Mon Feb 27, 12

# 119A - Competetive Exclusion



§6.4 Rabbits vs Sheep -

Q: what happens when two species compete over same foo supply?

- Simple Model - rabbits & sheep compete over grass = food supply
- Growth ~~rates~~ depend on
  - reproduction rates
  - food supply which depends on both feeding populations

- Growth rates $\frac{\dot{x}}{x}$, $\frac{\dot{y}}{y}$

$x(t)$ = pop of rabbits

$y(t)$ = pop of sheep

Eqvn: $\frac{\dot{x}}{x} = a$ $\frac{\dot{y}}{y} = b$

$a$ ↑ reproduction rate of rabbits

$b$ ↑ reproduction rate of sheep

## (4) Equations: (Simple Model)

$$\frac{\dot{x}}{x} = a - cx - dy$$

growth rate | reprod. rate | diminished food supply due to rabbits | diminished food supply due to sheep

$$\frac{\dot{y}}{y} = b - ex - fy$$

$a, b, c, d, e, f > 0$

Starting for any two species competing over the same resource –

$$\dot{x} = x(a - cx - dy) = f_1(x,y)$$
$$\dot{y} = x(b - ex - fy) = f_2(x,y)$$

Rest pts: $f_1(x,y) = 0 \iff x = 0,\ a - cx - dy = 0$

$f_2(x,y) = 0 \iff y = 0,\ b - ex - fy = 0$

Equations: (Simple model)

$$\frac{\dot{x}}{x} = a - cx - dy$$

growth rate

reproduction rate

diminished food supply due to pop of rabbits

diminished food supply due to pop of sheep

$$\frac{\dot{y}}{y} = b - ex - fy$$

Eg: Choose $a = 3$, $b = 2$, $c = 1$, $d = +2$, $e = +1$, $f = +1$ as typical numbers.

## Rest pts:

$x = 0$, $y = 0$ $\longrightarrow$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

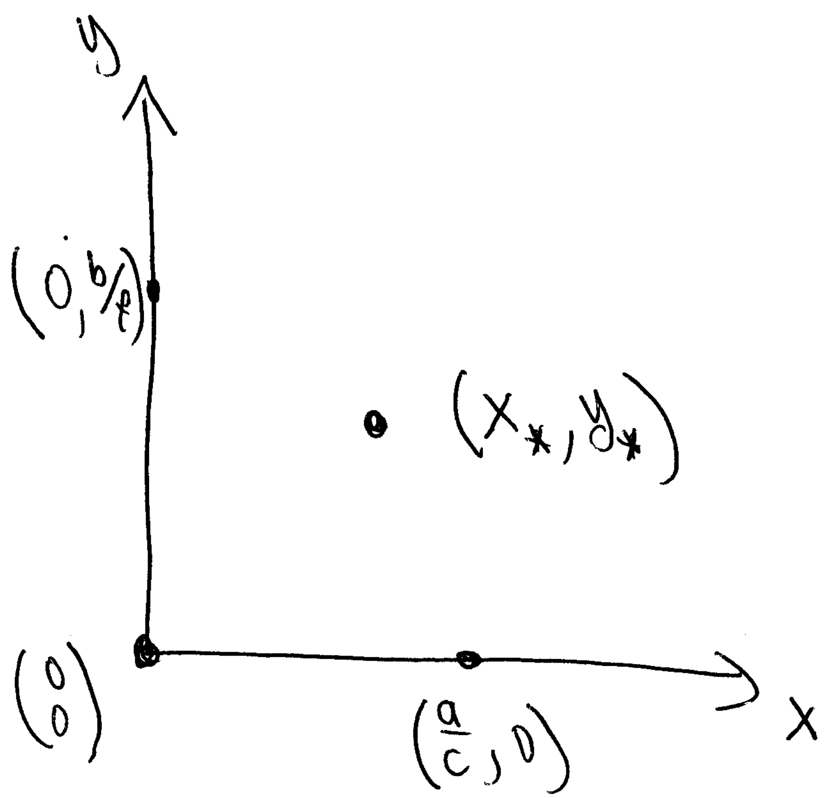
$x = 0$, $b - \cancel{ex} - fy = 0$ $\longrightarrow$ $\begin{pmatrix} 0 \\ b/f \end{pmatrix}$

$a - cx - \cancel{dy} = 0$, $y = 0$ $\longrightarrow$ $\begin{pmatrix} a/c \\ 0 \end{pmatrix}$

$a - cx - dy = 0$, $b - ex - fy = 0$ $\longrightarrow$

$$\underbrace{\begin{pmatrix} c & d \\ e & f \end{pmatrix}}_{A} \begin{pmatrix} x_* \\ y_* \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x_* \\ y_* \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$



Example: $\dot{x} = x(3-x-2y) = f_1(x,y)$

$$\dot{y} = y(2-x-y) = f_2(x,y)$$

Rest pts: $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$

$$\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \Rightarrow \begin{pmatrix}x_*\\y_*\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\nabla f_1 = (3-2x-2y, -2x)$$

$$\nabla f_2 = (-y, 2-x-2y)$$

$Df(0,0) = \begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}$ $\quad \lambda^2 - \tau\lambda + \Delta = \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$

unstable $\quad \lambda = 2, 3 > 0$

$Df\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}-1 & 0\\-2 & -2\end{pmatrix}$ $\quad \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$

stable $\quad \lambda = -1, -2$

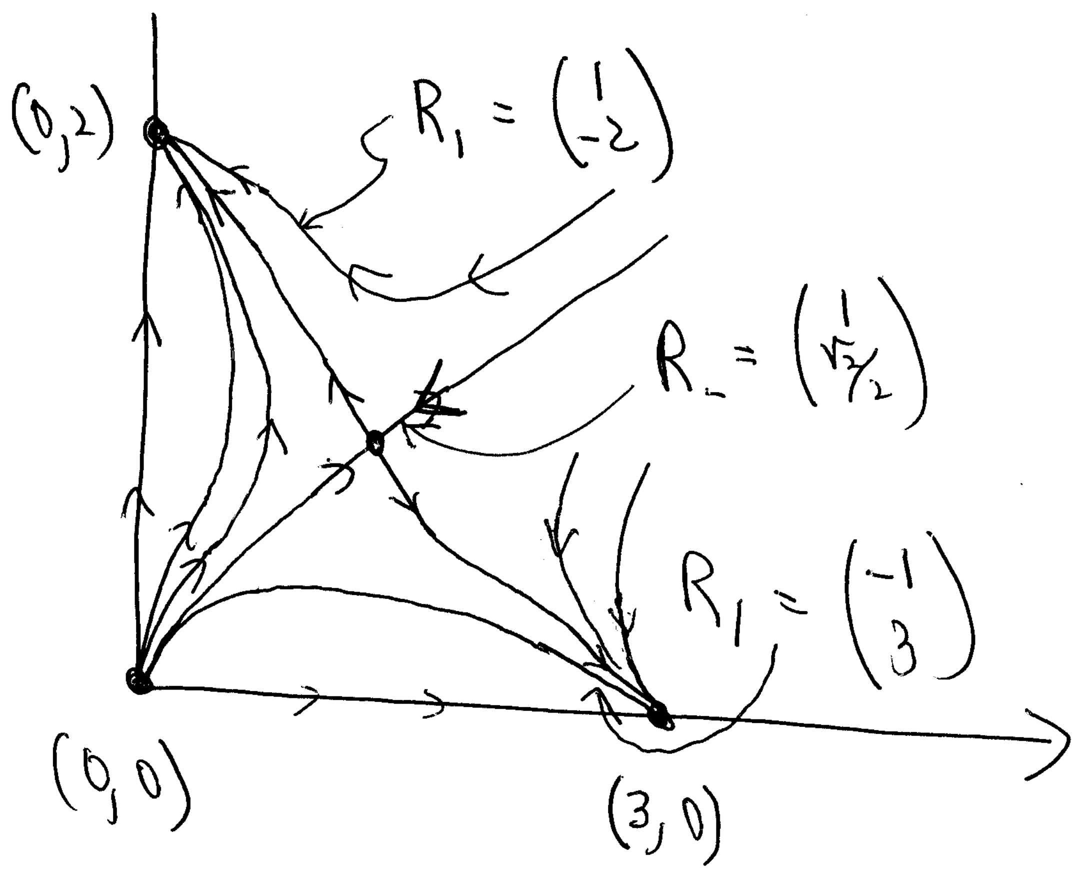
$Df\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}-3 & -6\\0 & -1\end{pmatrix}$ $\quad \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$

stable $\quad \lambda = -1, -3$

$$Df\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1 & -2\\ -1 & -1\end{pmatrix} \qquad \lambda^2 + 2 + (-1)$$

$$\lambda = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

saddle




Recall: $x(t) = c_1 R_1 e^{\lambda_1 t} + c_2 R_2 e^{\lambda_2 t}$ "trajectory tangent to eigenvector associated with smallest $|\lambda|$"

Comments –

- "Slow evector" at $\binom{0}{2}$

$$\begin{pmatrix} -1+1 & 0 \\ -2 & -2+1 \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = 0 \qquad \begin{aligned} -2-r &= 0 \\ r &= -2 \\ R_1 &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{aligned}$$

- "Slow e-vector" at $\binom{3}{0}$

$$\begin{pmatrix} -3+1 & -6 \\ -3 & -1+1 \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = 0 \qquad \begin{aligned} -2-6r &= 0 \\ r &= -1/3 \\ R_1 &= \begin{pmatrix} -3 \\ 1 \end{pmatrix} \end{aligned}$$

- Eigendirections @ $\binom{1}{1}$ : stable $\lambda = -1-\sqrt{2}$

$$\begin{pmatrix} \cancel{-1}+\cancel{1}+\sqrt{2} & -2 \\ -1 & \cancel{-1}+\cancel{1}+\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = 0 \qquad \begin{aligned} \sqrt{2}-2r &= 0 \\ r &= \frac{\sqrt{2}}{2} \\ R_- &= \begin{pmatrix} 1 \\ \sqrt{2}/2 \end{pmatrix} \end{aligned}$$

- $x=0$ & $y=0$ are trajectories $\Rightarrow$ solutions with $x>0, y>0$ cannot go to negative pop's (orbits can't cross)
- Conclude: one species goes extinct!

Defn: Basin of attraction

Defn: stable manifold = separatrices

Defn: Principl of competetive exclusion - "two species competing for the same limited resource cannot co-exist"

Q: why doesn't trajectories along axes come in along "slow" eval?

Ans : the 2nd evect @ $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ is $R_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

the 2nd evect @ $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is $R_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

check : $\begin{pmatrix} -1+2 & 0 \\ -2 & -2+2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$ ✓

$\begin{pmatrix} -3+3 & -6 \\ 0 & -1+3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ ✓