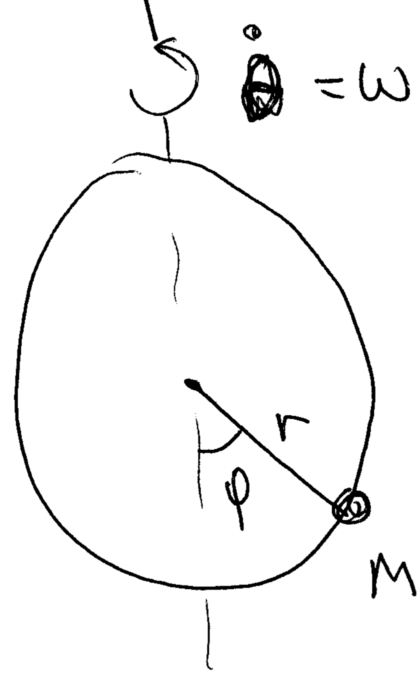
Equations of Physics —

Lagranges Principle of Least Action

- Recall bead on rotating hoop with friction —



- We derived equations from Newton's Laws —

$$m\vec{a} = \vec{F}$$

Derivation more complicated because of the <u>restoring</u> <u>force</u> that holds mass to hoop —

"Balance of forces" — $\vec{F}$ = gravity + centripidal force + force that holds mass to hoop —

Equations we derived –

$$\underbrace{mr\ddot{\varphi}}_{m\vec{a}} = \underbrace{-b\dot{\varphi}}_{F_f} \underbrace{- mg\sin\phi}_{F_g} + \underbrace{mr^2\omega^2\sin\varphi\cos\varphi}_{F_h}$$

$$m\vec{a} = F_f + F_g + F_h$$

F_f = Friction Force

F_g = Grav. Force

F_h = centripedal force + restoring force on hoop

Q: Is there a <u>systematic</u> way to derive the equations of motion <u>with</u> <u>constraints</u> (like hoop)

Ans: Yes when energy is conserved (holonomic constraints)

Not so easy otherwise (non-holonomic constraints)

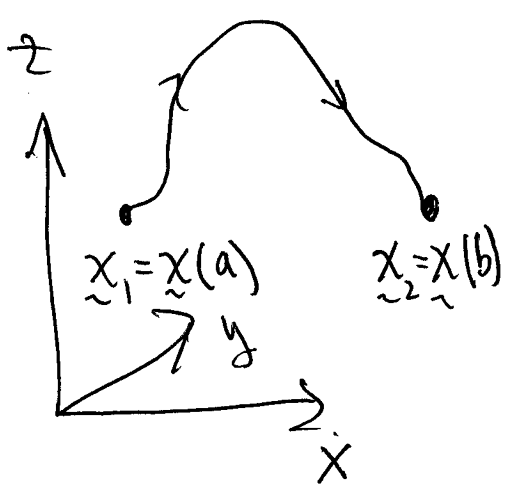
For us we need $b=0$ (no friction) to get cons of energy.

- Lagranges Idea – write Newton's Laws in a framework that is independent of coordinates. How? Express them as a variational = minimization principle.

- <u>Calculus of Variations</u>:

<u>Problem</u>: Given a function

$$L(\underset{\sim}{x}, \dot{\underset{\sim}{x}}) \equiv \text{"Lagrangian"}$$



find the equations for the curve $\underset{\sim}{x}(t)$ taking $\underset{\sim}{x}(a) = \underset{\sim}{x}_1$, $\underset{\sim}{x}(b) = \underset{\sim}{x}_2$ that "minimizes the action":

$$A[\underset{\sim}{x}(t)] = \int_a^b L(\underset{\sim}{x}(t), \dot{\underset{\sim}{x}}(t))\, dt$$

<u>Idea</u>: whatever curve minimizes the action in $\underset{\sim}{x} = (x,y,z)$ coordinates, will minimize the action in <u>any</u> <u>other</u> <u>coords</u>:

I.e., say $\underset{\sim}{x} = \Phi(\underset{\sim}{q}) = \begin{pmatrix} \varphi_1(\underset{\sim}{q}) \\ \varphi_2(\underset{\sim}{q}) \\ \varphi_3(\underset{\sim}{q}) \end{pmatrix}$

$$\Phi : (x,y,z) \longmapsto (q_1, q_2, q_3)$$

[Eg, $\underset{\sim}{q}$ could be spherical coords: $q_1 = \rho$, $q_2 = \phi$, $q_3 = \theta$]

Chain rule for curve — $\underset{\sim}{x} = \underset{\sim}{x}(t)$

$$\underset{\sim}{x}(t) = \Phi(\underset{\sim}{q}(t))$$

$$\dot{\underset{\sim}{x}}(t) = D\Phi \cdot \dot{\underset{\sim}{q}}(t) \qquad D\Phi = \begin{bmatrix} - \nabla\varphi_1 - \\ - \nabla\varphi_2 - \\ - \nabla\varphi_3 - \end{bmatrix}$$

Conclude:

$$A[\underset{\sim}{x}(t)] = \int_a^b L(\underset{\sim}{x}(t), \dot{\underset{\sim}{x}}(t))\,dt = \int_a^b L(\Phi(\underset{\sim}{q}(t)), D\Phi\,\mathring{\underset{\sim}{q}}(t))\,dt$$

$$= \int_a^b \bar{L}(\underset{\sim}{q}(t), \mathring{\underset{\sim}{q}}(t))\,dt = A[\underset{\sim}{q}(t)]$$

"The action is indepedent of coordinates
$\Rightarrow$ the curve that minimizes the action
is indept of coords".

<u>Theorem</u>: (Calculus of Variations) The curve $\underset{\sim}{x}(t)$ minimizes the action [really is a <u>critical point</u> of the action) iff

$$(EL)\qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \qquad \underset{\sim}{x} = (x^1, x^2, x^3)$$

Ie: (EL) is the ODE for the curve $\underset{\sim}{x}(t)$ that minimizes the action.

<u>Defn</u>: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$ are the Euler–Lagrange equations.

<u>Cor</u>: $\underset{\sim}{x}(t) = \Phi\,\underset{\sim}{q}(t)$ solves (EL) iff $\underset{\sim}{q}(t)$ satisfies

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^i} - \frac{\partial \bar{L}}{\partial q^i} = 0$$

Pf: $\underset{\sim}{q}(t)$ minimizes the action for $\bar{L}$ ✓

Theorem: (Calculus of Variations) The curve $\underset{\sim}{x}(t)$ minimizes the action [really is a critical point of action] iff $\underset{\sim}{x}(t)$ satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0, \quad \underset{\sim}{x} = (x^1, x^2, x^3) \qquad \text{(EL)}$$

More precisely: (EL) is the ODE for the curve $\underset{\sim}{x}(t)$ that is a critical point of the action in the sense that

$$0 = \frac{d}{d\varepsilon} A[\underset{\sim}{x}(t) + \varepsilon \eta(t)]\Big|_{\varepsilon=0} \quad \forall\, \eta(t)$$

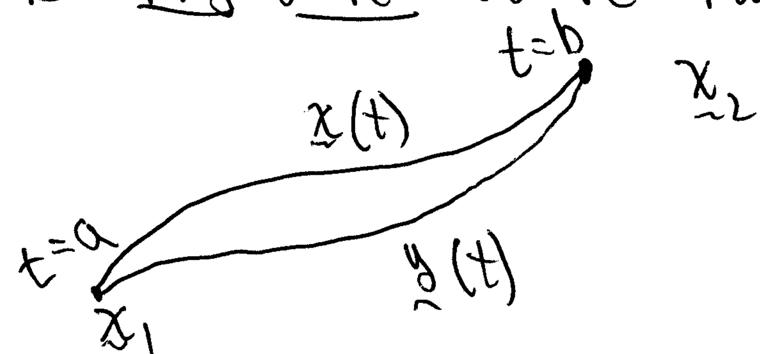
$\eta(t)$ any smooth curve $\quad \eta : [a,b] \longrightarrow \mathbb{R}^3$

so that $\quad \eta(a) = \eta(b) = 0$

$\underset{\sim}{y}(t) = \underset{\sim}{x}(t) + \varepsilon \eta(t)$ is any other curve taking

$\underset{\sim}{y}(a) = \underset{\sim}{x}_1$

$\underset{\sim}{y}(b) = \underset{\sim}{x}_2$



- Pf. Not so hard: See Modern Geometry — Methods & Applications Vol I Dubrovin Fomenko, Novikov

- **Defn:** $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$ are called the Euler-Lagrange Equations

- **Cor:** $\underset{\sim}{x}(t) = \Phi\,\underset{\sim}{q}(t)$ solves (E-L) iff $\underset{\sim}{q}(t)$ satisfies

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^i} - \frac{\partial \bar{L}}{\partial q^i} = 0$$

Pf. " $\underset{\sim}{q}(t)$ minimizes the action for $\bar{L}$ "

"The action $A[\underset{\sim}{x}(t)] = \bar{A}[\underset{\sim}{q}(t)]$ is indep of what coordinates you express the curve in"

<u>Conclude</u>: The (EL) equations are coordinate independent equations !

I.e. Given coordinates $\underset{\sim}{q}$, "find the Lagrangian $\bar{L}(\underset{\sim}{q}, \dot{\underset{\sim}{q}})$" and the equations are

$$\boxed{\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^i} - \frac{\partial \bar{L}}{\partial q^i} = 0}$$

Idea to make Newton's Laws coord independent - Find the Lagrangian.

<u>Newton</u>: $m\vec{a} = \vec{F}$ (too general)

<u>Assume Conservative</u>:

$$m\vec{a} = -\nabla U \iff \boxed{\ddot{\underset{\sim}{x}} = -\frac{1}{m}\nabla U(\underset{\sim}{x})}$$

<u>**Theorem**</u>: This works with

$$L(\underset{\sim}{x}, \dot{\underset{\sim}{x}}) = KE - PE = \frac{1}{2} m |\dot{\underset{\sim}{x}}|^2 - U(\underset{\sim}{x})$$

<u>**Proof**</u>:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^i}\left\{\frac{1}{2} m \left((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2\right) - \cancel{U(x^1, x^2, x^3)}\right\}$$

$$- \frac{\partial}{\partial x^i}\left\{\frac{1}{2} m \left(\cancel{(\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2}\right) - U(x^1, x^2, x^3)\right\}$$

$$= \frac{d}{dt}\left(m \dot{x}^i\right) - \frac{\partial U}{\partial x^i} = m \ddot{x}^i - \frac{\partial U}{\partial x^i}$$

<u>or</u>

$$\boxed{m \ddot{\underset{\sim}{x}} - \nabla U(\underset{\sim}{x}) = 0}$$ ✓

<u>Conclude</u>: Newton's Laws in a conservative force field are equivalent to

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{x}^i}}_{\text{"generalized momentum"}} - \underbrace{\frac{\partial L}{\partial x^i}}_{\text{"generalized force"}} = 0$$

<u>Conclude</u>: To get the equations of motion in a different coordinate system $\underset{\sim}{x} = \Phi \underset{\sim}{q}$, just find the Lagrangian

$$\bar{L}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}) = \frac{1}{2} m |D\Phi \dot{\underset{\sim}{q}}|^2 - U(\Phi(q))$$

$= KE - PE$ in $\underset{\sim}{q}$-coordinate

$\Rightarrow$ Equations: $\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^i} - \frac{\partial \bar{L}}{\partial q^i} = 0$ ✓

☐ Problem: What happens when there are constaints?

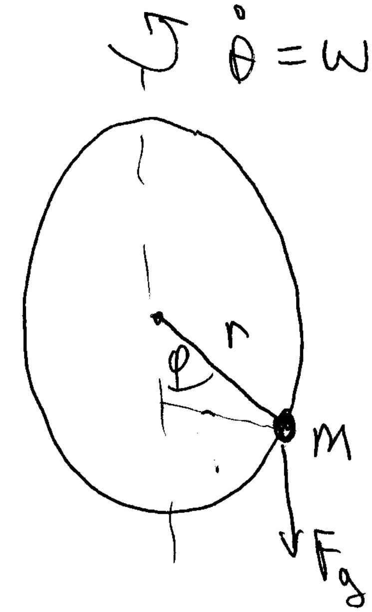
Defn: We call the constraints holonomic if the Lagrangian $L(\underline{q}, \dot{\underline{q}})$ is the constrained KE−PE & the equations are

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}^i} L - \frac{\partial}{\partial q^i} L = 0 .$$

Thm: "Whenever masses are constrained to move along surfaces with infinite restoring force in an otherwise conservative force field" the constraints are holonomic.

Ex: Pendulum, Bead on Rotating Hoop, ramps, ... (without friction!) [c.f. Arnold]

<u>Ex</u>: Derive equations for bead on rotating hoop w/o friction



- m moves in conservative gravitational forcefield constrained to rotating hoop $\Rightarrow$ holonomic.

- $KE = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m |\dot{\underset{\sim}{x}}|^2$ $\quad \underset{\sim}{x} = (x, y, z) = (x^1, x^2, x^3)$

  $PE = U(\underset{\sim}{x}) = mgz$

(i.e. $-\nabla U = (0, 0, -mg) = -mg \underset{\sim}{k}$ $\downarrow$ ✓)

- Coordinates: $(r, \varphi, \theta) = (q^1, q^2, q^3) = \underset{\sim}{q}$

(Spherical Coordinates)

- $x = R\cos\theta = r\sin\varphi\cos\theta$

  $y = R\sin\theta = r\sin\varphi\sin\theta$

  $z = -r\cos\varphi$ (z measured from center of hoop!)

$\Rightarrow \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{1}{2}mr^2(\dot{\varphi}^2 + \omega^2\sin^2\varphi)$

- $U(\underset{\sim}{q}) = mgz = -mgr\cos\varphi$

$KE - PE = \frac{1}{2}mr^2(\dot{\varphi}^2 + \omega^2\sin^2\varphi) + mgr\cos\varphi = L(q, \dot{q})$

Here constrained variable is $q = \varphi$

<u>Equations</u>: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \overset{\bullet}{\overbrace{mr^2\dot{\varphi}}} = mr^2\ddot{\varphi}$$

$$\frac{\partial L}{\partial q} = mr^2\omega^2\sin\varphi\cos\varphi - mgr\sin\varphi$$

Equations:

$$\boxed{m\cancel{r^2}\ddot{\varphi} = -mg\cancel{r}\sin\varphi + m\cancel{r^2}\omega^2\sin\varphi\cos\varphi}$$

"Never had to balance the forces"

Lagrangian Formalism:

Newton (Conservation Force Field)

$$m\ddot{\underset{\sim}{x}} + \nabla U = 0 \qquad \vec{F} = -\nabla U \ (N)$$

$$m\ddot{x}^i + \frac{\partial}{\partial x^i} U = 0 \qquad F_i = -\frac{\partial}{\partial x^i} U$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^i}\left(\frac{1}{2}m|\dot{\underset{\sim}{x}}|^2\right) + \frac{\partial}{\partial x^i} U = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^i}\overbrace{\left(\frac{1}{2}m|\dot{\underset{\sim}{x}}|^2 - U\right)}^{L} - \frac{\partial}{\partial x^i}\overbrace{\left\{\frac{1}{2}m|\dot{x}|^2 - U\right\}}^{L} = 0$$

$$\underbrace{\frac{d}{dt}\frac{\partial}{\partial \dot{x}^i} L}_{\text{generalized momentum}} - \underbrace{\frac{\partial}{\partial x^i} L}_{\text{generalized force}} = 0$$

Vector form: $\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{x}}} - \frac{\partial}{\partial \underset{\sim}{x}} L = 0}$

What about the energy?

For (N), $E = \frac{1}{2} m |\dot{\underset{\sim}{x}}|^2 + U(\underset{\sim}{x})$ ⇒ write in terms of L

$$= m|\dot{\underset{\sim}{x}}|^2 - \left(\frac{1}{2} m |\dot{\underset{\sim}{x}}|^2 - U(\underset{\sim}{x})\right)$$

$$= m|\dot{\underset{\sim}{x}}|^2 - L$$

$$m|\dot{\underset{\sim}{x}}|^2 = m\left((\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2\right)$$

$$= \dot{\underset{\sim}{x}} \cdot m\dot{\underset{\sim}{x}} = \dot{\underset{\sim}{x}} \cdot \frac{\partial L}{\partial \dot{\underset{\sim}{x}}}$$

Good Guess: Generalized Energy

$$E = \dot{\underset{\sim}{x}} \cdot \frac{\partial L}{\partial \dot{\underset{\sim}{x}}} - L = \left(\sum_{i=1}^{3} \dot{x}^i \frac{\partial L}{\partial \dot{x}^i}\right) - L$$

Theorem: $E = \dot{\underset{\sim}{x}} \cdot \frac{\partial L}{\partial \dot{\underset{\sim}{x}}} - L$ is constant along Soln's of (EL) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{x}}} - \frac{\partial L}{\partial \underset{\sim}{x}} = 0$.

Proof: Set $E(t) = \dot{\underset{\sim}{x}} \cdot \frac{\partial L}{\partial \dot{\underset{\sim}{x}}} - L(\underset{\sim}{x})$ on soln $\underset{\sim}{x}(t)$

$$\frac{dE}{dt} = \cancel{\ddot{\underset{\sim}{x}} \cdot \frac{\partial L}{\partial \dot{\underset{\sim}{x}}}} + \dot{\underset{\sim}{x}} \cdot \frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{x}}} \underbrace{- \frac{d}{dt} L(\dot{\underset{\sim}{x}}, \underset{\sim}{x})}_{-\cancel{\frac{\partial L}{\partial \dot{\underset{\sim}{x}}} \ddot{\underset{\sim}{x}}} - \frac{\partial L}{\partial \underset{\sim}{x}} \cdot \dot{\underset{\sim}{x}}}$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{x}}} - \frac{\partial L}{\partial \underset{\sim}{x}}\right) \cdot \dot{\underset{\sim}{x}} = 0 \checkmark$$

☒ Nice way to write Lagranges Equn's as a 1st order system:

- $E = \dot{x}\dfrac{\partial L}{\partial \dot{x}} - L \qquad p = \dfrac{\partial L}{\partial \dot{x}}$

- "Solve" $p = \dfrac{\partial L}{\partial \dot{x}}$ for $\dot{x} = f(x,p)$

- Then $E = f(x,p)\,p - L(x, f(x,p)) \equiv H(x,p)$

- $$\frac{\partial H}{\partial x} = \cancel{\frac{\partial f}{\partial x}p} - \frac{\partial L}{\partial x} - \cancel{\underbrace{\frac{\partial L}{\partial \dot{x}}}_{p}\frac{\partial f}{\partial x}} = -\frac{\partial L}{\partial x}$$

But $\dfrac{d}{dt}\underbrace{\dfrac{\partial L}{\partial \dot{x}}}_{p} = \dfrac{\partial L}{\partial x} \quad \Rightarrow \quad = -\dot{p}$

$$\frac{\partial H}{\partial p} = \cancel{\frac{\partial f}{\partial p}p} + f(x,p) - \cancel{\frac{\partial L}{\partial \dot{x}}\frac{\partial f}{\partial p}} = \dot{x}$$

Conclude: as 1st order system:

$$(*) \quad \boxed{\begin{aligned} \dot{x} &= H_p \\ \dot{p} &= -H_x \end{aligned}} \qquad \text{Hamilton's Eqn's}$$

Note: Just write the energy E as a fn of (x, p) & $(*)$ is the 1st order system

Eg: $m\ddot{x} = -\nabla U(x) \qquad E = \frac{1}{2}m\dot{x}^2 + U(x)$

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad\qquad = \frac{p^2}{2m} + U(x)$

$$\dot{x} = +\frac{\partial}{\partial p}\left(\frac{p^2}{2m} + U(x)\right) = +\frac{p}{m}$$

$$\dot{p} = -\frac{\partial}{\partial x}\left(\frac{p^2}{2m} + U(x)\right) = -U'(x)$$

(2) Newton's Laws in conservative force field has an energy that is conserved:

$$m\ddot{\underset{\sim}{x}} = -\nabla U(\underset{\sim}{x}) \Rightarrow E = \frac{1}{2}m|\dot{\underset{\sim}{x}}|^2 + U(\underset{\sim}{x})$$

$$\frac{dE}{dt}(\underset{\sim}{x}(t)) = 0 \quad \text{on solns of } m\ddot{\underset{\sim}{x}} = -\nabla U(\underset{\sim}{x})$$

Lagranges method shows that many other types of ODE's have an energy —

Ex: Set $L(x, \dot{x}) = x\dot{x}^2 - x^3$

(EL) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}(2x\dot{x}) - \dot{x}^2 + 3x^3$$

$$= 2x\ddot{x} + 2\dot{x}^2 - \dot{x}^2 + 3x^3$$

$$= 2x\ddot{x} + \dot{x}^2 + 3x^3 = 0$$

$$\boxed{2x\ddot{x} + \dot{x}^2 + 3x^3 = 0} \qquad (1)$$

• Solutions of (1) have the energy

$$E = \dot{x}\,\frac{\partial L}{\partial \dot{x}} - L. \qquad \frac{\partial L}{\partial \dot{x}} = 2x\dot{x}$$

$$= \dot{x}\,(2x\dot{x}) - (x\dot{x}^2 - x^3)$$

$$= 2x\dot{x}^2 - x\dot{x}^2 + x^3$$

$$\boxed{E = x\dot{x}^2 + x^3} \qquad (2)$$

<u>Thm</u>: "E is constant along soln's of (1)"

<u>Check</u>:

$$\frac{dE(x(t))}{dt} = 2x\dot{x}\ddot{x} + \dot{x}^3 + 3x^2\dot{x}$$

$$2x\ddot{x} = -\dot{x}^2 + 3x^3$$

$$\frac{dE}{dt} = \dot{x}(-\dot{x}^2 - 3x^3) + \dot{x}^3 + 3x^2\dot{x} = 0 \checkmark$$

- Write (1) as an equivalent (1st order) Hamiltonian system —

<u>Soln</u>: write E as a fn of (x,p) by solving $\dot{x} = f(x,p)$ and

$$E(x,\dot{x}) = E(x, f(x,p)) = H(x,p)$$

$$\dot{x} = H_p$$
$$\dot{p} = -H_x$$

I.e., $p = \frac{\partial L}{\partial \dot{x}} = 2x\dot{x} \Rightarrow \dot{x} = \frac{p}{2x}$ (3)

$$E(x,\dot{x}) = x\dot{x}^2 + x^3 = x\,\frac{p^2}{4x^2} + x^3 = \frac{p^2}{4x} + x^3$$

$$\boxed{H(x,p) = \frac{p^2}{4x} + x^3}$$

$$\boxed{\begin{aligned} \dot{x} &= H_p = \frac{2p}{4x} = \frac{p}{2x} \\ \dot{p} &= -H_x = +\frac{p^2}{4x^2} - 3x^2 \end{aligned}}$$

1st order System $\equiv$ (1)

(4)

Check: $2x\ddot{x} = -\dot{x}^2 - 3x^3$

$$\dot{x} = \frac{p}{2x} \Leftrightarrow (3)$$

$$\ddot{x} = \frac{\dot{p}}{2x} - \frac{p}{2x^2}\dot{x}$$

$$= \frac{\dot{p}}{2x} - \frac{p}{2x^2}\frac{p}{2x}$$

$$\cancel{2x}\left(\frac{\dot{p}}{\cancel{2x}} - \frac{p^2}{2x^2\cancel{2x}}\right) = -\left(\frac{p}{2x}\right)^2 - 3x^3$$

$$\dot{p} = \underbrace{\frac{p^2}{2x^2} - \frac{p^2}{4x^2}}_{\frac{p^2}{4x^2}} - 3x^3$$

✓