Feb 8 Wed 2012 Math 119A W02

Nondimensionalizing an Equation

☐ Principle: By re-scaling variables with respect to dimensional constants in the equation, you can transform a dimensional equation into an equivalent equation in which every variable and constant is <u>dimensionless</u>

i.e. $[u]=1$, $[c]=1$ ... $\Rightarrow$ variables & constants do <u>not</u> rescale when you change units.

- Dimensionless constants have an absolute size independent of units
- The qualitative behavior of solutions changes at <u>critical</u> values of dimensionless constants –
- The dimensionless equations typically have <u>fewer</u> constants $\Rightarrow$ <u>simpler</u>
- [General Thm] Buckingham Pi Theorem

Ex: Show the equation for bead on rotating hoop with friction

$$mr\ddot{\varphi} = -b\dot{\varphi} - mg\sin\varphi + mr\omega^2 \sin\varphi\cos\varphi \quad (1)$$

is equivalent to dimensionless equation

$$\varepsilon \frac{d^2\varphi}{d\tau^2} = -\frac{d\varphi}{d\tau} - \sin\varphi + \gamma\sin\varphi\cos\varphi$$

$$[\varepsilon] = [\varphi] = [\tau] = [\gamma] = 1$$

- Note: 5 constants m, r, b, g, ω reduced to two ε, γ

$$\varepsilon = \frac{m^2 g r}{b^2}$$

$$\gamma = \frac{r\omega^2}{g}$$

<u>Idea</u>: rescale the time wrt constants from the equation that have dimensions of time –

$$m r \ddot{\varphi} = - b\dot{\varphi} - mg \sin\varphi + m r \omega^2 \sin\varphi \cos\varphi$$

$[m] = M$

radius of hoop

$[r] = L$

accel of gravity

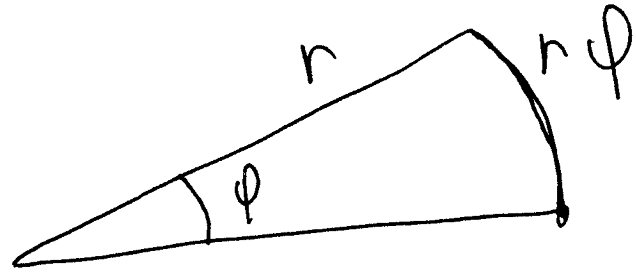
$[g] = \frac{L}{T^2}$

$[\omega^2] = [\omega]^2$

$= \left[\frac{d\theta}{dt}\right]^2 = \frac{1}{T^2}$

$[b\dot{\varphi}] = \frac{mL}{T^2}$ $[b] = \frac{mL}{T}$ ✓

Angles are dimensionless:

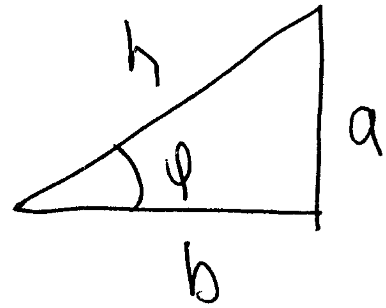


r = length of radius

rφ = length along circle of radius r

$$1 = \left[\frac{r\varphi}{r}\right] = \frac{[r\varphi]}{[r]} = 1 \Rightarrow [\varphi] = 1$$

$[\sin\varphi] = \left[\frac{a}{h}\right] = 1$



$[\cos\varphi] = \left[\frac{b}{h}\right] = 1$

$[\tan\varphi] = \left[\frac{a}{b}\right] = 1$ because they are "ratios of lengths"

<u>Defn:</u> $[c] = 1$ means c is dimensionless $\rightsquigarrow$ it does not rescale under change of unit.

- "Dimensionless quantities have an absolute size"
- "Qualitative ~~properties~~ behavior of solutions of ODE's change at critical values of dimensionless parameters – "

Eg:

Eg: Bead on rotating hoop — $\varepsilon = 0$

Pitchfork bifurcation occurs when $\gamma = \frac{r\omega^2}{g} = 1$

$$\left[\frac{r\omega^2}{g}\right] = \frac{[r][\omega]^2}{[g]} = \frac{L\left(\frac{1}{T}\right)^2}{L\left(\frac{1}{T^2}\right)} = 1 \checkmark$$

Nondimensionalizing an ODE by introduction of a time scale:

$$mr\ddot{\varphi} = -b\dot{\varphi} - mg\sin\varphi + mr\omega^2\sin\varphi\cos\varphi$$

"balance of forces $\Rightarrow$ each term is a force"

Rescale time: $\tau = \frac{t}{\alpha}$ $\quad\alpha$ to be chosen

$[\alpha] = T$

$$\dot{\varphi} = \frac{d\varphi}{d\tau}\frac{d\tau}{dt} = \frac{1}{\alpha}\frac{d\varphi}{d\tau}, \quad \ddot{\varphi} = \frac{1}{\alpha^2}\frac{d^2\varphi}{d\tau^2}$$

- $$\frac{mr}{\alpha^2}\frac{d^2\varphi}{d\tau^2} = -b\frac{1}{\alpha}\frac{d\varphi}{d\tau} - \underbrace{mg}_{\text{divide thru}}\sin\varphi + mr^2\omega^2\sin\varphi\cos\varphi$$

$$\frac{\cancel{m}r}{\cancel{m}g\alpha^2}\frac{d^2\varphi}{d\tau^2} = -\frac{b}{mg\alpha}\frac{d\varphi}{d\tau} - \sin\varphi + \frac{\cancel{m}r^2\omega^2}{\cancel{m}g}\sin\varphi\cos\varphi$$

dimensionless

$$\left[\frac{b}{mg\alpha}\right] = 1 = \left[\frac{b}{mg}\right]\frac{1}{[\alpha]} \qquad [\alpha] = \left[\frac{b}{mg}\right] = T$$

$$[b]\frac{1}{[mg]} = \frac{ML}{T}\cdot\frac{1}{M\frac{L}{T^2}} = T \checkmark$$

<u>Choose</u>: $\alpha = \dfrac{b}{mg}$

$$\Rightarrow \frac{r}{gT^2} = \frac{r}{\cancel{g}\frac{b^2}{m^2g^{\cancel{2}}}} = \frac{m^2gr}{b^2} = \varepsilon$$

$$[\varepsilon] = \left[\frac{m^2gr}{b^2}\right] = \frac{M^2\frac{L}{T^2}L}{\frac{M^2L^2}{T^2}} = 1 \checkmark$$

Our dimensionless equation:

$$\boxed{\varepsilon \frac{d^2\varphi}{d\tau^2} = -\frac{d\varphi}{d\tau} - \sin\varphi + \gamma \sin\varphi \cos\varphi}$$

- Overdamped applies when $\varepsilon \ll 1$

I.e. we say: Overdamped applies in limit

$$\varepsilon \longrightarrow 0$$

with $\varphi, \frac{d\varphi}{d\tau}, \gamma = O(1)$ as $\varepsilon \to 0$

__Defn__: (1) γ is $\overset{\text{"big"}}{O(1)}$ as $\varepsilon \to 0$ if

$|\gamma| <$ const indept of ε as $\varepsilon \to 0$

(2) γ is $\overset{\text{"little"}}{o(1)}$ as $\varepsilon \to 0$ if $\gamma \longrightarrow 0$ as $\varepsilon \to 0$. [~~Used~~ Terminology for asymptotics & analysis of limits]

<u>Note</u>: Fewer constants!

m, r, b, g, ω reduced to ε, γ

"Qualitative properties of solutions change at critical values of ε, γ"

- There is more than one way to non-dimensionalize $\Rightarrow$ different equivalent non-dimensional formulations —

Ex 3.5.7 logistic Eqn

- Equation for population growth:

$$\dot{N} = rN \qquad N(0) = N_0$$

"rate proportional to size" $\Rightarrow N(t) = N_0 e^{rt}$

$$\Rightarrow \frac{\dot{N}}{N} = \text{growth rate}$$

Simplest model that limits the growth — growth rate linear

$$\frac{\dot{N}}{N} = r - \frac{rN}{K} \qquad N(0) = N_0$$

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

(a) Find dimensions of the variables

Three dimensional parameters: r, K, N_0

$[N] = N$ = number dimension – eg,

number unit = 1 person, 1000 peoples
million peoples

$$[\dot{N}] = \frac{N}{T}$$

$$[rN] = [\dot{N}] = \frac{N}{T} \Rightarrow [r] = \frac{1}{T}$$

$$\left[\frac{rN^2}{K}\right] = \frac{N}{T} \Rightarrow [K] = \frac{T}{N}[rN^2] = N$$