Wed 2-22-'02 119A [4]

Linearizing About Rest Points – Phase Plane

Phase Portrait for nonlinear equations – Temple ①

- Setting: Autonomous Nonlinear 2x2 systems –

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \qquad \underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad f(\underset{\sim}{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

$$\dot{\underset{\sim}{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = f(\underset{\sim}{x})$$

autonomous: no dependence on t except thru unknowns $\underset{\sim}{x}$

- Important because: Newton $ma = F$

Eg:
$$m\ddot{x} = F(x, \dot{x}) \qquad x = x, \quad y = \dot{x}$$

$$m\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ F(x,y) \end{pmatrix}$$

$$f_1(x,y) = \frac{1}{m}y$$

$$f_2(x,y) = \frac{1}{m}F(x,y)$$

- Recall: (1) orbits/trajectories of soln's $\underset{\sim}{x}(t)$ cannot cross in xy-plane (2) $\underset{\sim}{x}(t+c)$ is a soln any c

$\Rightarrow$ "The qualitative structure of soln's in xy-plane can only change at rest pts"

$\Rightarrow$ Method:

(1) Find rest pts $\underset{\sim}{\bar{x}}$ where $f(\bar{\underline{x}}) = 0$

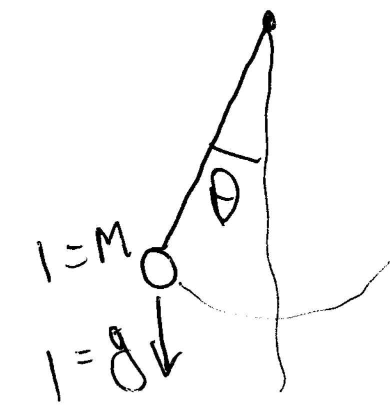
(2) Linearize around the rest points

(3) Solve linearized eqn $(\underset{\sim}{x} - \bar{\underline{x}})' = \underset{2\times 2}{A} (\underset{\sim}{x} - \bar{\underline{x}})$ at each rest pt

(4) Connect the orbits betw rest pts (orbits cannot cross)

Eg: We'll See: Nonlinear Pendulum (non-dim'd)

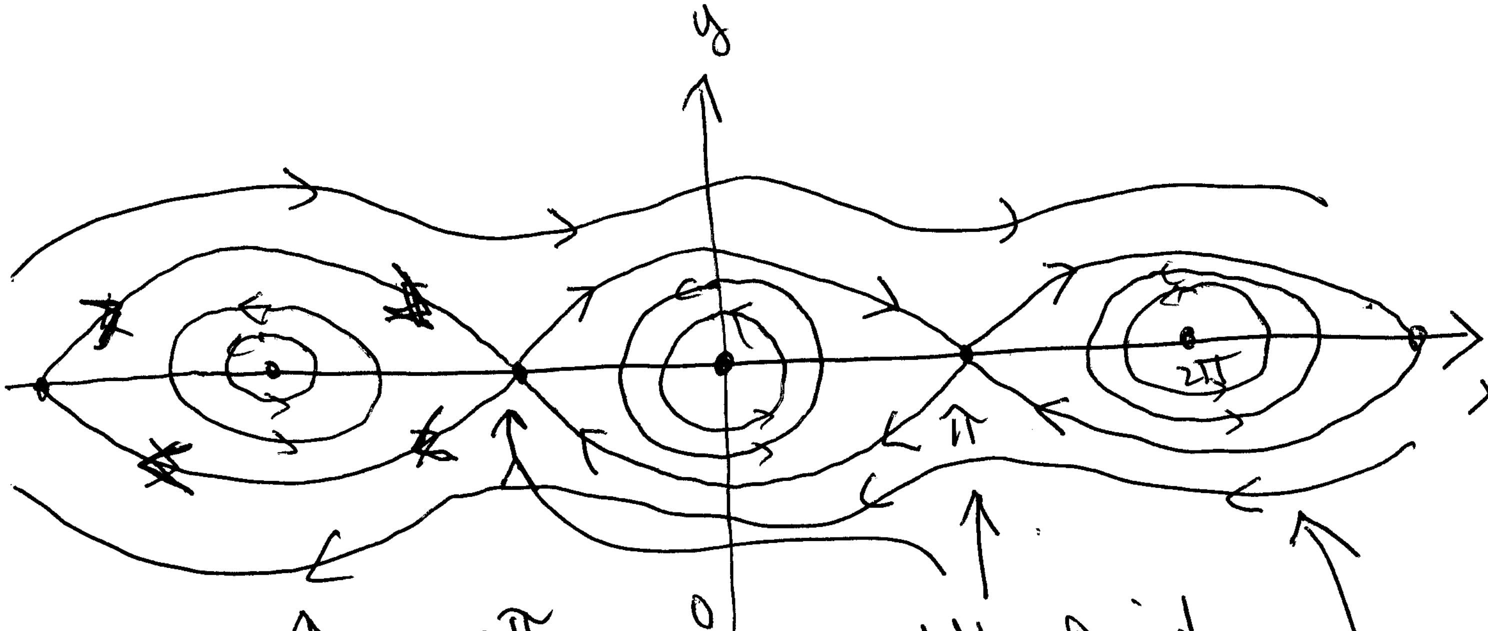
$$\ddot{\theta} + \sin\theta = 0 \qquad \theta = x, \quad \dot{\theta} = y$$



$$\dot{\underset{\sim}{x}} = \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} y \\ -\sin x \end{pmatrix} = f(\underset{\sim}{x})$$

$$f(\bar{\underset{\sim}{x}}) = 0 \Rightarrow y = 0,\ \sin x = 0 \Leftrightarrow x = n\pi$$

Phase Portrait –



Saddle Point

" $\lambda_1 < 0 < \lambda_2$ "

Centers

" $\lambda_\pm = \pm \lambda i$ "

Linearizing about rest point $\underset{\sim}{\bar{x}} = (\bar{x}, \bar{y})^T$

Assume: $\underset{\sim}{\dot{x}} = f(\underline{x})$, $f(\underline{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$

$$f(\underset{\sim}{\bar{x}}) = 0 = \begin{pmatrix} f_1(\bar{x},\bar{y}) \\ f_2(\bar{x},\bar{y}) \end{pmatrix}$$

- Taylor's Theorem:

$$f_1(x,y) = \underbrace{\cancel{f_1(\bar{x},\bar{y})}}_{0} + \frac{\partial f_1}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial f_1}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) + O(|\underline{x}-\bar{\underline{x}}|^2)$$

$$= \underbrace{\left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}\right)}_{\nabla f_1}\Bigg|_{(\bar{x},\bar{y})} \cdot \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} + \text{hot}$$

Similarly -

$$f_2(x,y) = \underbrace{\left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}\right)}_{\nabla f_2}\Bigg|_{(\bar{x},\bar{y})} \cdot \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} + \text{hot}$$

Putting these together –

$$\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} + \text{hot}$$

$$= \begin{pmatrix} -\nabla f_1 - \\ -\nabla f_2 - \end{pmatrix} \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} + \text{hot}$$

$$= Df\Big|_{\bar{\underset{\sim}{x}}} \cdot (\underset{\sim}{x} - \bar{\underset{\sim}{x}}) + \text{hot}$$

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix}$$

<u>Conclude</u>: if $Df(\bar{\underset{\sim}{x}})$ has <u>no</u> <u>kernel</u> $\Leftrightarrow$ $\det Df(\bar{\underset{\sim}{x}}) \neq 0$, then eqvations near $\bar{\underset{\sim}{x}}$ are approximated by $\dot{(\underset{\sim}{x} - \bar{\underset{\sim}{x}})} = Df(\bar{\underset{\sim}{x}})(\underset{\sim}{x} - \bar{\underset{\sim}{x}})$

<u>Example</u>: Linearize $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\sin x \end{pmatrix}$

about rest points $\bar{\underset{\sim}{x}} = (0,0)$ & $\bar{\underset{\sim}{x}} = (\pi, 0)$

<u>Soln</u>: $\bar{\underset{\sim}{x}} = (0,0) = \underset{\sim}{0}$

$$f(\underset{\sim}{x}) = \underbrace{\cancel{f(\underset{\sim}{0})}}_{0} + Df(\underset{\sim}{0}) \begin{pmatrix} x-0 \\ x-0 \end{pmatrix} \text{ that}$$

linearized system –

$$\dot{\underset{\sim}{x}} = Df(\underset{\sim}{0}) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$Df(\underset{\sim}{0}) = \begin{pmatrix} -\nabla f_1 - \\ -\nabla f_2 - \end{pmatrix}\Bigg|_{\underset{\sim}{0}} = \begin{pmatrix} \frac{\partial}{\partial x} y \,;\, \frac{\partial}{\partial y} y \\ -\frac{\partial}{\partial x} \sin x \,,\, \frac{\partial}{\partial y} \sin x \end{pmatrix}_{(x,y)=(0,0)}$$

$$= \begin{pmatrix} 0 & 1 \\ -\cos(0) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Linearized system at $\underset{\sim}{x} = \underline{\bar{x}}$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- evalues :

$$\lambda^2 - \tau\lambda + \Delta = 0 \quad \tau = 0 \,, \Delta = 1$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

Purely imaginary $\Rightarrow$ center ✓

Soln: $\underset{\sim}{\bar{x}} = (\pi, 0)$

Linearized equations near $\underset{\sim}{\bar{x}}$:

$$\begin{pmatrix} x-\pi \\ y-0 \end{pmatrix}^{\cdot} = \begin{pmatrix} -\nabla f_1- \\ -\nabla f_2- \end{pmatrix}\Bigg|_{\underset{\sim}{\bar{x}}} \begin{pmatrix} x-\pi \\ y-0 \end{pmatrix}$$

$$\nabla f_1 = \nabla y = (0, 1)$$

$$\nabla f_2 = \nabla(-\sin x) = (-\cos x, 0)$$

$$\begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix}\Bigg|_{\underset{\sim}{\bar{x}} = (\pi, 0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Linearized equations:

$$\begin{pmatrix} x-\pi \\ y \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-\pi \\ y \end{pmatrix}$$

Evalues: $\lambda^2 - \tau\lambda + \Delta = 0$ $\tau = 0$

$\Delta = -1$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

$-1 < 0 < 1$

saddle point ✓

Next:

1. Q: when do the linearized ~~equations~~ soln's describe the nonlinear solutions near the rest point?

Ans: When $\mathrm{Re}\,\lambda \neq 0$

Hartman-Grobman Thm: if $\mathrm{Re}\,\lambda \neq 0$, then the nonlinear orbits look qualitatively exactly like the linearized orbits

Defn: Structurally stable -

Precise: $\exists$ homomorphism taking $\mathbb{R}^2 \to \mathbb{R}^2$ that is cont, maps orbits of linearized eqvn's to ~~soln's~~ orbits of nonlinear eqvns.