Friday Feb 24, 02

[Nonlinear Stability of Rest Points] 119A W02 Temple (1)

Autonomous Systems —

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \qquad \underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad f(\underset{\sim}{x}) = \begin{pmatrix} f_1(\underset{\sim}{x}) \\ f_2(\underset{\sim}{x}) \end{pmatrix}$$

- Linearize about rest pt $\bar{\underset{\sim}{x}}$: $f(\bar{\underset{\sim}{x}}) = 0$

$$f(\underset{\sim}{x}) = \underbrace{\cancel{f(\bar{\underset{\sim}{x}})}}_{0} + \begin{bmatrix} -\nabla f_1- \\ -\nabla f_2- \end{bmatrix}_{\bar{\underset{\sim}{x}}} (\underset{\sim}{x} - \bar{\underset{\sim}{x}}) + \text{hot}$$

Linear const coeff system describes soln about/near rest point:

$$\dot{\overline{(\underset{\sim}{x}-\bar{\underset{\sim}{x}})}} = \dot{\underset{\sim}{x}} = \underbrace{\begin{bmatrix} -\nabla f_1- \\ -\nabla f_2- \end{bmatrix}_{\bar{\underset{\sim}{x}}}}_{A} (\underset{\sim}{x} - \bar{\underset{\sim}{x}})$$

$$\boxed{\dot{\underset{\sim}{z}} = A\underset{\sim}{z}} \qquad z = \underset{\sim}{x} - \bar{\underset{\sim}{x}}$$

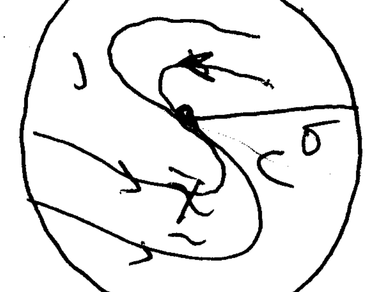
Describes soln near rest pt.

- **Defn**: $\bar{x}$ <u>Non-degenerate rest pt</u> if $\det A \neq 0$

- **Defn**: Rest point $\bar{\underset{\sim}{x}}$ is <u>asymptotically stable</u> if "trajectories sufficiently close to $\bar{\underset{\sim}{x}}$ tend to $\bar{\underset{\sim}{x}}$ as $t \to \infty$"

<u>Precisely</u>: $\exists\, \delta$ st if

$$|\underset{\sim}{x}_0 - \bar{\underset{\sim}{x}}| < \delta$$



"$|\underset{\sim}{x}_0 - \bar{\underset{\sim}{x}}|$ = dist from $\underset{\sim}{x}_0$ to $\bar{\underset{\sim}{x}}$"
$= \sqrt{(x_0-\bar{x})^2 + (y_0-\bar{y})^2}$

then the soln $\underset{\sim}{x}(t)$ of

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$
$$\underset{\sim}{x}(0) = \underset{\sim}{x}_0$$

satisfies $\lim_{t \to \infty} \underset{\sim}{x}(t) = \bar{\underset{\sim}{x}}$.

• <u>Defn</u>: Rest point $\bar{\underset{\sim}{x}}$ is Liapunov <u>stable</u> if "trajectories suffiently close to $\bar{\underset{\sim}{x}}$ stay in a correspondingly small nbhd of $\bar{\underset{\sim}{x}}$"

<u>Precisely</u>: $\forall\ \delta$ suff small $\exists$ $\varepsilon > 0$ st if $|\underset{\sim}{x}_0 - \bar{\underset{\sim}{x}}| < \varepsilon$, then the soln of $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$, $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ satisfies $|\underset{\sim}{x}(t) - \bar{\underset{\sim}{x}}| < \delta$ for all $t \geq 0$.

• <u>Defn</u>: Rest point is <u>unstable</u> if "there exist trajectories that leave the rest pt"

<u>Precisely</u>: $\exists\ \delta$ s.t. $\forall\ \varepsilon > 0$ $\exists$ $\underset{\sim}{x}_0, t_*$ with $|\underset{\sim}{x}_0 - \bar{\underset{\sim}{x}}| < \varepsilon$ s.t. $|\underset{\sim}{x}(t_*) - \bar{\underset{\sim}{x}}| > \delta$ some $t_* > 0$.

($\underset{\sim}{x}(t)$ solves $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$, $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$)

- Recall soln's of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$

① Two real evals $\lambda_1 < \lambda_2$

$\Rightarrow$ soln $\boxed{\underset{\sim}{x}(t) = C_1 R_1 e^{\lambda_1 t} + C_2 R_2 e^{\lambda_2 t}}$

<u>Conclude</u>: Asymptotically Stable iff $\lambda_1, \lambda_2 < 0$ & Unstable if $\lambda_1 > 0$ or $\lambda_2 > 0$

Q: what about $\lambda_i = 0$? Ans: $\det A = 0$ ✓

② One ~~real~~ eval $\lambda_1 = \lambda = \lambda_2 \Rightarrow$ <u>real</u>

(i) $A = \lambda I \Rightarrow$ star

$$\underset{\sim}{x}(t) = C_1 R_1 e^{\lambda t} + C_2 R_2 e^{\lambda t}$$

(ii) $A \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \sim$ one eigenvector $R_1 \Rightarrow \exists R_2$

s.t. $\underset{\sim}{x}(t) = C_1(R_1 + R_1 t)e^{\lambda t} + C_2 R_2 e^{\lambda t}$ $(R_1 = (A-\lambda)R_2)$

Conclude again: Asymptotically stable iff $\lambda < 0$, Unstable iff $\lambda > 0$

3 Complex evals $\lambda = \alpha \pm i\omega$

- Choose one epair (λ, R) $\quad \lambda = \alpha + i\omega$

  $R = U + iV$

- $\Rightarrow$ complex soln: $\underset{\sim}{x}(t) = R e^{\lambda t}$

  $\Rightarrow$ real & imag parts are real solns –

Eg: $\underset{\sim}{x}(t) = (U + iV) e^{\alpha t} e^{i\omega t} = e^{\alpha t}(U + iV)(\cos\omega t + i \sin\omega t)$

$$= \underbrace{(U\cos\omega t - V\sin\omega t)}_{I} e^{\alpha t} + \underbrace{(V\cos\omega t + U\sin\omega t)}_{II} e^{\alpha t}$$

Gen real soln: $\underset{\sim}{x}(t) = c_1 \underbrace{(\quad)}_{I} e^{\alpha t} + c_2 \underbrace{(\quad)}_{II} e^{\alpha t}$

<u>Conclude</u>: Asymptotically stable iff $\alpha = \text{Re}\,\lambda < 0$, Unstable $\alpha = \text{Re}\,\lambda > 0$

<u>But</u>: One more case $\alpha = \text{Re}\,\lambda = 0$

($\text{Re}\,\lambda = 0 \not\Leftrightarrow \det A = 0$!)

<u>Finally</u>: the only case of Liapunov Stable but <u>not</u> Asymptotically stable is $\text{Re}\,\lambda = 0$, λ complex

<u>Thm</u>: Linearized Stability: $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, $\bar{\underset{\sim}{x}} = 0$ $\quad |A| \neq 0$

(i) Asymptotically stable iff $\text{Re}\,\lambda < 0$ all λ

(ii) Unstable iff $\text{Re}\,\lambda > 0$ some λ

(iii) Liapunov Stable ~ Marginally Stable iff $\text{Re}\,\lambda = 0$

Summary:

- Nonlinear Equations: $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$
- Rest Point $\bar{\underset{\sim}{x}}$: $f(\bar{\underset{\sim}{x}}) = 0$
- T-Thm: $f(\underset{\sim}{x}) = \underbrace{\cancel{f(\bar{\underset{\sim}{x}})}}_{0} + A(\underset{\sim}{x} - \bar{\underset{\sim}{x}})$ + hot

$$A = \begin{bmatrix} - \nabla f_1 - \\ - \nabla f_2 - \end{bmatrix}$$

- Linearized Equations: $\dot{\overline{(\underset{\sim}{x} - \bar{\underset{\sim}{x}})}} = A\underbrace{(\underset{\sim}{x} - \bar{\underset{\sim}{x}})}_{\underset{\sim}{z}}$

$$\dot{\underset{\sim}{z}} = A\underset{\sim}{z}$$

- Q: when does stabity of linearized equations imply stabity of nonlinear Equations —

$\boxed{\text{~~Theorem~~ C(Ha}}$

<u>Thm</u>: (Hartman - Grobman)

(i) Asymptotically stable iff $Re\{\lambda\} < 0$ for all λ

(ii) Unstable iff $Re\{\lambda\} > 0$ for some λ

(iii) $Re\{\lambda\} = 0$ Anything could happen

<u>Actually</u>: H-G says more - it says when $Re\{\lambda\} \neq 0$, the nonlinear ~~linearized~~ trajectories look qualitatively <u>exactly</u> like the linearized trajectories. "$\exists$ a 1-1 cont mapping between them"

Example: (linearized stability $\not\Rightarrow$ nonlinear stability when $\mathrm{Re}\,\lambda = 0$)

$$\dot{x} = -y + ax(x^2+y^2) = f_1(x,y)$$
$$\dot{y} = x + ay(x^2+y^2) = f_2(x,y)$$

$$\left.\begin{array}{l} f_1(x,y) = 0 \text{ iff } x = y = 0 \\ f_2(x,y) = 0 \text{ iff } x = y = 0 \end{array}\right\} \Rightarrow \bar{\underset{\sim}{x}} = 0 \text{ rest pt}$$

$$\nabla f_1(0,0) = (3ax^2 + ay^2, -1 + 2axy)\Big|_{x=0=y} = (0,-1)$$

$$\nabla f_2(0,0) = (1,0)$$

Linearized Equations: $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Evals: $\lambda^2 - \tau\lambda + \Delta = \lambda^2 + 1 = 0$

$\lambda = \pm i \quad Re\{\lambda\} = 0 \Rightarrow$ case (iii)

Conclude - Rest point $\underset{\sim}{x} = 0$ is Liapunov stable for linearized equations.

Q: What about nonlinear eqn's?

To Solve nonlinear eqn's outright:

Polar coords: $r^2 = x^2 + y^2$

$$\cancel{2}r\dot{r} = \cancel{2}x\dot{x} + \cancel{2}y\dot{y}$$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}$$

$\dot{x} = -y + axr^2 \quad \dot{y} = x + ayr^2$

$$\dot{r} = \frac{x(-\cancel{y} + axr^2) + y(\cancel{x} + ayr^2)}{r} = \frac{ar^2(x^2+y^2)}{r} = ar^3$$

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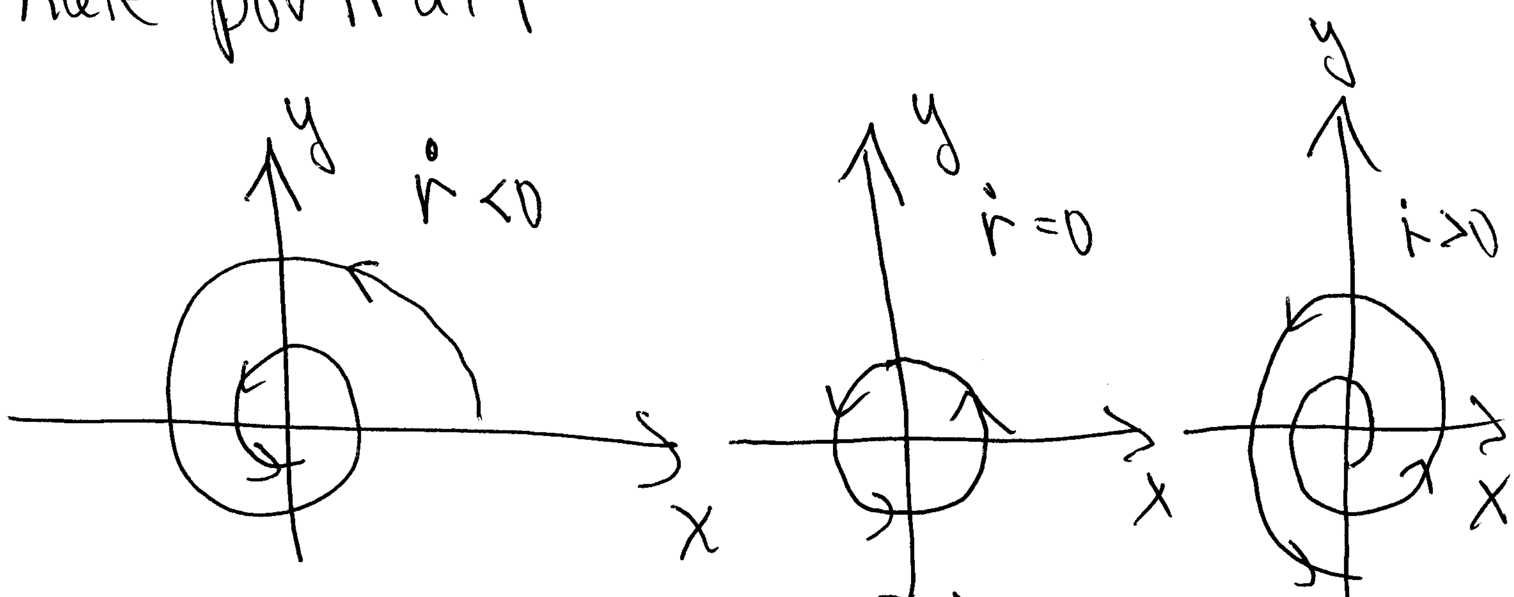
$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1$$

$\Rightarrow$ transformed ODE:

$$\dot{r} = ar^3$$
$$\dot{\theta} = 1$$

Phase portrait



Spiral in $a < 0$
asymptotically stable

Circles
$a = 0$
Liapunov
Stable

Spiral out
unstable
$a > 0$