PRACTICE
MIDTERM EXAM
Math 116

5pt (#1) Let $\alpha(t) = \cos t\, \underset{\sim}{i} + \sin t\, \underset{\sim}{j} + t k$

(i) Find $s = g(t)$ where s denotes arclength along α measured from $t=0$.

(ii) Find $T(s)$, $N(s)$, and $B(s)$

(iii) Find $\kappa(s)$, $\tau(s)$.

(iv) Find the parametric equation of the tangent line at $t = \frac{\pi}{2}$

(v) Find the equation of the osculating plane at $t = \frac{\pi}{2}$

(#2) Recall the Frenet-Serret equations:

5pt) $T' = \kappa N$, $N' = -\kappa T + \tau B$, $B' = -\tau N$. Show that $N'' \cdot B = \tau'$

#3 Let $L\vec{v} = \overrightarrow{(2,1,-3)} \times \vec{v}$ define a linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{v} = \overrightarrow{(a^1, a^2, a^3)}$.

(5pt) Find the 3×3 matrix that represents L.

#4 Assume: $\alpha'(t) = -\nabla U(\alpha(t))$, where

(15pt)
$$\underset{\sim}{x} = \alpha(t) = (x(t), y(t), z(t))$$
$$U \equiv U(x, y, z).$$

Prove: $E(t) = \frac{1}{2}|\alpha'(t)|^2 + U(\alpha(t))$ is constant all along $\alpha(t)$.

#5 Let $\alpha_1(t) = (x_1(t), y_1(t))$ and $\alpha_2(t) = (x_2(t), y_2(t))$

Solve

(15pt)
$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & f(t) \\ -f(t) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

where $f(t)$ is a smooth non-zero function, and assume $\alpha_1(0) = \alpha_2(0)$. Prove <u>directly</u> that $\alpha_1(t) = \alpha_2(t)$ for all t. (Do <u>not</u> use the theory of ODE's.)

#6 Assume: $[\bar{e}_1, \bar{e}_2] = [e_1, e_2] \begin{bmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{bmatrix}$

15pt) where A^i_α are the entries of matrix A.

(i) Let a^i, g_{ij}, T^i_j be the $\{e_1, e_2\}$-components of a vector $\vec{v}$, an inner product g and a transformation T, respectively. Write a formula for $\bar{a}^\alpha$, $\bar{g}_{\alpha\beta}$, $\bar{T}^\alpha_\beta$, the corresponding components wrt $\{\bar{e}_1, \bar{e}_2\}$, using the summation convention, and translate these into matrix notation.

(ii) Using the summation convention, what is the transformation law for the components T^i_{jk} of a $\binom{1}{2}$-tensor?

Math 116 Midterm Solutions

#1 $\alpha'(t) = -\sin t \, \underset{\sim}{i} + \cos t \, \underset{\sim}{j} + k$

$$|\alpha'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} = \frac{ds}{dt}$$

(i) $ds = \sqrt{2}\, dt \Rightarrow s = \sqrt{2}\, t$

(ii) $$T(s) = \frac{\alpha(t(s))}{|\alpha(t(s))|} = \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right)$$

$$T'(s) = KN = \left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$= \frac{1}{2}\left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$N(s) = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

$$B(s) = T(s) \times N(s) = \frac{1}{\sqrt{2}} \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -\sin t & \cos t & 1 \\ -\cos t, & -\sin t & , 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}\, \underset{\sim}{i} - \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\, \underset{\sim}{j} + \frac{1}{\sqrt{2}}\, \underset{\sim}{k}$$

(iii) $K(s) = |T'(s)| = \frac{1}{2}$

$$B'(s) = -\tau(s)\, N$$

$$\Downarrow \quad \frac{1}{2}\cos\frac{s}{\sqrt{2}}\, \underset{\sim}{i} + \frac{1}{2}\sin\frac{s}{\sqrt{2}}\, \underset{\sim}{j} = -\frac{1}{2} N$$

$$\Rightarrow \tau(s) = \frac{1}{2}$$

(iv) $t = \pi/2 \Rightarrow \alpha(\pi/2) = (0, 1, \frac{\pi}{2})$

$$\vec{V} = (-\sin t, \cos t, 1)$$

$$(x, y, z) - (0, 1, \tfrac{\pi}{2}) = t\,(-\sin t, \cos t, 1)$$

(v) Normal $B \parallel$ to $(\sin t, -\cos t, 1)\big|_{\pi/2} = (1, 0, 1)$

$$\left[(x, y, z) - (0, 1, \tfrac{\pi}{2})\right] \cdot (\overset{1}{\cancel{\sin}\, t}, \overset{0}{-\cancel{\cos}\, t}, 1) = 0$$

soln $(x + z - \frac{\pi}{2}) = 0$

#2 $$N'' = -\kappa' T - \kappa T' + \tau' B + \tau B'$$

$$= -\kappa' T - \kappa^2 N + \tau' B - \tau^2 N$$

$$N'' \cdot B = \tau' \checkmark$$

#3 $$L\vec{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 1 & -3 \\ a^1 & a^2 & a^3 \end{vmatrix} = (1\cdot a^3 + 3\cdot a^2)\underset{\sim}{i} - (2\cdot a^3 + 3a^1)\underset{\sim}{j} + (2a^2 - 1\cdot a^1)\underset{\sim}{k}$$

$$= \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}$$

matrix

#4 $$\frac{dE}{dt} = \cancel{|\alpha'(t)|} \frac{\alpha'(t)}{\cancel{|\alpha'(t)|}} \cdot \alpha''(t) + \nabla U \cdot \alpha'(t)$$

$$= \alpha'(t) \cdot \alpha''(t) - \alpha''(t) \cdot \alpha'(t) = 0 \checkmark$$

$\Rightarrow$ $E(t)$ constant

#5 $\|\alpha_2(t) - \alpha_1(t)\|^2 = (x_2(t) - x_1(t), y_2(t) - y_1(t))$

$$\cdot(\qquad\qquad)$$

$$\frac{d}{dt}\|\alpha_2(t) - \alpha_1(t)\|^2 = \frac{d}{dt}(\alpha_2 - \alpha_1)\cdot(\alpha_2 - \alpha_1)$$

$$= 2(\alpha_2' - \alpha_1')^t \cdot (\alpha_2 - \alpha_1)$$

$$\cancel{2\begin{bmatrix} 0 & f(t) \\ -f(t) & 0 \end{bmatrix}} =$$

$$= 2\,(\Delta x(t), \Delta y(t))\begin{bmatrix} 0 & -f(t) \\ f(t) & 0 \end{bmatrix}\begin{bmatrix} \Delta x(t) \\ \Delta y(t) \end{bmatrix}$$

$= 0 \Rightarrow \|\alpha_2 - \alpha_1\|^2 = \text{const in time}$

But $\alpha_2 - \alpha_1 = 0$ at $t = 0 \Rightarrow \text{const} = 0$ ✓

(#6) $\bar{a}^{\alpha} = A^{\alpha}_{i} a^{i} \qquad \bar{g}_{\alpha\beta} = g_{ij} A^{i}_{\alpha} A^{i}_{\beta}$

$$\bar{T}^{\alpha}_{\beta} = T^{i}_{j} A^{j}_{\beta} A^{\alpha}_{i} \qquad A^{\alpha}_{i} = \left(A^{i}_{\alpha}\right)^{-1}$$

$$\Leftrightarrow \quad \bar{a} = A^{-1} a \;, \quad \bar{g} = A^{t} g A \;, \quad \bar{T} = A^{-1} T A$$

(ii) $\bar{T}^{\alpha}_{\beta\gamma} = T^{i}_{jk} A^{j}_{\beta} A^{k}_{\gamma} A^{\alpha}_{i}$