MATH119AW02
3-5-12

Reversibility

①

§6.6 Reversible Systems – Have properties a lot like conservative systems

- Consider Newton: $m\ddot{x} = f(x) \quad x \in \mathbb{R}$

1st order system –

$$\dot{x} = y$$
$$\dot{y} = \frac{1}{m} f(x)$$

- If $f(x) = -U'(x)$ (just integrate $U(x) = \int_{x_0}^{x} f$)

then $E(t) = \frac{1}{2}\dot{x}^2 + U(x)$ energy const along solns –

- Another property = time-reversal symmetry

"Replace t by $-t$ & $\dot{x} = y$ by $-y$ and equations stay same"

- $s = -t \quad z = -y$

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = -\frac{dx}{dt} = -y = z$$

$$\frac{dz}{ds} = \frac{dz}{dt}\frac{dt}{ds} = -\frac{dy}{dt}(-1) = \frac{dy}{dt} = \frac{1}{m}f(x)$$

$' \equiv \frac{d}{ds} \Rightarrow \boxed{\begin{aligned} x' &= z \\ z' &= \frac{1}{m}f(x) \end{aligned}}$ same equations with different variables

(*) Same as replacing $t \to -t$ & $y \to -y$ ✓

Defn: A general 2×2 system

$$\dot{x} = f_1(x,y)$$
$$\dot{y} = f_2(x,y)$$

is time reversible if equations are invariant under $t \mapsto -t$, $y \mapsto -y$

Ex 1 $\dot{x} = y$

$\dot{y} = \frac{1}{m} f(x)$

Newton's eqn's are time reversible (no friction)

Ex 2 If f_1 is odd in y

$$f_1(x,-y) = -f_1(x,y)$$

and f_2 is even in y

$$f_2(x,-y) = f_2(x,y)$$

$$\frac{dx}{ds} = -\dot{x} = f_1(x,y) = -f_1(x,-y)$$

$$\boxed{x' = f_1(x,z)}$$

$s = -t$

$z = -y$

$$\frac{dy}{ds} = -\frac{dy}{dt} = -\left(-\frac{d\cancel{z}}{dt}\right) = f_2(x,y) = f_2(x,-y)$$

$$= f_2(x,z)$$

$$\boxed{\cancel{y}' = f_2(x,z)}$$

□ <u>Note</u>: In $\mathbb{R}^3$, Newton's Laws can be time reversible but <u>not</u> conservative.

<u>Eg</u> $m\vec{a} = \vec{F}$ $\quad \underset{\sim}{x}(t) = (x(t), y(t), z(t))$

$$m\ddot{\underset{\sim}{x}} = \vec{F}(\underset{\sim}{x}) \leftarrow \text{(no friction)}$$

1st order system:

6 equn's in 6 unknowns.
$$\begin{cases} \dot{\underset{\sim}{x}} = \underset{\sim}{y} \\ \dot{\underset{\sim}{y}} = \frac{1}{m}\vec{F}(\underset{\sim}{x}) \end{cases}$$

Conservative $\Leftrightarrow$ $\vec{F} = -\nabla U(\underset{\sim}{x}) \Leftrightarrow$ "$\vec{F}$ is a conservative vector field" $\Leftrightarrow$ Curl $\vec{F} = 0$

$\ni$ Most $\vec{F}$ <u>not</u> conservative in $\mathbb{R}^3$

- Time reversible: $t \mapsto -t \quad \underset{\sim}{y} \mapsto -\underset{\sim}{y}$

$$\begin{aligned} \dot{\underset{\sim}{x}} &= \underset{\sim}{y} \\ \dot{\underset{\sim}{y}} &= \tfrac{1}{m}\vec{F}(\underset{\sim}{x}) \end{aligned} \quad \longmapsto \quad \begin{aligned} -\dot{\underset{\sim}{x}} &= -\underset{\sim}{y} \\ -(\dot{-\underset{\sim}{y}}) &= \tfrac{1}{m}\vec{F}(\underset{\sim}{x}) \end{aligned}$$

same

"Most $\vec{F}$ are not conservative, but equ's are time reversible"

Back to 2x2 systems:

Defn: The 2x2 system

$$\dot{x} = f_1(x,y) \qquad (1)$$
$$\dot{y} = f_2(x,y)$$

is time reversible if it is invariant under the replacement $t \to -t$ $y \to -y$

Thm: (1) is time reversible if f_1 is odd in y and f_2 is even in y.

Ie $t \to -t$ $y \to -y$

$$-\dot{x}(t) = \dot{x}(-t) = f_1(x,-y) = -f_1(x,y)$$

cancels if $f_1(x,-y) = -f_1(x,y)$ odd

$$-(-\dot{y}(t)) = -\dot{y}(-t) = f_2(x,-y) = f_2(x,y)$$

no minus if $f_2(x,-y) = f_2(x,y)$ even

✓

<u>Cor</u>: If (1) is time-reversible, then

$(x(t), y(t))$ solves (1) iff $(x(-t), -y(-t))$ does
$a \le t \le b$ $\qquad\qquad\qquad\qquad$ $-b \le t \le -a$

Pf. Be careful: set $s = -t$. then

$$\bar{x}(s) = \bar{x}(-t) = x(t)$$
$$\bar{y}(s) = \bar{y}(-t) = y(t)$$

we show $(x(t), y(t))$ solves (1) iff $(\bar{x}(s), -\bar{y}(s))$ does:

$$\dot{x}(t) = \frac{d}{dt}x(t) = \frac{ds}{dt}\frac{d}{ds}x(-s) = -\frac{d}{ds}\bar{x}(s)$$

$$f_1(x(t), y(t)) = f_1(x(-s), y(-s)) = f_1(\bar{x}(s), \bar{y}(s)) = -f_1(\bar{x}(s), -\bar{y}(s))$$

$$\Rightarrow \dot{x}(t) = f_1(x(t), y(t)) \quad \underset{s=-t}{\text{iff}} \quad \dot{\bar{x}}(s) = f_1(\bar{x}(s), -\bar{y}(s))$$

Similarly:

$$\dot{y}(t) = \frac{d}{dt}y(t) = \frac{ds}{dt}\frac{d}{ds}y(-s) = -\frac{d}{ds}\bar{y}(s)$$

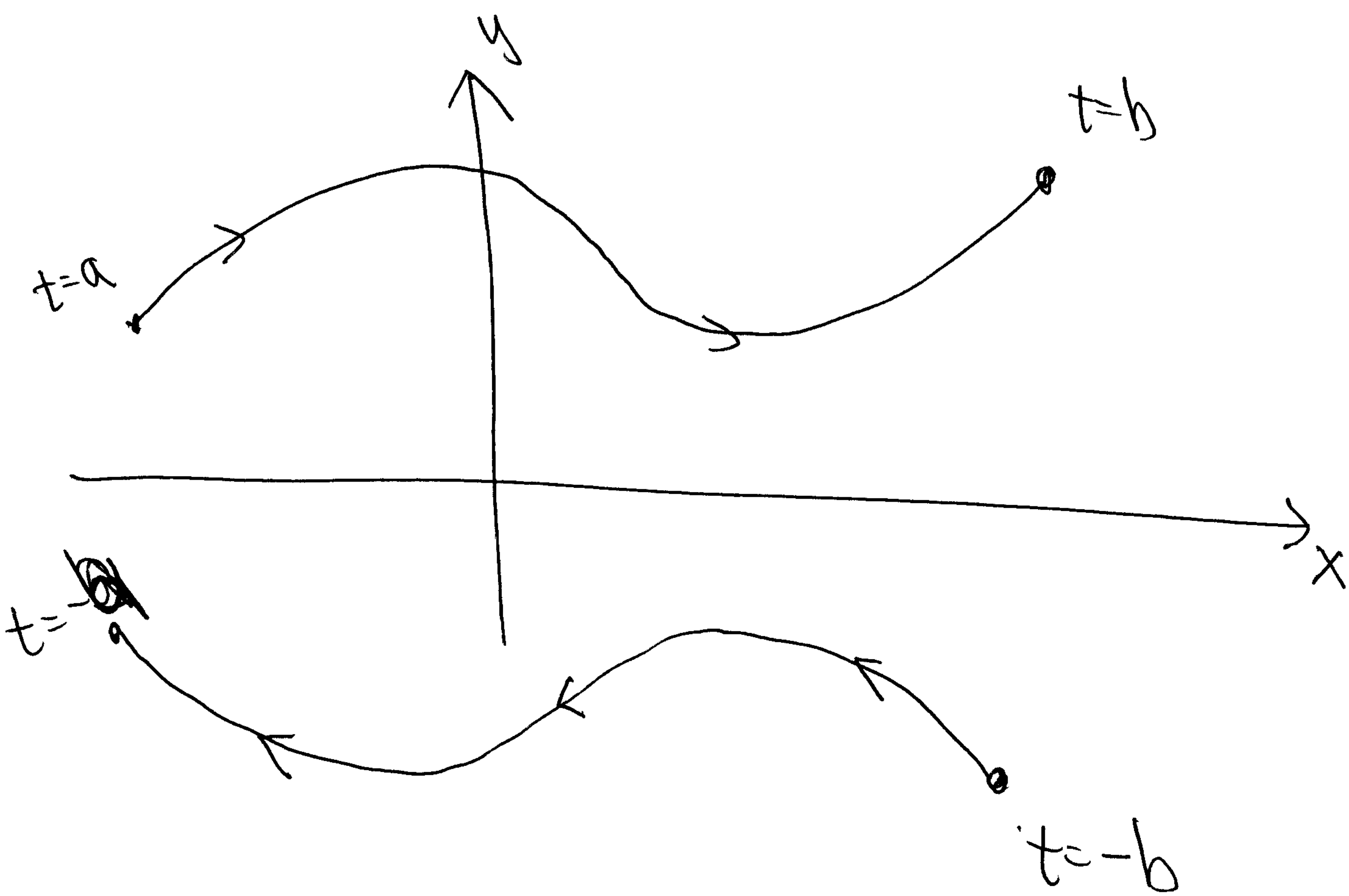
$$f_2(x(t), y(t)) = f_2(x(-s), y(-s)) = f_2(\bar{x}(s), -\bar{y}(s))$$

$$\Rightarrow \dot{y}(t) = f_1(x(t), y(t)) \quad \underset{s=-t}{\text{iff}} \quad -\dot{\bar{y}} = f_2(\bar{x}(s), -\bar{y}(s))$$

Picture: $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ is reversible $\Leftrightarrow$

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ solves (1) iff $\begin{pmatrix} x(-t) \\ -y(-t) \end{pmatrix}$ does

$a \le t \le b$ $\qquad\qquad$ $-b \le -t \le -a$



"Same solution running backwards"

<u>Theorem:</u> (Nonlinear Centers perturb)

if $\bar{\underset{\sim}{x}} = \underset{\sim}{0}$ is an isolated rest pt for $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$

& linearized equations $\dot{\underset{\sim}{x}} = Df(\underset{\sim}{0})$

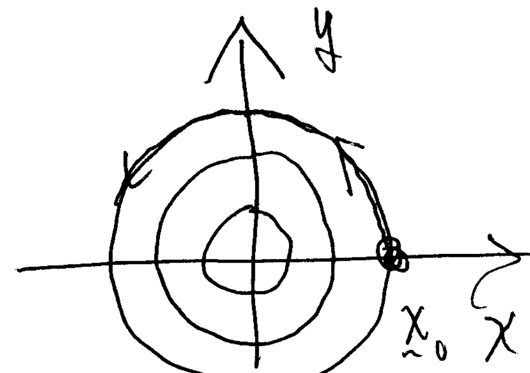
is a center at $\bar{\underset{\sim}{x}} = \underset{\sim}{0}$, then

nearby nonlinear trajectory are

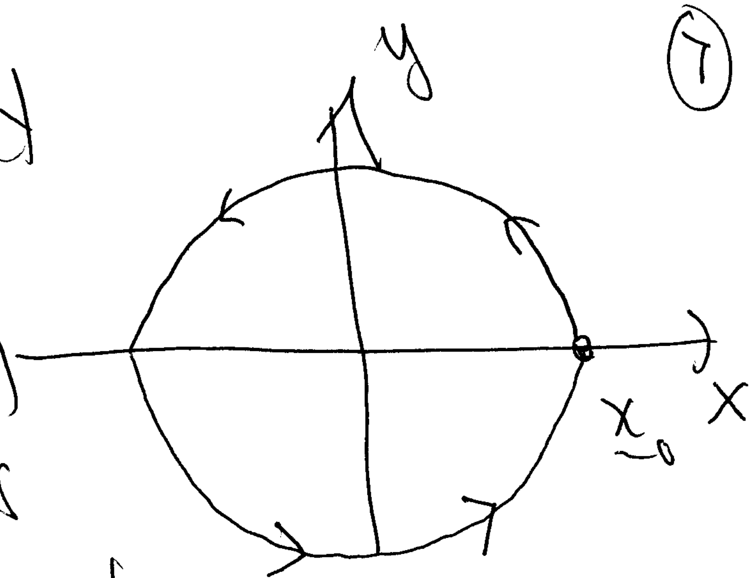
closed orbits.

Pf. Linearized Equations

$\lambda = \pm ib \Rightarrow$ closed orbits



• Start with $\underset{\sim}{x}_0$ on x-axis near $\bar{\underset{\sim}{x}} = \underset{\sim}{0}$. Since linearized equation dominates near $\underset{\sim}{x} = \underset{\sim}{0}$, if x_0 is suff close to $\underset{\sim}{0}$, then orbit must approx the linearized orbit –



$\Rightarrow$ orbit must intersect the negative x-axis.
(Plausible, not hard to prove)

- But the reflection is also an orbit, & no rest pts
- Orbit at $y>0$ must continue to orbit at $y<0$ because otherwise two distinct orbits intersect $\#$

Ex: Prove solutions of $\dot{x} = y - y^3$, $\dot{y} = -x - y^2$ form closed solutions near $(x,y) = \underset{\sim}{0}$.

Soln: $t \mapsto -t \quad y \mapsto -y \Rightarrow$

$$-\dot{x} = (-y) - (-y)^3$$

$$\dot{x} = y - y^3 \checkmark$$

$$-(-\dot{y}) = -x - (-y)^2$$

$$\dot{y} = -x - y^2$$

Equations time-reversible.

Linearize at $(0,0)$: $Df(0,0) = \begin{pmatrix} 0, & 1-3y^2 \\ -1, & -2y \end{pmatrix}_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\lambda^2 - \tau\lambda + \Delta = \lambda^2 + 1 = 0 \quad \lambda = \pm i$ center

& apply Thm ✓

Ex: Show system $\dot{x} = y$, $\dot{y} = x - x^2$ has a homoclinic orbit at $(0,0)$.

Defn: homoclinic orbit starts & ends at same rest pt.

Defn: heteroclinic orbit starts at one rest pt ends at another.

Soln: $\begin{aligned} -\dot{x} &= -y \\ -(-\dot{y}) &= x - x^2 \end{aligned} \iff \begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^2 \end{aligned} \Rightarrow$ reversible.

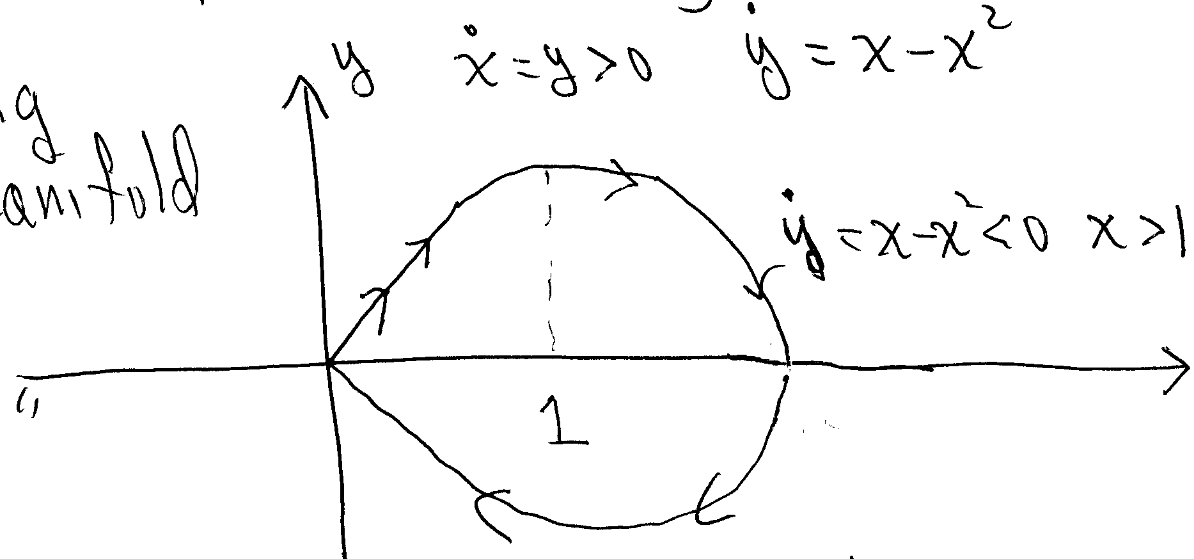
Linearized equations: $\dot{\underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underset{\sim}{x}$

$$\lambda^2 - \tau\lambda + \Delta = \lambda^2 - 1 \qquad \lambda = \pm 1$$

$\left(-1, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ & $\left(+1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ are e-pairs

- Linearized eqn's near $\begin{pmatrix}0\\0\end{pmatrix}$:

"orbit along unstable manifold must hit x-axis..."



- Reversible $\Rightarrow$ $\exists$ reflected solution <u>must be</u> the continuation of the orbit.