VIII

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Geodesics:

- We have that L_{ij} & Γ_{ij}^{k} are defined by

$$\underset{\sim}{x}_{ij} = L_{ij}\,\vec{n} + \Gamma_{ij}^{k}\,\underset{\sim}{x}_k \qquad (1)$$

- Consider an arclength parameterized curve $\gamma(s)$ in M with $\underset{\sim}{x}$-coord representation

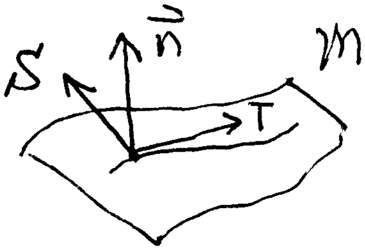
$$\gamma(s) = \underset{\sim}{x}\left(\gamma^1(s), \gamma^2(s)\right).$$

- $T(s) = \dot{\gamma}(s) \Rightarrow \|T\| = 1$ & $T(s) \in T_{\gamma(s)}M$

$$\ddot{\gamma}(s) = \dot{T}(s) = K(s)N(s), \quad \boxed{N \perp T}$$

- <u>Define</u>: $$K(s)N(s) = \underbrace{K_n(s)}_{\text{normal curvature}}\vec{n}(s) + \underbrace{K_g(s)}_{\text{geodesic curvature}} S(s) \qquad (2)$$

$$\boxed{S = \vec{n} \times T}$$



$$S(s) \in T_{\gamma(s)}M, \quad \boxed{S \perp T}$$

(K_n & K_g <u>not</u> related to Gaussian curvature)

- **Lemma 1**: $L_{ij} = \langle \underset{\sim}{x}_{ij}, \vec{n} \rangle$

Pf. dot both sides of (1) with $\vec{n}$

- **Lemma 2**: $\Gamma_{ij}^{k} = g^{k\ell} \langle \underset{\sim}{x}_{ij}, \underset{\sim}{x}_{\ell} \rangle$

Pf. $\langle \underset{\sim}{x}_{ij}, \underset{\sim}{x}_{\ell} \rangle = \langle \Gamma_{ij}^{\sigma} \underset{\sim}{x}_{\sigma}, \underset{\sim}{x}_{\ell} \rangle$

$$= \Gamma_{ij}^{\sigma} \langle \underset{\sim}{x}_{\sigma}, \underset{\sim}{x}_{\ell} \rangle$$

$$= \Gamma_{ij}^{\sigma} g_{\sigma\ell}$$

$$\therefore \; g^{k\ell} \langle \underset{\sim}{x}_{ij}, \underset{\sim}{x}_{\ell} \rangle = \Gamma_{ij}^{\sigma} \underbrace{g_{\sigma\ell} g^{k\ell}}_{\delta_{\sigma}^{k} \,\equiv\, k=\sigma} = \Gamma_{ij}^{k}$$

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<u>Note</u>: you could take L1 & L2 as the defn of L_{ij} & Γ_{ij}^{k} as book does...

• **Lemma 3** : $K_n = L_{ij}\,\dot{\gamma}^i\,\dot{\gamma}^j$

$$K_g \underset{\sim}{S} = \left[\ddot{\gamma}^k + \Gamma^k_{ij}\,\dot{\gamma}^i\,\dot{\gamma}^j\right] \underset{\sim}{x}_k$$

Proof : $\ddot{\gamma}(s) = K_n(s)\,\vec{n}(s) + K_g(s)\,\underset{\sim}{S}(s)$

But

$$\gamma(s) = \underset{\sim}{x}(\gamma^1(s), \gamma^2(s))$$

$$\dot{\gamma}(s) = \underset{\sim}{x}_1\,\dot{\gamma}^1 + \underset{\sim}{x}_2\,\dot{\gamma}^2 = \dot{\gamma}^i(s)\,\underset{\sim}{x}_i(s)$$

$$\ddot{\gamma}(s) = \ddot{\gamma}^i(s)\,\underset{\sim}{x}_i(s) + \dot{\gamma}^i(s)\,\dot{\underset{\sim}{x}}_i(s)$$

$$\dot{\underset{\sim}{x}}_i(s) = \frac{d}{ds}\underset{\sim}{x}_i(\gamma^1(s), \gamma^2(s))$$

$$= \underset{\sim}{x}_{i1}\,\dot{\gamma}^1(s) + \underset{\sim}{x}_{i2}\,\dot{\gamma}^2(s) = \underset{\sim}{x}_{ij}\,\dot{\gamma}^j(s)$$

$$\therefore\ \ddot{\gamma}(s) = \ddot{\gamma}^k(s)\,\underset{\sim}{x}_k(s) + \dot{\gamma}^i(s)\,\underset{\sim}{x}_{ij}\,\dot{\gamma}^j(s)$$

$$= \ddot{\gamma}^k(s)\,\underset{\sim}{x}_k(s) + (L_{ij}\,\vec{n} + \Gamma^k_{ij}$$

$$= L_{ij}\,\dot{\gamma}^i\,\dot{\gamma}^j\,\vec{n} + \left[\ddot{\gamma}^k + \Gamma^k_{ij}\,\dot{\gamma}^i\,\dot{\gamma}^j\right] \underset{\sim}{x}_k$$

So conclude:

$$\ddot{\gamma}(s) = \underbrace{L_{ij}\dot{\gamma}^i\dot{\gamma}^j \vec{n}}_{K_n} + \underbrace{\left[\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j\right] \underset{\sim}{X}_k}_{K_g S}$$

<u>Defn</u>: A geodesic is a unit speed curve along which $K_g = 0$ $\Rightarrow$ geodesics satisfy

$$\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$$

(a 2nd order ODE for $\gamma(s)$, unique locally given i-conds $\gamma(0)$, $\dot{\gamma}(0)$.)

- Intuition: "an ant stuck in the surface feels no tangential ≡ surface acceleration" (thinking of acceleration $\propto K_g$)

__Theorem__: $\Gamma^k_{ij} = \frac{1}{2} g^{kl} \{ -g_{ij,l} + g_{li,j} + g_{jl,i} \}$

$g_{ij,k} = \frac{\partial}{\partial u^k} g_{ij}(u^1, u^2)$, etc.

This says geodesics are __intrinsic__.

__Proof__: (Gauss)

$$g_{ij}(u^1,u^2) = \langle \underset{\sim}{x}_i(u^1,u^2), \underset{\sim}{x}_j(u^1,u^2) \rangle$$

$$\frac{\partial}{\partial u^k} g_{ij} = g_{ij,k} = \langle \underset{\sim}{x}_{ik}, \underset{\sim}{x}_j \rangle + \langle \underset{\sim}{x}_i, \underset{\sim}{x}_{jk} \rangle$$

or $$-g_{ij,k} = -\cancel{\langle \underset{\sim}{x}_{ik}, \underset{\sim}{x}_j \rangle} \mp \cancel{\langle \underset{\sim}{x}_{jk}, \underset{\sim}{x}_i \rangle}$$

Cyclicly permute: $$g_{ki,j} = \cancel{\langle \underset{\sim}{x}_{kj}, \underset{\sim}{x}_i \rangle} + \langle \underset{\sim}{x}_{ij,k} \rangle$$

$$g_{jk,i} = \langle \underset{\sim}{x}_{ji}, \underset{\sim}{x}_k \rangle + \cancel{\langle \underset{\sim}{x}_{ki,j} \rangle}$$

$$\Rightarrow -g_{ij,k} + g_{ki,j} + g_{jk,i} = 2 \langle \underset{\sim}{x}_{ij,k} \rangle$$

$$\therefore \Gamma^k_{ij} = g^{kl} \langle \underset{\sim}{x}_{ij,l} \rangle = \frac{1}{2} g^{kl} \{ -g_{ij,l} + g_{li,j} + g_{jl,i} \}$$