Linear Rest Points

Linear Systems: $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ $\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

- Rest Point: $\bar{\underset{\sim}{x}} = 0$ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Thm: If (λ, R) is an eigenpair $\Leftrightarrow AR = \lambda R$ then

$$\underset{\sim}{x}(t) = R e^{\lambda t}$$

solves $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

If (λ_1, R_1), (λ_2, R_2) are eigenpairs & R_1, R_2 is a basis ($R_1 \not\parallel R_2$ in $\mathbb{R}^2$ is suff.) then all solutions are given by

$$\boxed{\underset{\sim}{x}(t) = c_1 R_1 e^{\lambda_1 t} + c_2 R_2 e^{\lambda_2 t}} \quad (*)$$

- The Thm applies equally to real & complex solutions (evals/evects can be complex)

- **Thm**: if $\underset{\sim}{x}(t) = \underset{\sim}{v}(t) + \underset{\sim}{w}(t)\,\underset{\sim}{i}$ is a complex valued soln of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, A real, then the real and complex parts $\underset{\sim}{v}(t)$ & $\underset{\sim}{w}(t)$ are two <u>real</u> soln's of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

Pf. $\dot{\underset{\sim}{x}} = \dot{\underset{\sim}{v}} + \dot{\underset{\sim}{w}}\underset{\sim}{i} = A(\underset{\sim}{v} + \underset{\sim}{w}i)$

$$= A\underset{\sim}{v} + (A\underset{\sim}{w})i$$

$$\Leftrightarrow (\dot{\underset{\sim}{v}} - A\underset{\sim}{v}) + (\dot{\underset{\sim}{w}} - A\underset{\sim}{w})\underset{\sim}{i} = 0 \quad \forall t$$

$$\Leftrightarrow \dot{\underset{\sim}{v}} - A\underset{\sim}{v} = 0 \ \& \ \dot{w} - A\underset{\sim}{w} = 0 \ \checkmark$$

- The character of solutions depends on eigenpairs of A.

Useful Results from Linear Algebra –

(1) If $A_{n \times n}$ has n distinct evalues, then A has a basis of eigenvectors.

(2) A symmetric matrix $(A^T = A)$ has real evals & an ON basis of evectors.

(3) Complex eigenvalues come in complex conjugat pairs $\lambda = a \pm ib$. I.e.

(λ, R) solves $AR = \lambda R$ iff $\overline{A}\overline{R} = \overline{\lambda}\overline{R} \Leftrightarrow A\overline{R} = \overline{\lambda}\overline{R}$

iff $(\overline{\lambda}, \overline{R})$ is eigenpair

[$\overline{\lambda} = \overline{a+ib} = a - ib$ is the complex conj of λ]

◈ The cases for 2x2 A:

1. Two real distinct evals $\lambda_2 < \lambda_1 < 0$: stable node

   Canonical Case: $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

2. Two real distinct evals $0 < \lambda_1 < \lambda_2$: unstable node

   Canonical Case: $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

3. Two real evals of different sign : $\lambda_1 < 0 < \lambda_2$

   Canonical Case: $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ Saddle

4. Two complex evals $\lambda, \bar{\lambda}$: $\lambda = \alpha + i\omega$

   Canonical Case: $A = \begin{bmatrix} \alpha & -\omega \\ +\omega & \alpha \end{bmatrix}$ spiral/center

5 Degenerate Cases - <u>one</u> evalue

(a) Every vector is Eigenvector: $A\underset{\sim}{v} = \lambda \underset{\sim}{v}$ $\forall \underset{\sim}{v} \in \mathbb{R}^2$

Canonical Case: $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

- stable star $\lambda < 0$
- unstable star $\lambda > 0$

(b) One eval & one eigenvector

Canonical Case: $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

- stable degenerate node $\lambda < 0$
- unstable degenerate node $\lambda > 0$.

- Two ways to go –

<u>Thm</u> (Linear Algebra): $\forall$ 2×2 matrix A, $\exists$ transformation $T_{2\times 2}$ such that

$$T^{-1}AT = B$$

where B is one of the canonical cases:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_{\text{real}}, \quad \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}, \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Then write: $\underset{\sim}{x} = T\underset{\sim}{y}$

$$T\dot{\underset{\sim}{y}} = \dot{\underset{\sim}{x}} = A\underset{\sim}{x} = AT\underset{\sim}{y}$$

$$\boxed{\dot{\underset{\sim}{y}} = T^{-1}AT\underset{\sim}{y}}$$

Reduce all problems to canonical case ✓

Eg, if R_1, R_2 basis of eigenvectors, then

$$T = \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix}.$$

- It is easier just to find the eigenpairs an construct solutions <u>directly</u>.

Ex: given A, find e-values –

<u>Soln</u>: $AR = \lambda R \Leftrightarrow (A - \lambda I)R = 0 \Leftrightarrow$

R in ker $(A - \lambda I) \Leftrightarrow \det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

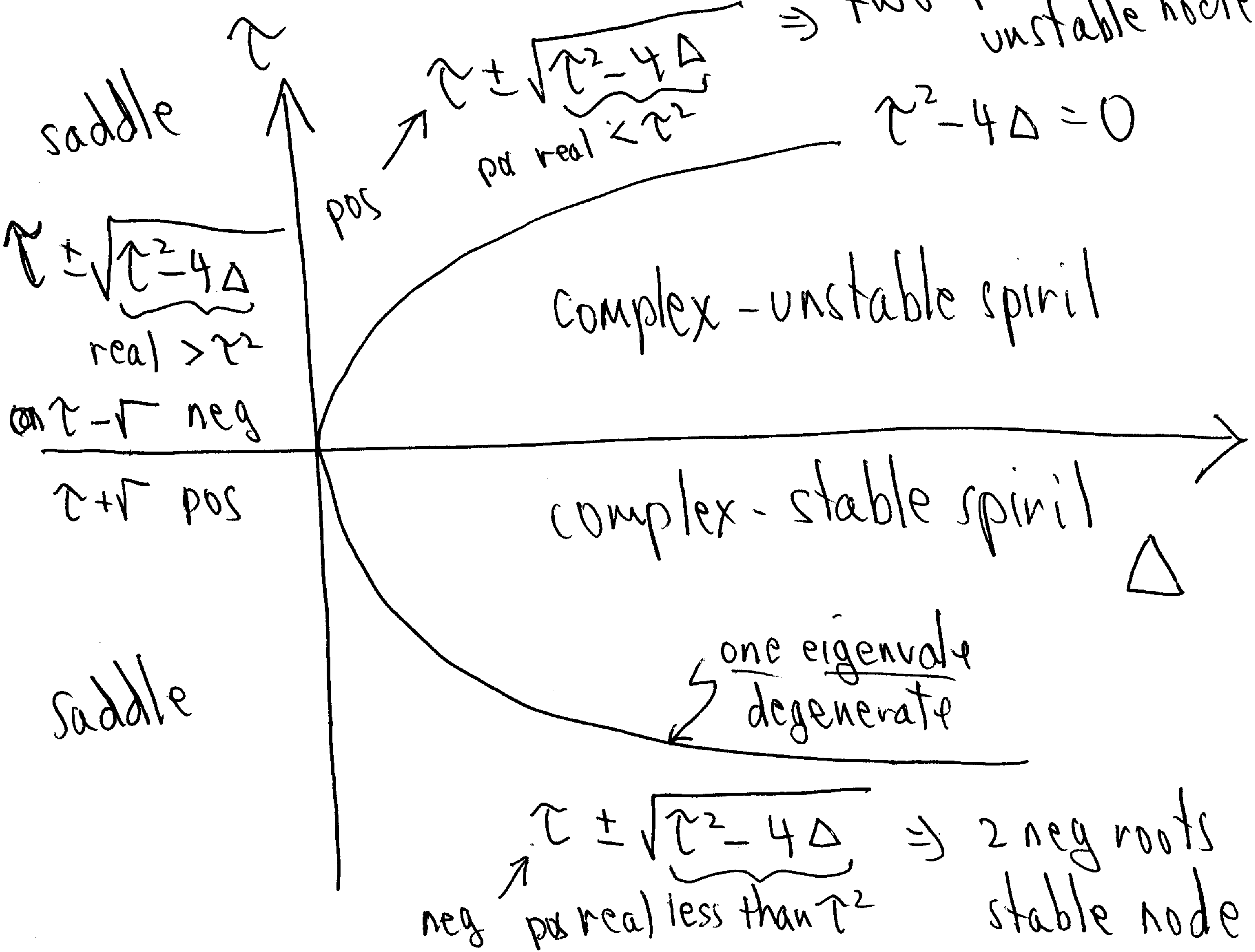
$$\lambda^2 - \underbrace{(d+a)}_{\text{Trace} A = \tau}\lambda + \underbrace{(ad-bc)}_{\text{Det} A = \Delta} = 0$$

$$\lambda^2 - \tau\lambda + \Delta = 0$$

$$\boxed{\lambda_{\pm} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

Structure of evals depends on discriminant

$$\boxed{D = \tau^2 - 4\Delta}$$



- **Note**: Degenerate nodes are unlikely & rare...

Examples:

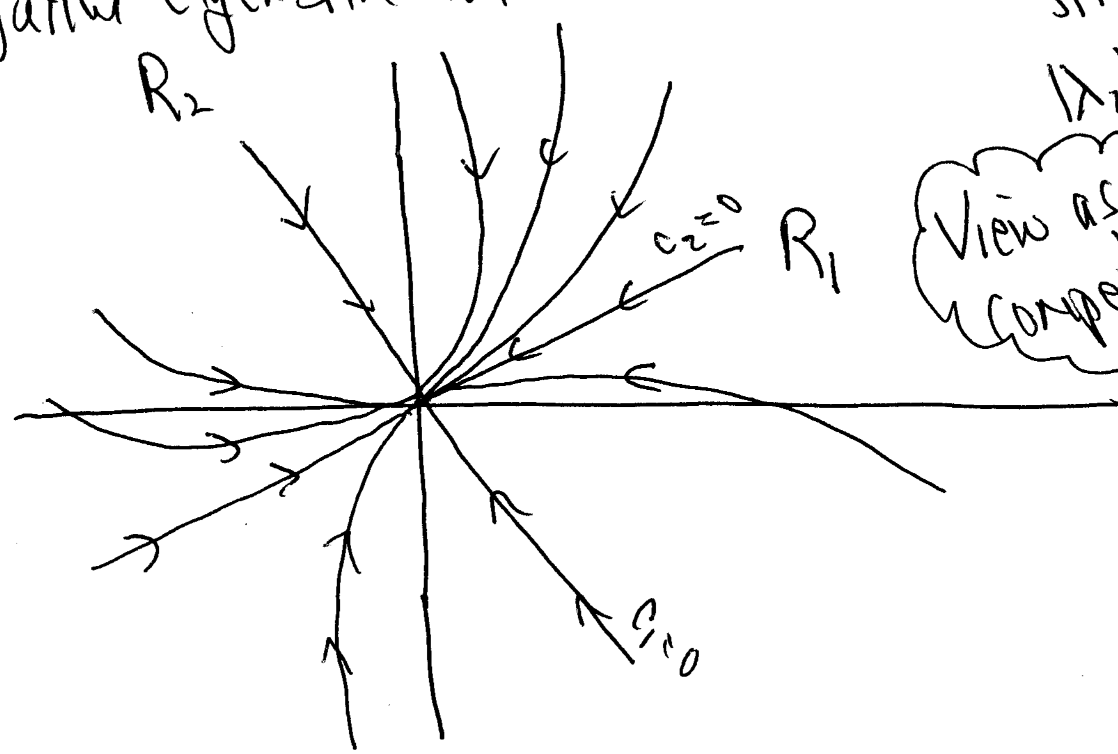
① $\lambda_2 < \lambda_1 < 0$ $(\lambda_1, R_1), (\lambda_2, R_2)$ epairs
stable node

Soln: $\underset{\sim}{x}(t) = C_1 R_1 e^{\lambda_1 t} + C_2 R_2 e^{\lambda_2 t}$

faster decay since $|\lambda_2| > |\lambda_1|$

View as two competing terms

Stable: solutions come in tangent to least negative eigendirection

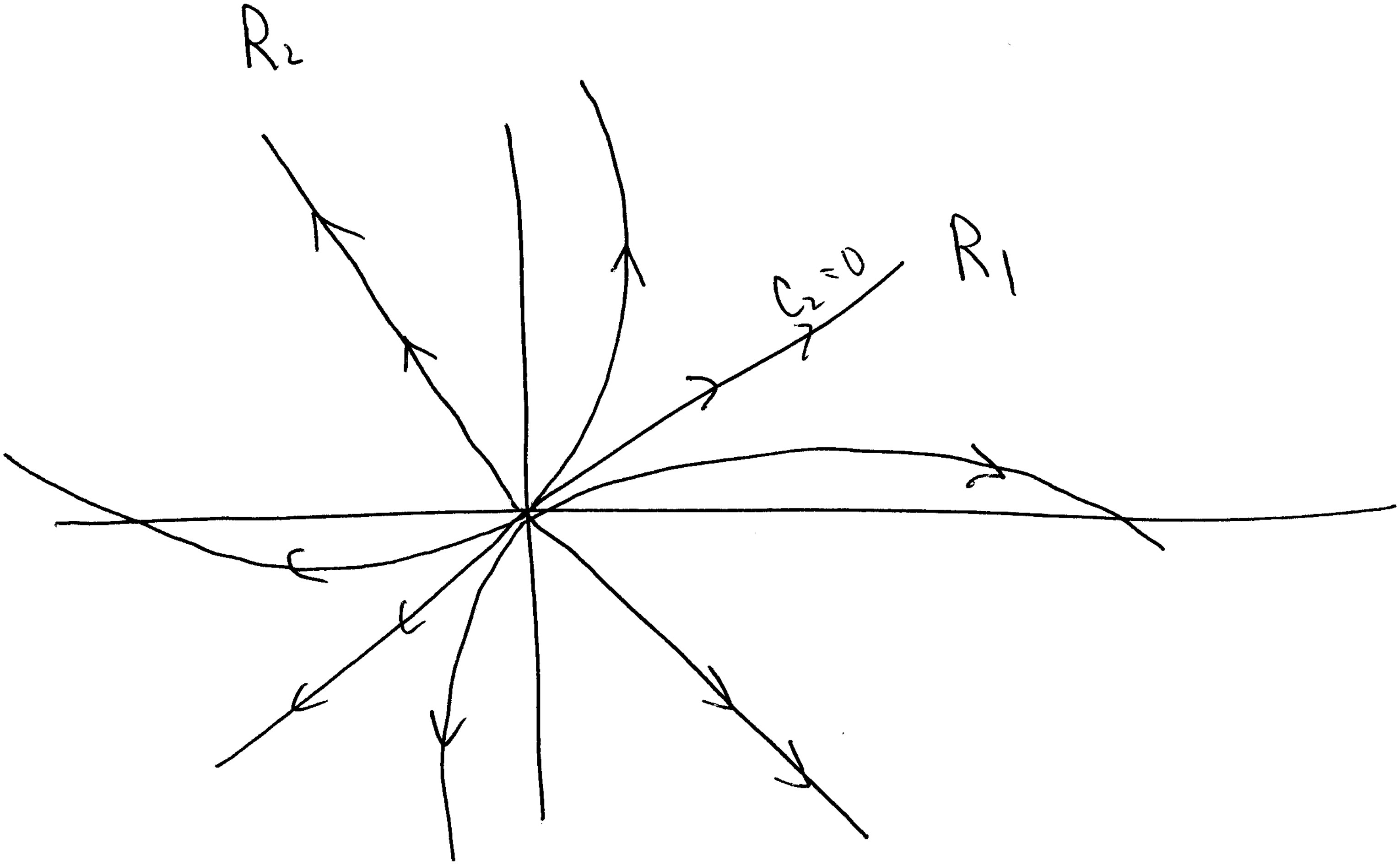


(2) $0 < \lambda_1 < \lambda_2$ Unstable node

$$\underset{\sim}{X}(t) = C_1 R_1 e^{\lambda_1 t} + C_2 R_2 e^{\lambda_2 t}$$

wins in backward time

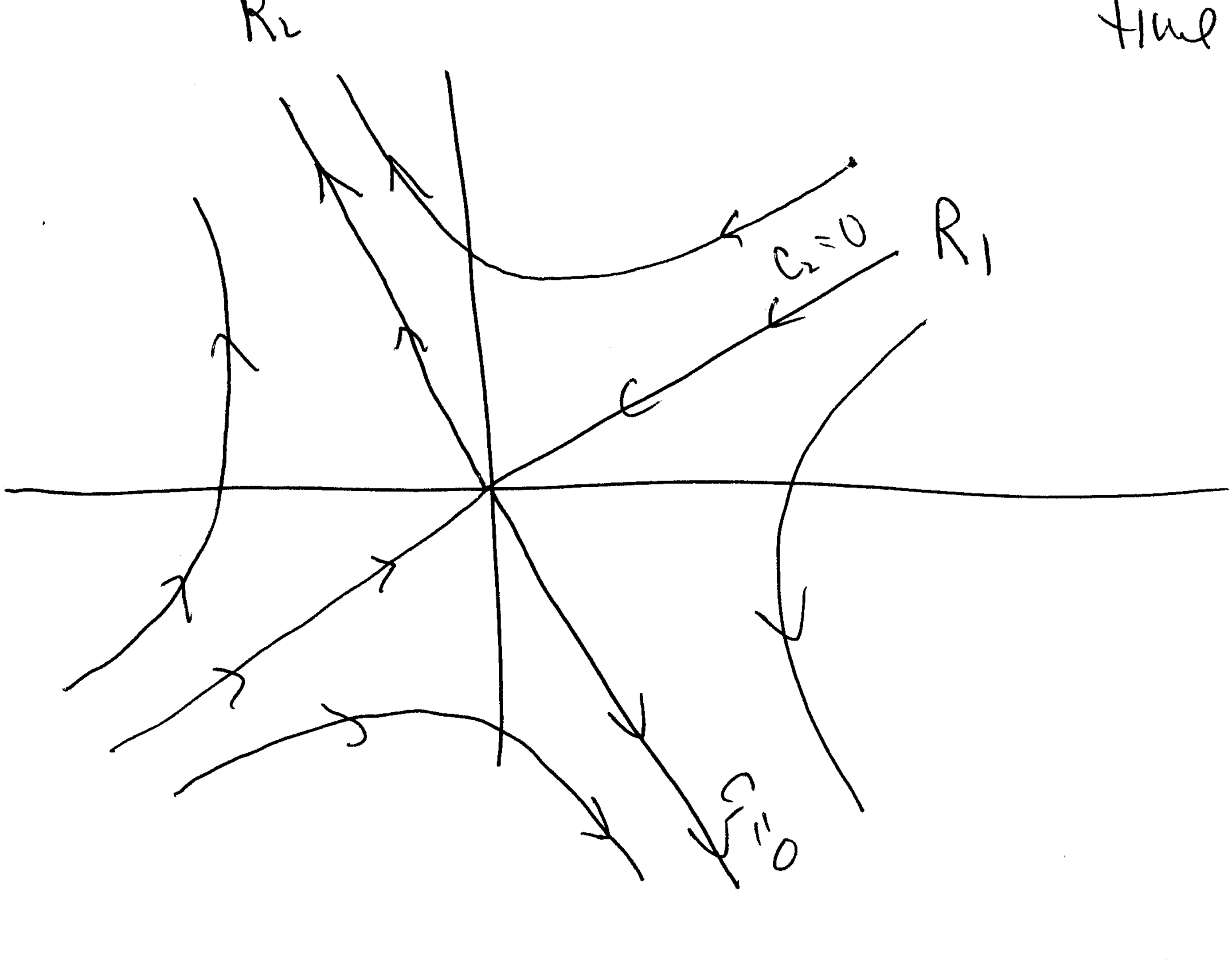
wins in forward time



3 $\lambda_1 < 0 < \lambda_2$ saddle

$$\underset{\sim}{x}(t) = c_1 R_1 e^{\lambda_1 t} + c_2 R_2 e^{\lambda_2 t}$$

wins in forward time
loses in backward time



(4) Complex evals – $\lambda_{1,2} = \alpha \pm i\omega$

Find one complex evector –

$$R_1 + R_2 i = \lambda = \alpha + i\omega$$

$\Rightarrow$ one complex solution:

$$\underset{\sim}{x}(t) = (R_1 + iR_2)\, e^{(\alpha + i\omega)t}$$

Real & Imag parts are soln's –

$$\underset{\sim}{x}(t) = (R_1 + iR_2)\, e^{(\alpha + i\omega)t} \quad \text{separate real/im}$$

$$= (R_1 + iR_2)\, e^{\alpha t} e^{i\omega t} = e^{\alpha t}(R_1 + iR_2)(\cos\omega t + i\sin\omega t)$$

$$= e^{\alpha t}(R_1\cos\omega t - R_2\sin\omega t) + i\, e^{\alpha t}(R_2\cos\omega t + R_1\sin\omega t)$$

$\Rightarrow$ 2 real solutions

$$\underset{\sim}{x}_1(t) = e^{\alpha t}(R_1\cos\omega t - R_2\sin\omega t)$$

$$\underset{\sim}{x}_2(t) = e^{\alpha t}\underbrace{(R_2\cos\omega t + R_1\sin\omega t)}_{\text{periodic}}$$

$$\underset{\sim}{X}(t) = c_1\underset{\sim}{x}_1(t) + c_2 x_2(t)$$

- Conclude: solution is product of $e^{\alpha t}$ times (periodic of period $\frac{2\pi}{\omega}$)

decay if $\alpha < 0$, amplify grows if $\alpha > 0$

$$\Rightarrow \text{spiral} \begin{cases} \alpha < 0 & \text{stable} \\ \alpha > 0 & \text{unstable} \end{cases}$$

Eg: $A = \begin{bmatrix} \alpha & -\omega \\ \omega & \alpha \end{bmatrix}$ $\lambda = \alpha + i\omega$

$$A - \lambda I = \begin{bmatrix} \alpha - (\alpha + i\omega) & -\omega \\ \omega & \alpha - (\alpha + i\omega) \end{bmatrix} = \begin{bmatrix} -i\omega & -\omega \\ \omega & -i\omega \end{bmatrix}$$

$$\begin{bmatrix} -i\omega & -\omega \\ \omega & -i\omega \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0 \Rightarrow R = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

$$\underset{\sim}{x}_1(t) = e^{\alpha t}\begin{pmatrix}\cos\omega t\\ -\sin\omega t\end{pmatrix} \quad \underset{\sim}{x}_2(t) = e^{\alpha t}\begin{pmatrix}\sin\omega t\\ \cos\omega t\end{pmatrix}$$

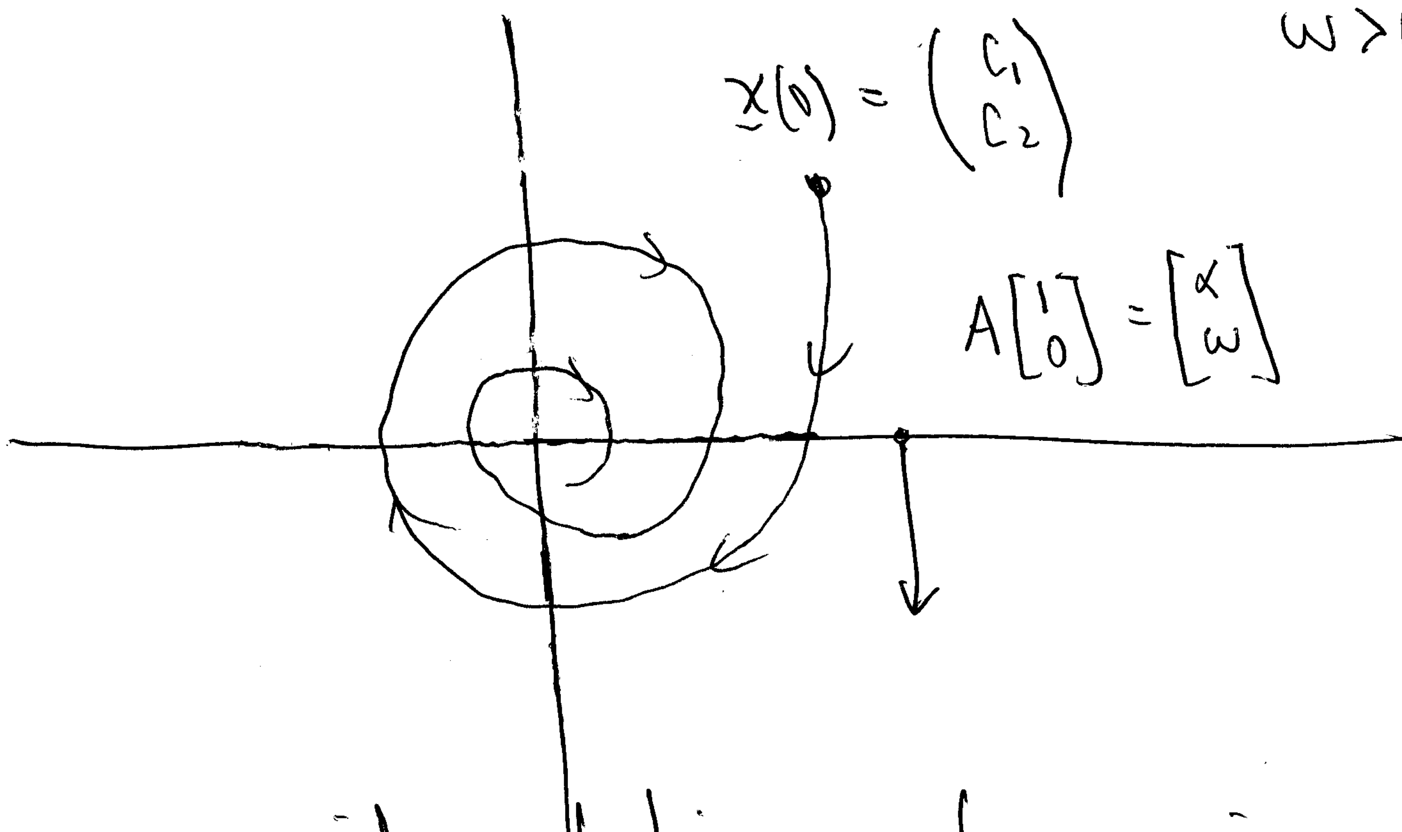
$$\underset{\sim}{x}(t) = e^{\alpha t}\Big\{\underbrace{c_1\begin{pmatrix}\cos\omega t\\ -\sin\omega t\end{pmatrix} + c_2\begin{pmatrix}\sin\omega t\\ \cos\omega t\end{pmatrix}}_{\text{periodic}}\Big\}$$

$\alpha < 0$

$\omega > 0$

$\underset{\sim}{x}(0) = \begin{pmatrix}c_1\\ c_2\end{pmatrix}$

$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\alpha\\ \omega\end{bmatrix}$



$\alpha = 0$ periodic solutions $\approx$ harmonic osc.

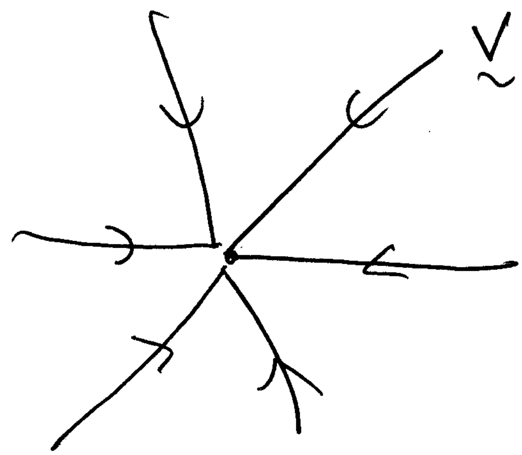
(14b)

⑤ Degenerate nodes —

(a) $\forall\ \underset{\sim}{v}$, $(\lambda, \underset{\sim}{v})$ eigenpair $\Rightarrow$

$$\underset{\sim}{x}(t) = \underset{\sim}{v}\, e^{\lambda t} \quad \text{eg } \lambda < 0 \text{ stable}$$

$\Rightarrow$ <u>star</u>



(b) Assume $AR_1 = \lambda R_1$.
Look for soln of form

$$\underset{\sim}{x}(t) = (R_1 t + R_2)\, e^{\lambda t}$$

What must R_2 satisfy?

$$\dot{\underset{\sim}{x}} = R_1 e^{\lambda t} + (R_1 t + R_2)\lambda e^{\lambda t}$$

$$A\underset{\sim}{x} = (AR_1 t + AR_2) e^{\lambda t}$$

$$= (\lambda R_1 t + AR_2) e^{\lambda t}$$

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \Rightarrow \boxed{R_1 + \lambda R_2 = AR_2}$$

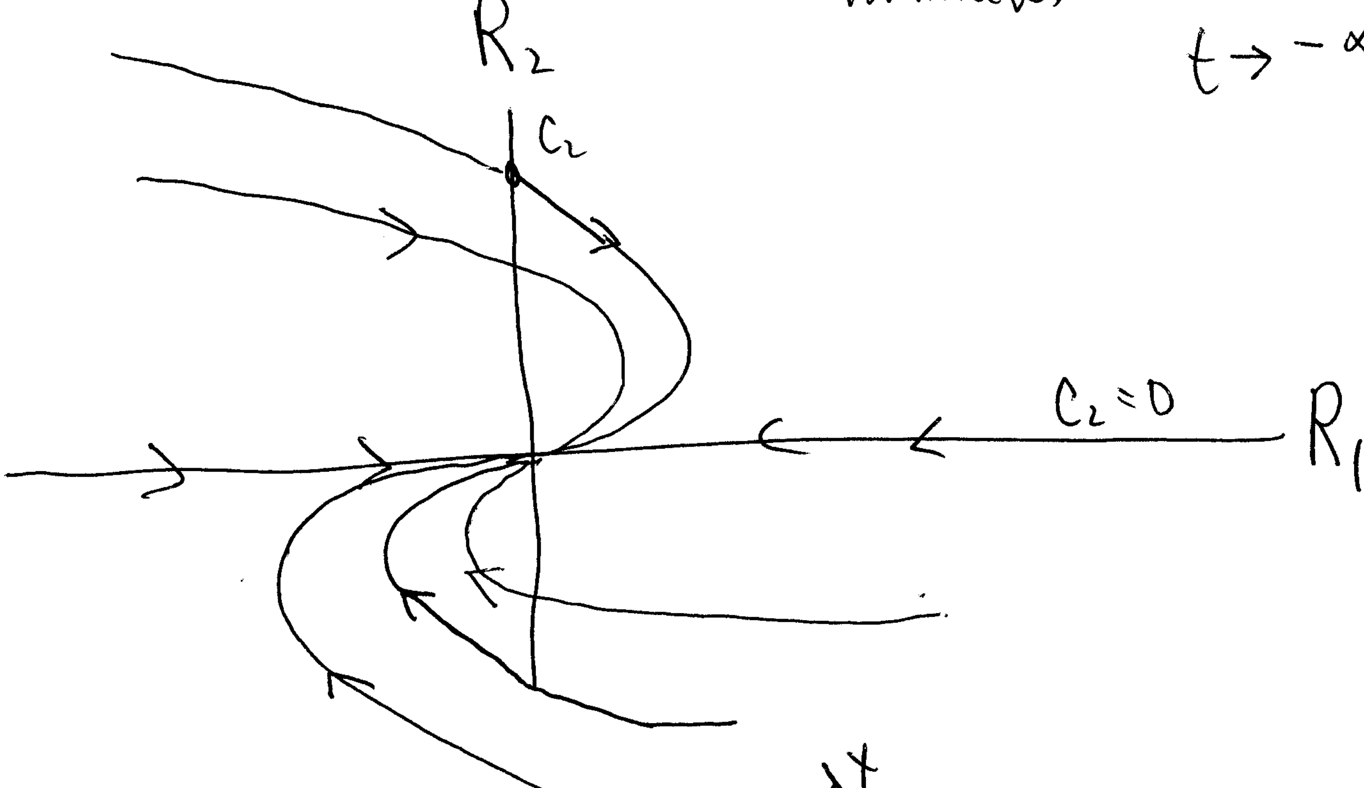
Turns out – can alway solve for R_2 in case (b)

<u>Eg</u>:
$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{R_1} + \lambda \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{R_2}$$

Soln : $\underset{\sim}{x}(t) = \{ C_1 R_1 + C_2 (\underbrace{R_1 t}_{} + R_2) \} e^{\lambda t}$

dominates as $t \to \infty$
$t \to -\infty$



$C_1 = 0$, $t = 0$ $\quad \underset{\sim}{x}(0) = C_2 R_2 e^{\lambda t}$

$\dot{\underset{\sim}{x}}(0) = C_2 (R_1 + \lambda R_2) e^{\lambda t}$

neg