(Add #8.8)

IX

1

Parallel Translation:

We first describe ∥-translation along geodesics

Defn: $X = X^i \underset{\sim}{x}_i$ is a vector field on $\mathcal{M}$ if $X^i = X^i(u^1, u^2)$ is a smooth function.

Defn: a Vector field is parallel along a geodesic $\gamma(s)$ if the length of X is constant along $\gamma(s)$, and $X(\gamma^1(s), \gamma^2(s))$ maintains a constant angle with the tangent vector T.

Conclude ① $T = \dot{\gamma}(s)$ is parallel along geodesic $\gamma(s)$.

- Let $\dot{X} = \frac{d}{ds} X(s) = \frac{d}{ds}\left[X^i(s) \underset{\sim}{x}_i(s)\right]$ viewed as vector deriv in $\mathbb{R}^3$

Lemma ①: X is parallel along geodesic $\gamma(s)$ iff $\dot{X}$ is parallel to $\vec{n}$ (no tangengential change in X along γ).

Proof: If $X = T$ along $\gamma(s)$, then clearly $\dot{X} = \dot{T} = KN = K_n\vec{n}$ (γ is geodesic) so then $\dot{X}$ is parallel to $\vec{n}$. Assume $X \neq T$. Then $X \parallel$ along $\gamma \Rightarrow$

(1) $0 = \frac{d}{ds}\langle X, T\rangle = \langle \dot{X}, T\rangle + \cancel{\langle X, \dot{T}\rangle}$

$\Rightarrow \dot{X} \perp T$ along γ

(2) $0 = \frac{d}{ds}\langle X, X\rangle = 2\langle \dot{X}, X\rangle$

$\Rightarrow \dot{X} \perp X$ along γ

$\therefore$ $\dot{X}$ is parallel to $\vec{n}$.

Conversely: If $\dot{X}$ is parallel to $\vec{n}$, then

(1) $0 = 2\langle \dot{X}, X\rangle = \frac{d}{ds}\langle X, X\rangle = 0$

$\Rightarrow$ length of X is constant

(2) $0 = \langle \dot{X}, T\rangle + \langle X, \dot{T}\rangle = \frac{d}{ds}\langle X, T\rangle$

$\Rightarrow \langle X, T\rangle = \|X\|\|T\|\cos\theta = \cos\theta = \text{const}$

$\Rightarrow$ X maintains a const angle with $\dot{\gamma} = T$.

Defn: X is $\parallel$ along an arbitrary curve $\gamma(s)$ iff $\dot{X}$ is parallel to $\underset{\sim}{n}$.

Note: $X = X^k \underset{\sim}{x}_k$ so along $\gamma(s)$, $X = X^k(s)\,\underset{\sim}{x}(s)$
$\equiv X(\gamma(s))$.

Let $\dot{X} = \frac{d}{ds}\left(X^k \underset{\sim}{x}_k\right) = \dot{X}^k \underset{\sim}{x}_k + X^k \underset{\sim}{x}_{k\ell}\,\dot{\gamma}^\ell$

where $\dot{X}^k \equiv \frac{d}{ds} X^k(s)$ (derivative of the component!)

Theorem: X is parallel along ~~geodesic~~ curve γ iff

$$\frac{dX^k}{dt} + \Gamma^k_{ij} \dot{\gamma}^i X^j = 0,$$

$$X = X^k \underset{\sim}{x}_k$$

Proof: X is $\parallel$ along $\gamma(s)$ iff

$$0 = \langle \dot{X}, \underset{\sim}{x}_i \rangle = \langle \dot{X}^j \underset{\sim}{x}_j, \underset{\sim}{x}_i \rangle$$

$$+ \langle X^j \dot{\underset{\sim}{x}}_j, \underset{\sim}{x}_i \rangle$$

$$= \dot{X}^j g_{ji} + X^j \langle \underset{\sim}{x}_{jl} \dot{\gamma}^l, \underset{\sim}{x}_i \rangle$$

$$= \dot{X}^j g_{ji} + X^j \dot{\gamma}^l \langle \underset{\sim}{x}_{jl}, \underset{\sim}{x}_i \rangle$$

But: $\underset{\sim}{x}_{jl} = L_{jl} \underset{\sim}{n} + \Gamma^k_{jl} \underset{\sim}{x}_k$

$$\Rightarrow \langle \underset{\sim}{x}_{jl}, \underset{\sim}{x}_i \rangle = \Gamma^k_{jl} \langle \underset{\sim}{x}_k, \underset{\sim}{x}_i \rangle = \Gamma^k_{jl} g_{ki}$$

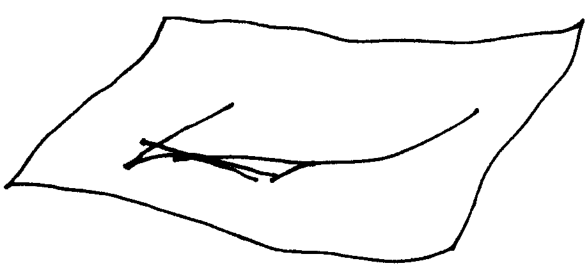
$$\Rightarrow \quad 0 = \dot{X}^j g_{ji} + X^j \dot{\gamma}^l \Gamma^k_{jl} g_{ki}$$

$$0 = \left(\dot{X}^k + X^j \dot{\gamma}^l \Gamma^k_{jl}\right) g_{ki}$$

$$\Rightarrow \quad \boxed{\dot{X}^k + \Gamma^k_{jl} X^j \dot{\gamma}^l = 0}$$

"geodesic equation"

<u>Note</u>: $\parallel$ translation along an arbitrary curve does <u>not</u> ~~in general~~ preserve the ~~the~~ $\angle$ betw T and X.



"approx by geodesics & $\parallel$ translate."

Eg. flat space

Proposition ① Let $\gamma(s)$ be an arbitrary curve, and assume X & Y are parallel along γ. Then

① Length of X & Y are const.

② angle betw X and Y is const.

check: $\frac{d}{ds}\langle X, X\rangle = 0$ ✓ ($\dot{X} \parallel \vec{n} \Rightarrow \langle \dot{X}, X\rangle = 0 = \langle \dot{X}, Y\rangle$)

$\frac{d}{ds}\langle X, Y\rangle = 0$ ✓

Proposition ② $\gamma(s)$ is a geodesic iff T is parallel along γ.

Pf: $T = \dot{\gamma}$ $\parallel$ along $\gamma(s)$ $\Leftrightarrow$ $X = \dot{\gamma}^i \underset{\sim}{x}_i$ is $\parallel$ along γ

$$\ddot{\gamma}^k + \Gamma^k_{j\ell}\,\dot{\gamma}^j \dot{\gamma}^\ell = 0$$

the geodesic equation!

<u>Pf</u> <u>of</u> <u>Prop</u> ① : (direct –)

$$\frac{d}{ds}\langle X, Y\rangle = \frac{d}{ds}\langle X^k \underset{\sim}{x}_k, Y^i \underset{\sim}{x}_i\rangle$$

$$= \underbrace{\langle \dot{X}^k \underset{\sim}{x}_k + X^k \dot{\underset{\sim}{x}}_k, Y^i \underset{\sim}{x}_i\rangle}_{\mathrm{I}} + \underbrace{\langle X^k \underset{\sim}{x}_k, \dot{Y}^i \underset{\sim}{x}_i + Y^i \dot{\underset{\sim}{x}}_i\rangle}_{\mathrm{II}}$$

$$\mathrm{I} = \langle \dot{X}^k \underset{\sim}{x}_k + X^{k} \underset{\sim}{x}_{kl}\dot{\gamma}^l, Y^i \underset{\sim}{x}_i\rangle$$

$$= Y^i \dot{X}^k \underbrace{\langle \underset{\sim}{x}_k, \underset{\sim}{x}_i\rangle}_{g_{ki}} + X^j \dot{\gamma}^l Y^i \underbrace{\langle \underset{\sim}{x}_{kl}, \underset{\sim}{x}_i\rangle}_{\Gamma^k_{jl}\, g_{ki}}$$

$$= Y^i g_{ki}\left\{ \dot{X}^k + \Gamma^k_{jl} X^j \dot{\gamma}^l \right\} = 0$$

$\underline{\mathrm{II}} = 0$ (interchange X & Y!)

Covariant Derivative. Given vector field X and curve $\gamma(s)$, $\dot{\gamma}(0) = Y$. Then X is $\parallel$ along γ iff

$$\dot{X}^k + \Gamma^k_{j\ell} X^j \dot{\gamma}^\ell = 0.$$

Thus: $\dot{X}^k + \Gamma^k_{j\ell} X^j Y^\ell$ measures the rate at which X deviates from the vector $X_\parallel$ which is $\parallel$ along γ.

Defn:

$$\nabla_Y X = \dot{X}^k + \Gamma^k_{j\ell} X^j Y^\ell$$

"Covariant Derivative of X in direction Y"

$\equiv$ "rate at which X diverges from $X_\parallel$ in direction Y"

<u>Theorem</u>: $\nabla_y \underline{X}$ transforms like a $\binom{0}{1}$-tensor, ie like a vector @ each point $p \in M$

Pf $\quad X^k = \frac{\partial x^k}{\partial y^\gamma} X^\gamma \quad . \quad y^l = \frac{\partial x^l}{\partial y^\beta} y^\beta \quad$ so

$$\nabla_y X = \frac{d}{ds}\left(\frac{\partial x^k}{\partial y^\gamma} X^\gamma\right) + \Gamma^k_{jl}\left(\frac{\partial x^j}{\partial y^\alpha} X^\alpha\right)\left(\frac{\partial x^l}{\partial y^\beta} y^\beta\right)$$

$$= \underbrace{\frac{\partial}{\partial y^\delta}\left(\frac{\partial x^k}{\partial y^\gamma}\right) X^\gamma y^\delta}_{\mathrm{I}} + \underbrace{\frac{\partial x^k}{\partial y^\gamma} \dot{X}^\gamma + \Gamma^k_{jl}(\;)(\;)}_{\mathrm{II}}$$

$$\mathrm{II} = \frac{\partial x^k}{\partial y^\gamma}\left\{ \dot{X}^\gamma + \frac{\partial y^\gamma}{\partial x^m} \Gamma^m_{jl} \frac{\partial x^j}{\partial y^\alpha} \frac{\partial x^l}{\partial y^\beta} X^\alpha y^\beta \right\}$$

$$\mathrm{I} = \frac{\partial x^k}{\partial y^\gamma}\left\{ \frac{\partial y^\gamma}{\partial x^j} \frac{\partial}{\partial y^\beta}\left(\frac{\partial x^j}{\partial y^\alpha}\right) X^\alpha y^\beta \right\}$$

which gives

$$\nabla_Y X = \frac{\partial x^k}{\partial y^\gamma} \left\{ \dot{X}^\gamma + \bar{\Gamma}^\gamma_{\alpha\beta} X^\alpha Y^\beta \right\}$$

with

$$\bar{\Gamma}^\gamma_{\alpha\beta} = \underbrace{\frac{\partial y^\gamma}{\partial x^k} \Gamma^k_{j\ell} \frac{\partial x^j}{\partial y^\alpha} \frac{\partial x^\ell}{\partial y^\beta}}_{\binom{1}{2}\text{-tensor}} + \underbrace{\frac{\partial y^\gamma}{\partial x^j} \frac{\partial}{\partial y^\beta} \frac{\partial x^j}{\partial y^\alpha}}_{\text{2nd deriv correction}}$$

<u>Lemma</u>: this is <u>exactly</u> how Γ transforms!

(Omit)

☒ Geodesics as curves that minimize length on $\mathcal{M}$.

<u>Thm</u>: a curve $\gamma(s)$ that minimizes the distance between two points on $\mathcal{M}$ satisfies

$$\ddot{\gamma}^{l} + \Gamma^{l}_{ij}\,\dot{\gamma}^{i}\dot{\gamma}^{j} = 0$$

$$\Gamma^{l}_{ij} = \frac{1}{2} g^{l\sigma}\left\{ g_{\sigma i,j} + g_{\sigma j,i} - g_{ij,\sigma} \right\}$$

<u>Theorem</u> (Euler-Lagrange) Let $L(\dot{q}, q)$ be a function on $\mathbb{R}^n$, $q = (q^1, \dots, q^n)$. Let $q(t) = (q_1(t), \dots, q_n(t))$ be a curve st $q(t_1) = q_1$, $q(t_2) = q_2$ fixed. Then if $q(t)$ minimizes

$$\int_{t_1}^{t_2} L(\dot{q}(t), q(t))\,dt$$

among all curves taking q_1 to q_2, then $q(t)$ satisfies the Euler Lagrange equations:

(E-L) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \qquad i = 1, \dots, n.$$

Ref: Spivak Vol I
pg 434

<u>Cor</u>: Let $\gamma(t) = \underset{\sim}{x}(\gamma^1(t), \gamma^2(t))$ be a curve on M that minimizes the distance from $P_1 = \underset{\sim}{x}(\underset{\sim}{u}_1)$ to $P_2 = \underset{\sim}{x}(\underset{\sim}{u}_2)$.
Then $(\gamma^1(t), \gamma^2(t)) = \gamma(t)$ satisfies

$$\ddot{\gamma}^\ell + \Gamma^\ell_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$$

Pf. : A curve that minimizes the distance from P_1 to P_2, minimizes the length

$$L_a^b = \int_{t_a}^{t_b} \sqrt{g_{ij}(\gamma^1, \gamma^2)\, \dot{\gamma}^i \dot{\gamma}^j}\, dt.$$

It turns out that minimizers of L_a^b are exactly the minimizers of

$$E_a^b = \frac{1}{2}\int_{t_a}^{t_b} g_{ij}(\gamma^1, \gamma^2)\, \dot{\gamma}^i \dot{\gamma}^j\, dt.$$

(Spivak, Vol I p 440)

We show minimizers of E_a^b satisfy the geodesic equation. Here, $q = (\gamma^1, \gamma^2) \in \mathbb{R}^2$, $\dot{q} = (\dot{\gamma}^1, \dot{\gamma}^2) \in \mathbb{R}^2$

$$L(q,\dot q) = \frac{1}{2} g_{ij}\, \dot\gamma^i \dot\gamma^j \qquad q^i = \gamma^i ,\ \dot q^i = \dot\gamma^i$$

$$\frac{\partial L}{\partial q^l} = \frac{1}{2} g_{ij,l}\, \dot\gamma^i \dot\gamma^j$$

$$\frac{\partial L}{\partial \dot q^l} = g_{il}\, \dot\gamma^i$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^l} = g_{il,j}\, \dot\gamma^j \dot\gamma^i + g_{il}\, \ddot\gamma^i$$

(E-L) $$g_{il}\, \ddot\gamma^i + g_{il,j}\, \dot\gamma^j \dot\gamma^i - \frac{1}{2} g_{ij,l}\, \dot\gamma^i \dot\gamma^j = 0$$

$$\Leftrightarrow\ g_{il}\, \ddot\gamma^i + \left\{ \frac{1}{2} g_{il,j}\, \dot\gamma^i \dot\gamma^j + \frac{1}{2} g_{jl,i}\, \dot\gamma^i \dot\gamma^j - \frac{1}{2} g_{ij,l}\, \dot\gamma^i \dot\gamma^j \right\}:$$

mutl by g^{kl} & sum on l:

$$\ddot\gamma^k + \frac{1}{2} g^{kl} \left\{ g_{il,j} + g_{jl,i} - g_{ij,l} \right\} \dot\gamma^i \dot\gamma^j = 0$$

$$\ddot\gamma^k + \Gamma^k_{ij}\, \dot\gamma^i \dot\gamma^j = 0$$

Example: Let M be the 2-d plane

$$ax + by + cz = d.$$

Consider the Monge patch $x = u^1$, $y = u^2$

$$\underset{\sim}{x}(u^1, u^2) = \left(u^1, u^2, \frac{d - au^1 - bu^2}{c}\right)$$

$$\underset{\sim}{x}_1 = \left(1, 0, -\frac{a}{c}\right)$$

$$\underset{\sim}{x}_2 = \left(0, 1, -\frac{b}{c}\right)$$

$$g_{ij} = \langle \underset{\sim}{x}_i, \underset{\sim}{x}_j \rangle = \begin{bmatrix} 1 + \left(\frac{a}{c}\right)^2 & \frac{ab}{c^2} \\ \frac{ab}{c^2} & 1 + \left(\frac{b}{c}\right)^2 \end{bmatrix}_{ij}$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{\sigma k} \{ g_{i\sigma,j} + g_{j\sigma,i} - g_{ij,\sigma} \}$$

$$\equiv 0$$

$\Rightarrow$ geodesics: $\ddot{\gamma}^k = 0 \Rightarrow \gamma^1(s) = As + C_1$

$$\gamma^2(s) = Bs + C_2$$

$(\gamma^1(s), \gamma^2(s)) = (A, B)s + (C_1, C_2)$ st line

$$\gamma(s) = \underset{\sim}{x}(\gamma^1(s), \gamma^2(s))$$

$$= \left(As + C_1, Bs + C_2, \frac{d - a(As + C_1) - b(Bs + C_2)}{c}\right)$$

$$= \left(A, B, -\frac{aA - bB}{c}\right)s + \left(C_1, C_2, \frac{d - aC_1 - bC_2}{c}\right)$$

a straight line ✓ (shortest dist betw two pts in plane is st. line.)