Jan 11 2022

Math 119A : Harmonic Oscillator

Intro to ODE's

Defn: An ODE is an equation for an unknown function $x(t)$ that involves x & derivatives of x.

- $x(t) \in \mathbb{R}$ scalar
- $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$ system

Three fundamental ODE's:

(1) $\frac{dx}{dt} = kx$ $\left(\frac{dx}{dt} \equiv x'(t) \equiv \dot{x}(t)\right.$: for ODE's t is "usually" time$\left.\right)$

(Exponential)

(2) $\frac{dx}{dt} = kx^2$ (Ricotti Equation - simplest nonlinear equation)

(3) $\frac{d^2x}{dt^2} + kx = 0$ (Harmonic oscillator (when $k > 0$))

Note: the fundamental theorem of calculus tells us how to solve only the simplest ODE's —

$$\frac{dx}{dt} = f(t) \quad \text{where } f(t) \text{ is known fn.}$$

(*) $$\boxed{x(t) = x(0) + \int_0^t f(\xi)\, d\xi}$$ formula

- All of 21B was about getting an expression for $\int_0^t f(\xi)\, d\xi$ for special $f(\xi)$

Ex $$\frac{dx}{dt} = \sin t \Rightarrow x(t) = x(0) + \int_0^t \sin\xi \, d\xi$$

$$= x(0) - \cos\xi \Big]_0^t = x(0) - \cos t + 1$$

In general (*) IS the formula!

- The important trick of separation of variables tells how to solve <u>some fundamental</u> ODE's

(1) $\frac{dx}{dt} = kx \iff \int_{x_0}^{x} \frac{dx}{kx} = \int_{0}^{t} dt$

$$\boxed{x(t) = x_0 e^{kt}}$$

Also works for

(2) $\frac{dx}{dt} = kx^2$ (Done last time)

$$\boxed{x(t) = \frac{1}{\frac{1}{x_0} - ct}}$$

Nonlinear $\Rightarrow$ solution cannot be defined for all t : $x(t) \to \infty$ as $t \to \frac{1}{cx_0}$

Often: the ODE a function satisfies is the most important property of a function:

Example: $x(t) = \cos t$, $y(t) = \sin t$.

Q: Why is trigonometry such an important subject?

Fake Ans: Surveying, Navigation, construction ...

Real Ans: Everything in the universe is vibrating, and every vibration, if sufficiently weak, looks like sines & cosines –

<u>Justification</u>: What ODE does $x(t) = \cos t$ satisfy?

$$x(t) = \cos t$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{d^2x}{dt^2} = -\cos t$$

$$\therefore \boxed{\frac{d^2x}{dt^2} + x(t) = 0} \qquad \text{(ODE (3))}$$

Similarly: $x(t) = \sin t$, $x''(t) = -\sin t \Rightarrow$ $\cos t$, $\sin t$ solves same equation.

Similarly - the general soln of

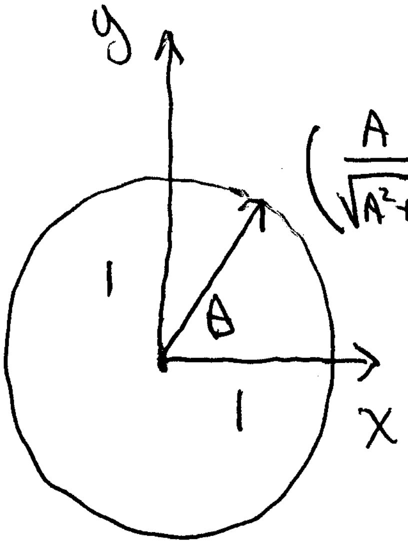
$$\frac{d^2x}{dt^2} + kx = 0 \quad \text{is}$$

$$\boxed{x(t) = A\cos\sqrt{k}\,t + B\sin\sqrt{k}\,t}$$

<u>Claim</u>: $x(t)$ is just a shifted sinusoidal oscillation:

$$x(t) = \sqrt{A^2+B^2}\left\{\frac{A}{\sqrt{A^2+B^2}}\cos\sqrt{k}t + \frac{B}{\sqrt{A^2+B^2}}\sin\sqrt{k}t\right\}$$

unitvector



$$\left(\frac{A}{\sqrt{A^2+B^2}}, \frac{B}{\sqrt{A^2+B^2}}\right) = (\cos\theta, \sin\theta)$$

for some θ

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \Rightarrow$$

$$x(t) = \sqrt{A^2+B^2}\cos\left(\theta - \sqrt{k}t\right)$$

<u>Also</u>: $\cos\alpha = \sin\left(\pi/2 - \alpha\right)$

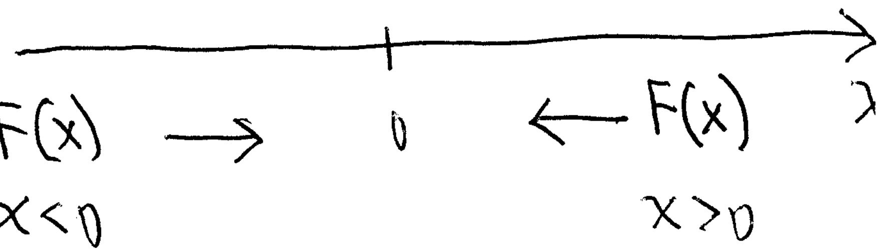
$$\boxed{x(t) = \sqrt{A^2+B^2}\sin\left(\frac{\pi}{2} - \theta + \sqrt{k}t\right)}$$

<u>Conclude</u>: Every solution of $\frac{d^2x}{dt^2} + kx = 0$ is a (rescaled & shifted) sinusoidal oscillation

"Any thing that oscillates in a restoring force, must oscillate sinusoidally when it is sufficiently weak" —

"Proof:" Newton: $m\frac{d^2x}{dt^2} = \vec{F}$

Restoring force:



$$F(x) = \underbrace{\cancel{F(0)}}_{0} + \underbrace{F'(0)}_{-k<0}\,x + \text{Error} \qquad |\text{Error}| \le Cx^2$$

For x sufficiently small, Error $\ll 1$
$\Rightarrow$ neglect it. Thus Newton gives:

$$m\frac{d^2x}{dt^2} = -kx \quad (k>0 \text{ so "restoring"})$$

$$\boxed{\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0} \Rightarrow$$

Solutions of the harmonic oscillator oscillate sinusoidally!

Q: How do you know that
$x(t) = A\cos\sqrt{k}t + B\sin\sqrt{k}t$ gives all soln's
of $\frac{d^2x}{dt^2} + kx = 0$? We need a theory