FRI Feb 10 2012

① The phase portrait in 2-D
Autonomous Systems in the plane

- Notation: $\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

Autonomous ODE

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \qquad f(\underset{\sim}{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

no explicit dependence on t

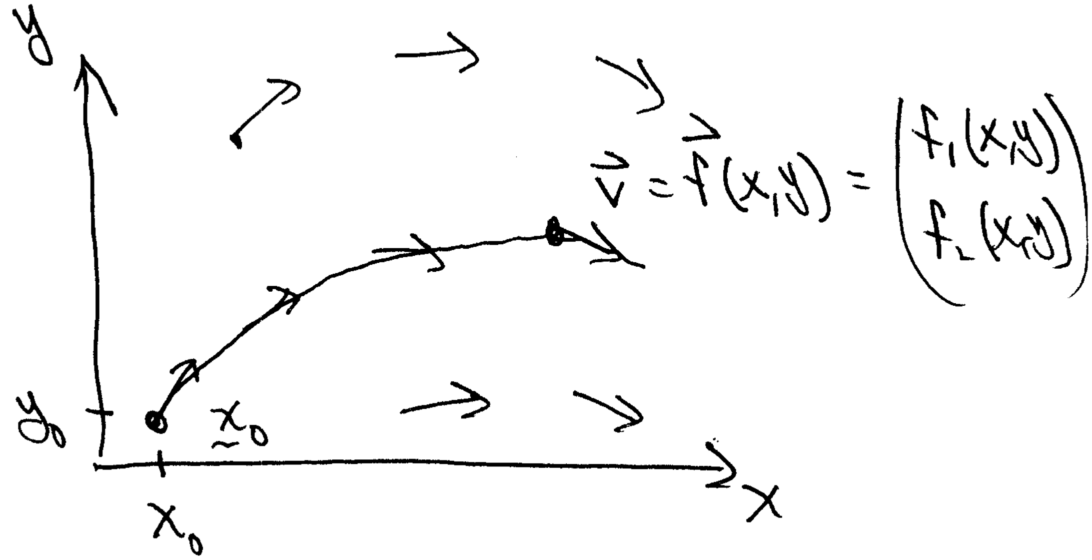
- Look for solutions $\underset{\sim}{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the ivp

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$

$$\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0 \in \mathbb{R}^2$$

5.1.1, 5.1.9, 5.2.1, 5.2.2, 5.2.4, 5.2.13

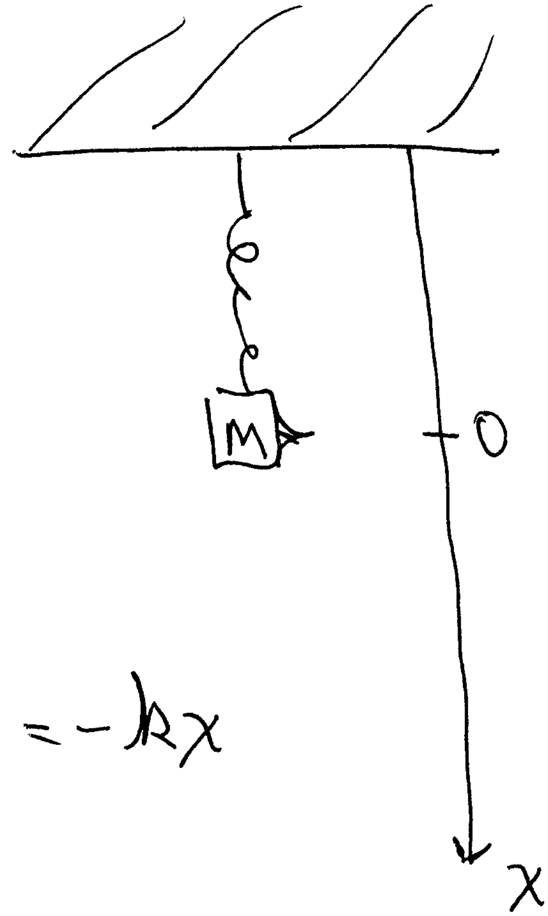
- Visualize: $f(\underset{\sim}{x})$ is a vector field on the plane – look for $\underset{\sim}{x}(t)$ tangent to $f(\underset{\sim}{x})$ @ each point.



$\underset{\sim}{x}(t)$ solves $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ if:

(1) $\vec{v} = \dot{\underset{\sim}{x}}$ is tangent to $f(\underset{\sim}{x}(t))$ at each $\underset{\sim}{x}(t)$

(2) The speed $\|\vec{v}\| = \frac{ds}{dt} = \|f(\underset{\sim}{x})\|$ at each pt.



Ex: Force $= -kx$

Spring constant

Equation: $ma = \text{force} = -kx$

$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\boxed{\ddot{x} + \omega^2 x = 0} \qquad \omega = \sqrt{\frac{k}{m}}$$

Harmonic oscillator has periodic soln's

$$x(t) = A\cos\omega t + B\sin\omega t$$

constants A, B.

- Write as a first order system:

$$x = x \quad \Rightarrow \quad \dot{x} = y$$

$$y = \dot{x} = v \quad \Rightarrow \quad \dot{y} = \ddot{x} = -\omega^2 x$$

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} y \\ -\omega^2 x \end{pmatrix}}_{} \equiv f(\underset{\sim}{x})$$

$$f(\underset{\sim}{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \qquad \begin{matrix} f_1(x,y) = y \\ f_2(x,y) = -\omega^2 x \end{matrix}$$

Because it is linear we can write it in matrix form:

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{\substack{\text{const coeff} \\ \text{matrix } A}} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Solution to the initial value problem:

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \qquad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\omega^2 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\Longleftrightarrow$$

$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0 \qquad \begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

Since we know: $x(t) = A\cos\omega t + B\sin\omega t$

$$y(t) = \dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

we can choose A, B to meet *any* i-cond.

$$x_0 = A\cos(0) + B\sin(0) = A$$

$$y_0 = -A\omega\sin(0) + B\omega\cos(0) = B\omega$$

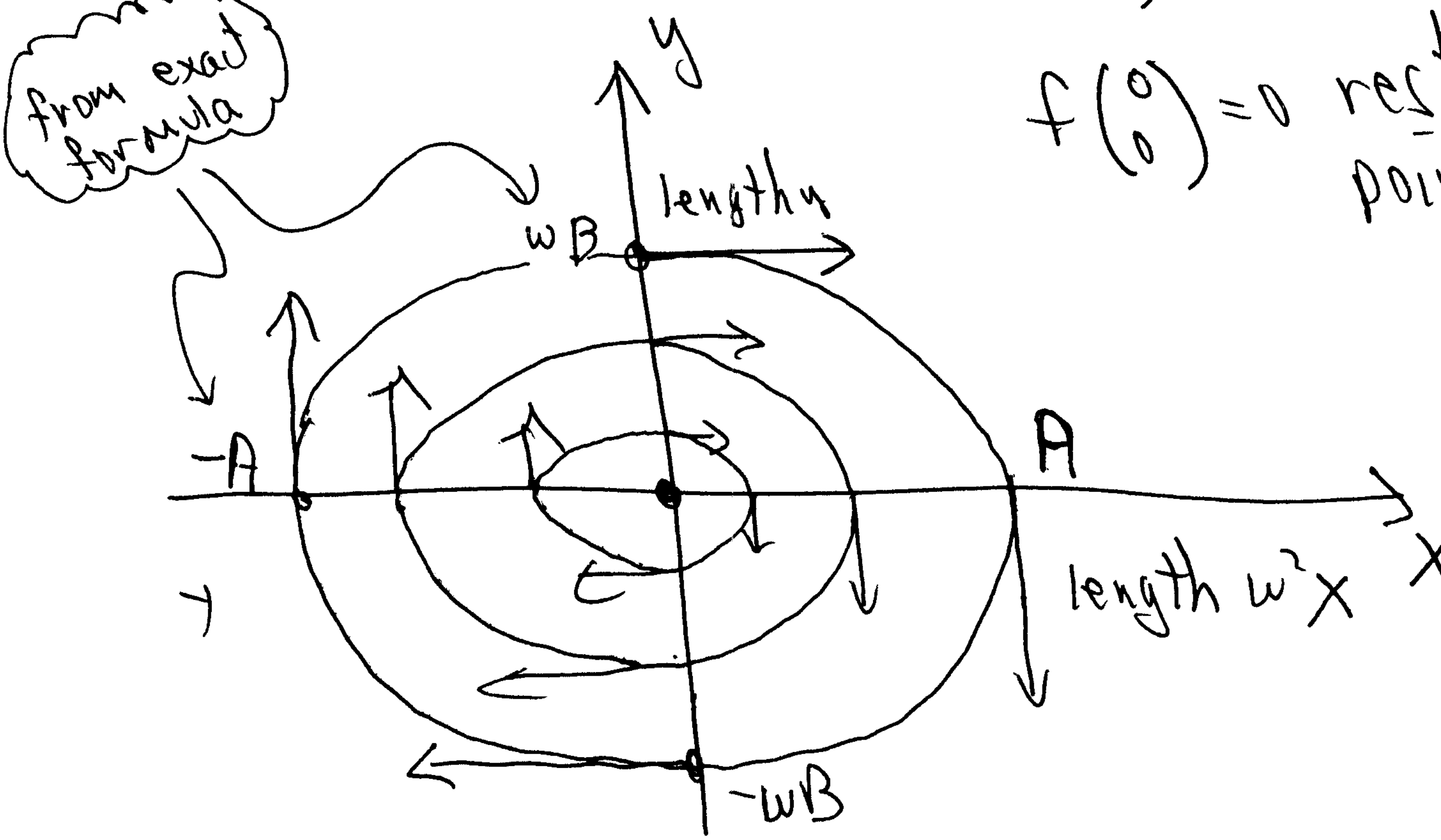
$$\boxed{\begin{aligned} A &= x_0\\ B &= y_0/B \end{aligned}} \quad \text{solve i.v.p.}$$

$\underset{\sim}{x}(t+2\pi) = \underset{\sim}{x}(t) \Rightarrow$ <u>periodic</u> <u>orbits</u>

- Phase portrait:

Vector field: $f(\underline{x}) = \begin{pmatrix} y \\ -\omega^2 x \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ rest point



Not hard to show are ellipses —
They all circle around the rest point (a)
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Big Picture: general 2x2 autonomous system/nonlinear

$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$

↖ nonlinear vector field

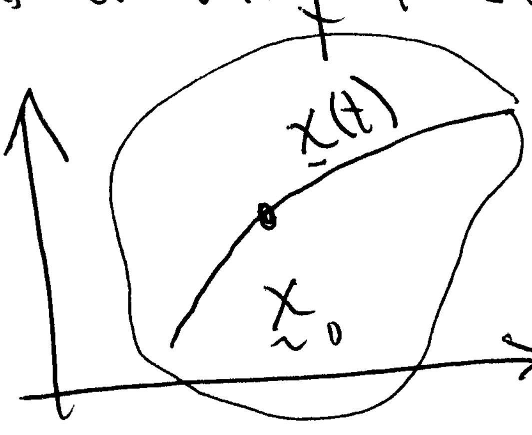
- Assume f Lipschitz Continuous in $\underset{\sim}{x}$

$\exists\, K > 0$ such that

$$\|f(\underset{\sim}{x}_2) - f(\underset{\sim}{x}_1)\| \le K \|\underset{\sim}{x}_2 - \underset{\sim}{x}_1\| \quad (*)$$

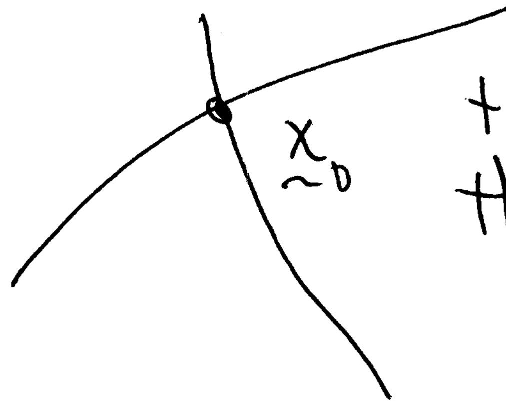
for all $\underset{\sim}{x}_1$ & $\underset{\sim}{x}_2$ in the region where you look for solutions

$\Rightarrow$ The ivp has a unique solution $\forall\, \underset{\sim}{x}_0$



defined until it goes out of region where (*) holds

- Conclude: solution orbits/trajectories cannot cross:



two solutions with the same initial condition!

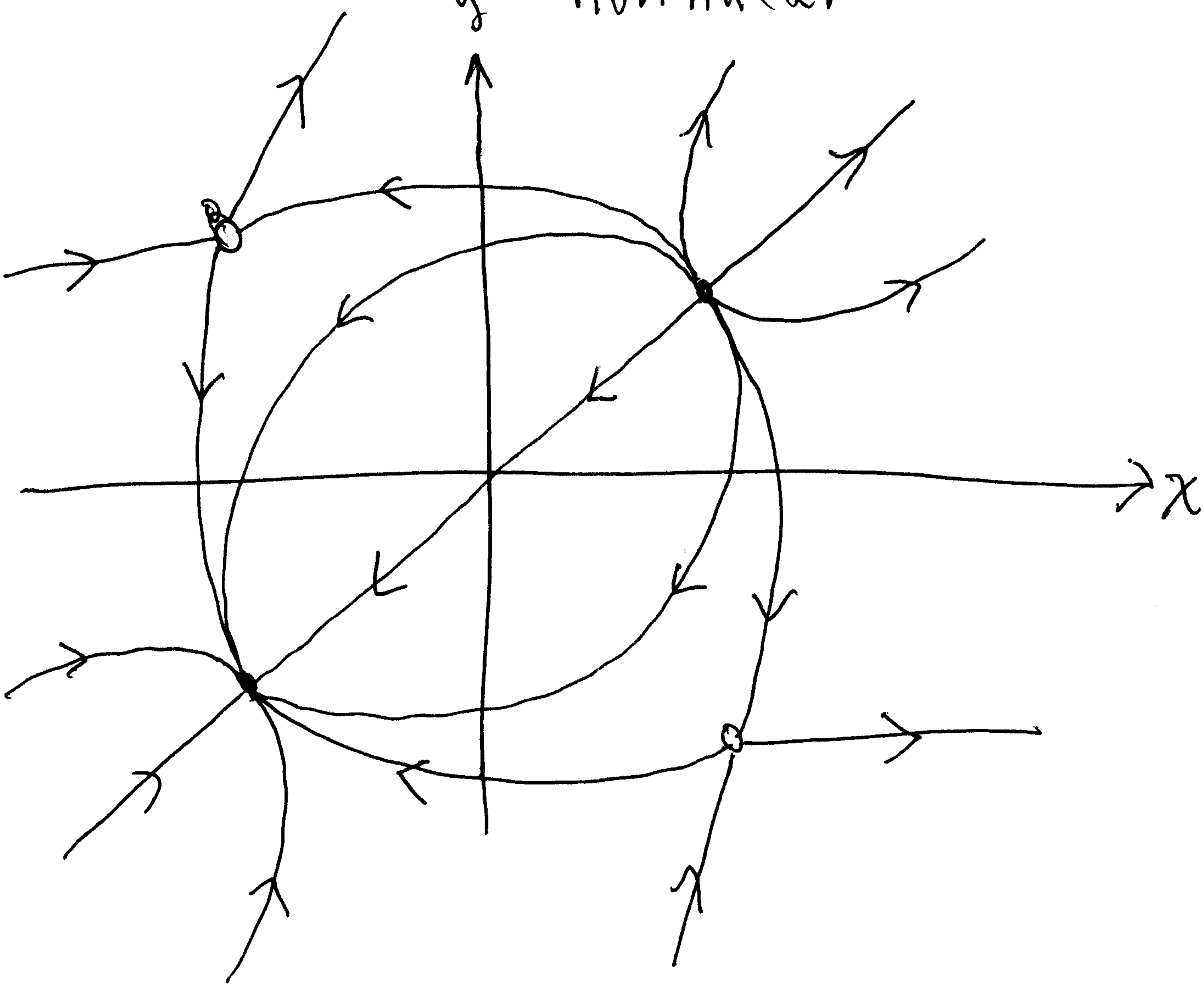
- $\rightsquigarrow$ The qualitative feature of solutions is determined by the structure of solutions around the rest points - when

$$f(\bar{\underset{\sim}{x}}) = 0$$

Ex 6.6.3

$$\dot{x} = -2\cos x - \cos y$$
$$\dot{y} = -2\cos y - \cos x$$

nonlinear



stable/unstable rest points
or nodes (like harmonic oscillator)
(marginally stable)

<u>Program</u>: Linearize the equation around the rest points where $f(\bar{\underset{\sim}{x}})=0$.

Get 2x2 constant coeff system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Determine the structure of solutions by eigenvalue methods at each rest point
- Connect the orbits so "no trajectories cross"