FINAL EXAM
Math 116
Temple

#1.(36pts) Let $\alpha(s)$ be a regular curve in $\mathbf{R}^3$, parameterized by arclength.

i) Give the definitions of $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}(s)$, $\kappa(s)$, and $\tau(s)$.

ii) Give the 3x3 matrix $\mathbf{A}$ such that

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

iii) Define the osculating plane, normal plane and rectifying plane at a point P on the curve α.

#2. (12pts) Let $\alpha(t)$ denote a regular curve in $\mathbf{R}^3$, and assume that $|\alpha'(t)|$ = const. independent of t. Show that $<\alpha',\alpha''>=0$, where $< , >$ denotes the dot product on $\mathbf{R}^3$.

#3. (25pts) Let $\alpha(t)=(\cos(\pi t),\sin(\pi t),t))$. Find explicit formulas for $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$, $\kappa(t)$, and $\tau(t)$.

#4. (16pts) Let $\alpha(s)$ be a unit speed curve in $\mathbf{R}^3$. Find a vector $\mathbf{w}(s)$ such that $\mathbf{T}'=\mathbf{w}X\mathbf{T}$, $\mathbf{N}'=\mathbf{w}X\mathbf{N}$, and $\mathbf{B}'=\mathbf{w}X\mathbf{B}$. The vector $\mathbf{w}$ is called the Darboux vector.

#5. Let **x** and **y** be two different coordinate systems on a surface ***M*** ,

$$\mathbf{x}: U \to \mathbf{R}^3 ,\ \mathbf{x}(u^1,u^2) \varepsilon M ;$$

$$\mathbf{y}: V \to \mathbf{R}^3 ,\ \mathbf{y}(v^1,v^2) \varepsilon M .$$

A) (16pts) Let $J^i_\alpha = \frac{\partial u^i}{\partial v^\alpha}$ denote the Jacobian of the transformation that takes (v^1,v^2) to (u^1,u^2), and let $J^\alpha_i = \frac{\partial v^\alpha}{\partial u^i}$ denote its inverse. Let $X = X^i \mathbf{x}_i = X^1 \mathbf{x}_1 + X^2 \mathbf{x}_2$ be a vector field on ***M*** , and let $g_{ij} = <\mathbf{x}_i, \mathbf{x}_j>$ denote the metric and L^i_j denote the second fundamental form on ***M***. Write the coordinate transformation laws for X^i , $\mathbf{x}_i$, g_{ij} and L^i_j in terms of $\frac{\partial u^i}{\partial v^\alpha}$ and $\frac{\partial v^\alpha}{\partial u^i}$ using the summation convention.

B) (48pts) Assume that

$$\mathbf{x}(u^1,u^2) = (\sin(u^1u^2),\ \cos(u^1u^2), \ln(u^1),$$

and

$$\mathbf{y}^{-1} \circ \mathbf{x} = (v^1(u^1,u^2), v^2(u^1,u^2)) = (u^2\cos(u^1), e^{u^1u^2}).$$

i) Find $\mathbf{x}_1$ and $\mathbf{x}_2$ in terms of (u^1,u^2).

ii) What vector $X^i\mathbf{x}_i = X^1\mathbf{x}_1 + X^2\mathbf{x}_2$ in ***TM*** corresponds to the vector $3\frac{\partial}{\partial u^1} + 2\frac{\partial}{\partial u^2}$ at the point $(u^1,u^2) = (\pi,1)$ in **u**-space.

iii) What vector in the **y**-coordinates corresponds to the vector $3\frac{\partial}{\partial u^1} + 2\frac{\partial}{\partial u^2}$ at the point $(u^1,u^2) = (\pi,1)$ in **u**-space.

iv) If

$$g_{ij}=\langle \mathbf{x}_i,\mathbf{x}_j\rangle=\begin{bmatrix}1&0\\0&1\end{bmatrix}_{ij},$$

at the point $(u^1,u^2)=(\pi,1)$ in **u**-coordinates, find the metric entries at the corresponding point in **y**-coordinates.

#6. (15pts) Let **X** and **Y** be vector fields on ***M*** that are parallel along a curve γ in ***M***. Show that the angle between **X** and **Y** is constant along γ.

#7. Recall that γ is a geodesic on M if

$$(*)\qquad (\gamma'')^k+\Gamma^k{}_{ij}(\gamma')^i(\gamma')^j=0,$$

all along γ, where

$$\Gamma^k{}_{ij}=\frac{1}{2}g^{k\sigma}\{\frac{\partial}{\partial u^j}g_{i\sigma}+\frac{\partial}{\partial u^i}g_{\sigma j}-\frac{\partial}{\partial u^\sigma}g_{ij}\}.$$

(We assume the summation convention.)

i) (12pts) Show that if $\gamma(s)$ solves (*), then so does $\gamma(cs)$, where c is any real constant.

ii) (20pts) If $g_{ij}=\langle \mathbf{x}_i,\mathbf{x}_j\rangle=\begin{bmatrix}u^1&0\\0&1\end{bmatrix}_{ij}$, find $\Gamma^k{}_{ij}$ for all i,j,k=1,2, and write the equations (*) in this case.