle \underset{\sim}{n}_i, \underset{\sim}{n}_j \rangle = \langle -L^k_i\, \underset{\sim}{x}_k, -L^\ell_j\, \underset{\sim}{x}_\ell \rangle$$
$$= L^k_i \langle \underset{\sim}{x}_k, \underset{\sim}{x}_\ell \rangle L^\ell_j = L^k_i\, g_{k\ell}\, L^\ell_j$$

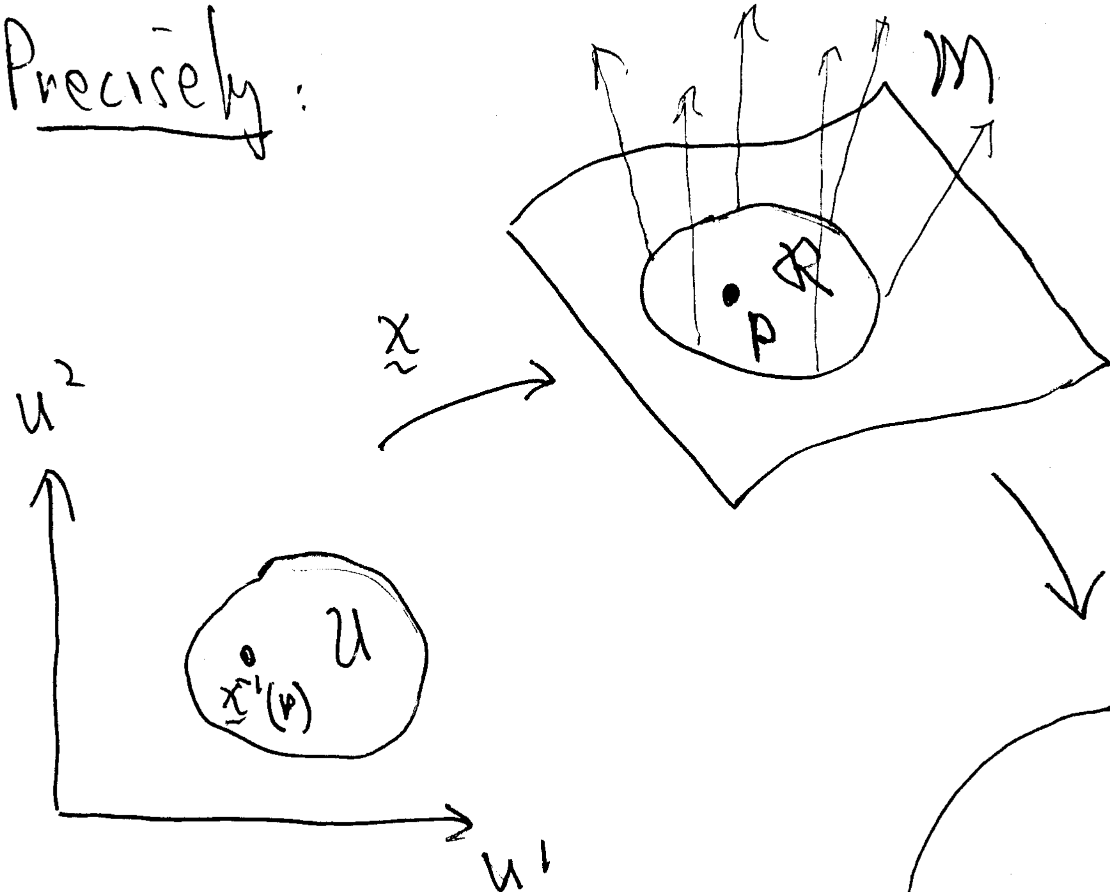
Thus:
$$\hat{g} = \det|g_{ij}| = (\det L)^2 \det|g_{ij}|$$
$$= \underline{K}^2 g$$

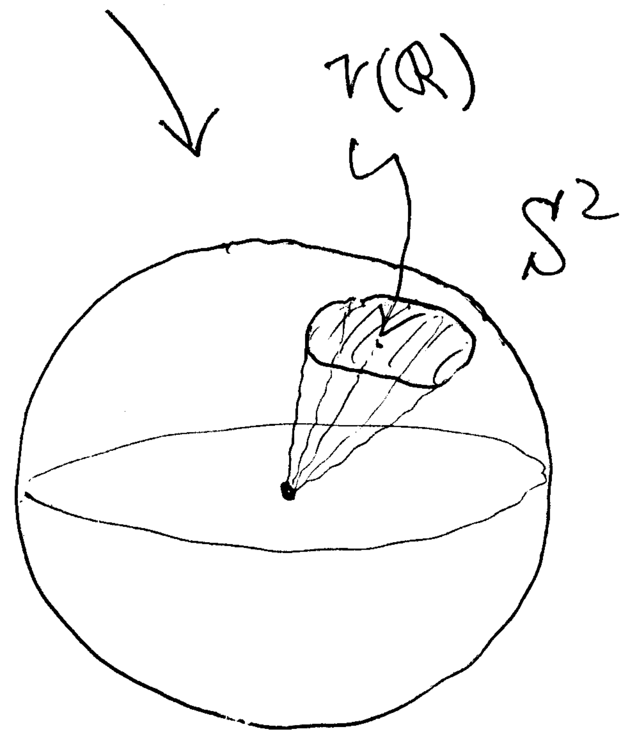
<u>Conclude</u>: The amplification factor for areas for $\underset{\sim}{x}$-map is $\sqrt{g}$, & the amp. factor for areas for $\underset{\sim}{n}$-map is $\sqrt{g}\,\mathbb{K}$ ($\mathbb{K}$ times larger).

Said Differently –

"$\mathbb{K}$ is the <u>ratio</u> of the area $A(\mathcal{V}(\mathcal{R}))$ of the region $\mathcal{V}(\mathcal{R})$ swept out on $\mathcal{S}^2$ by normals $\vec{n}$ on $\mathcal{R}$, to the area $A(\mathcal{R})$ on $\mathcal{M}$, <u>in the limit</u> $A(\mathcal{R}) \to 0$"

Precisely:





$$A(R) = \iint_R dS = \iint_U \sqrt{g}\, du^1 du^2$$

$$A(\nu(R)) = \iint_{\nu(R)} dS = \iint_U \sqrt{\hat{g}}\, du^1 du^2 = \iint_U \sqrt{g}\, K\, du^1 du^2$$

So

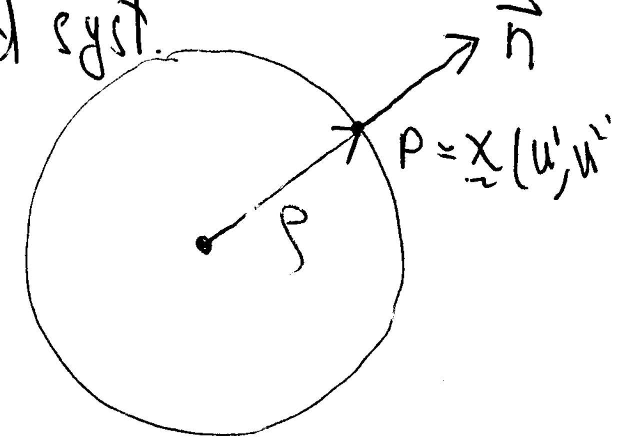
$$\lim_{A \to 0} \left\{ \frac{A(\nu(R))}{A(R)} = \frac{\iint_U \sqrt{g}\, K\, du^1 du^2}{\iint_U \sqrt{g}\, du^1 du^2} \right\} = \frac{A(U)\sqrt{g(p)}\, K(p)}{A(U)\sqrt{g(p)}} = K(p)$$

✓

~~• Result: $K = \kappa_1\kappa_2 < 0 \Rightarrow$ saddle shape: convex up in direction N_i, convex down in othe direction $N_{\bar{i}}$~~

~~• $K = \kappa_1\kappa_2 > 0 \Rightarrow$ convexity the same in all directions $\approx$ sphere~~

$\underline{\underline{Ex}}$ Let $M = S^2_\rho \equiv$ sphere of radius ρ, center $(0,0,0) \in \mathbb{R}^3$, $\underset{\sim}{x}$ a coord syst.



• $\underset{\sim}{x}(u^1, u^2) = (x(u^1,u^2), y(u^1,u^2), z(u^1,u^2))$

$$\underset{\sim}{n}(u^1,u^2) = \frac{1}{\rho}\underset{\sim}{x}(u^1,u^2)$$

$$\Rightarrow \underset{\sim}{x}_1 = \rho\underset{\sim}{n}_1, \quad \underset{\sim}{x}_2 = \rho\underset{\sim}{n}_2$$

• Choose $X_p \in T_pM$ $\quad X_p = a^i\frac{\partial}{\partial u^i} \equiv a^i\underset{\sim}{x}_i$

$$X_p(\vec{n}) = a^i\frac{\partial}{\partial u^i}\underset{\sim}{n}(u^1,u^2) = \frac{1}{\rho}a^i\frac{\partial}{\partial u^i}\underset{\sim}{x}(u^1,u^2) = \frac{1}{\rho}a^i\underset{\sim}{x}_i$$

$$= \frac{1}{\rho}X_p = -LX_p \Rightarrow L = -\frac{1}{\rho}\,\mathrm{Id} = K = |L| = \frac{1}{\rho}$$

(1

• Conclude: $\kappa_1 = \kappa_2 = -\frac{1}{\rho}$ ($\kappa_1 = \kappa_2 \Rightarrow P$ called umbilic point)

$$\underline{K} = \left(-\frac{1}{\rho}\right)\left(-\frac{1}{\rho}\right) = \frac{1}{\rho^2} > 0 \quad \checkmark$$

Thm: S^2_ρ has constant Gaussian curvature $K > 0$. (It's the only space of const $\underline{K} = \underline{K}_0 > 0$)

• Note: $L \to -L$, $\kappa_i \to -\kappa_i$ if we switch $\vec{n} \to -\vec{n}$.

• The hyperbolic plane = space of constant negative curvature

$$ds^2 = \frac{du^2 + dv^2}{(1-u^2-v^2)^2} \qquad 0 \le u^2, v^2 < 1$$

$$g_{ij} = \begin{bmatrix} \frac{1}{(1-u^2-v^2)^2} & 0 \\ 0 & \frac{1}{(1-u^2-v^2)^2} \end{bmatrix}$$

Problem - cannot be embedded in $\mathbb{R}^3 \Rightarrow$ can't get L from $\underline{X}(\vec{n}) = -L\underline{X} \Rightarrow$ need an intrinsic formula for K to compute it! (This metric meets Euclid 1-4 but not 5)

• **Theorem**: (Gauss's Thm Egregium)

① Gaussian Curvature is intrinsic —

$$K = R_{1221}$$

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial u^k} - \frac{\partial \Gamma^i_{jk}}{\partial u^l} + \Gamma^\sigma_{jl}\Gamma^i_{\sigma k} - \Gamma^\tau_{jk}\Gamma^i_{\tau k}$$

(curl) + (commutator)

$$\Gamma^i_{jh} = \tfrac{1}{2} g^{i\sigma}\{-g_{jh,i} + g_{ij,h} + \underbrace{g_{ki,j}}\}$$

$$\frac{\partial}{\partial u^j} g_{ki}(u^1,u^2) \text{ etc}$$

K can be computed from $g_{ij}(u^1,u^2)$ alone (intrinsic)

② If two surfaces are locally isometric, their Gaussian curvatures are equal.

<u>Defn</u>: An <u>isometry</u> $f: M \to N$ is a 1-1, onto, differentiable map such that the lengths of all curves are the same; i.e. $\gamma: [a,b] \to M \Rightarrow f\circ\gamma$ has same length as γ.

$$\int_a^b \|\dot{\gamma}(t)\|_M \, dt = \int_a^b \|\dot{\overline{f\circ\gamma}}(t)\|_N \, dt$$

$\forall$ curve.

③ Not every smooth positive definite metric determines a surface in $\mathbb{R}^3$: (Hyperbolic plane is one.) However, if:

ⓐ g_{ij} pos det & $L_{ij} = L_{ji}$ given fn's of (u^1, u^2) s.t.

ⓑ
$$R^{\ell}_{ijk} = L_{ik} L^{\ell}_{j} - L_{ij} L^{\ell}_{k} \quad \text{(Gauss Eqn)}$$

$$\frac{\partial L_{ij}}{\partial u^k} - \frac{\partial L_{ik}}{\partial u^j} = \Gamma^{\ell}_{ik} L_{\ell j} - \Gamma^{\ell}_{ij} L_{\ell k}$$

(Codazzi-Mainardi)

Then $\exists$ surface with given g_{ij}, L_{ij} & all such surfaces are isometric & in fact $\exists$ rigid motion that takes one to other.

Pf (omitted) Rather than go thru pf, time better spent learning about Riemann Curvature Tensor $R^{i}_{jk\ell}$ ...