Mon 3-12-12 1199

Bifurcation in rotating hoop

Bead on rotating hoop - phase portrait —

$$mr\ddot{\varphi} = mg\sin\varphi + mr\omega^2\sin\varphi\cos\varphi - \underbrace{b\dot{\varphi}}_{\text{friction}}$$

Non-dim: $\tau = \frac{t}{T} \Rightarrow \dot{} = \frac{d}{d\tau}$

$$m\frac{r}{T^2}\ddot{\varphi} = -mg\sin\varphi + mr\omega^2\sin\varphi\cos\varphi - \frac{b}{T}\dot{\varphi}$$

$$\ddot{\varphi} = -\frac{gT^2}{r}\sin\varphi + \frac{\cancel{r}\omega^2 T^2}{\cancel{r}}\sin\varphi\cos\varphi - \frac{bT^{\cancel{2}}}{mr\cancel{T}}\dot{\varphi}$$

Choose: $\frac{gT^2}{r} = 1 \Leftrightarrow T = \sqrt{\frac{r}{g}}$

$$k = \omega^2T^2 = \frac{mr\omega^2}{mg} = \frac{\text{"centripidal force"}}{\text{grav force}} \; ; \; \mu = \frac{bg}{r}$$

$$\Leftrightarrow \boxed{\ddot{\varphi} = \sin\varphi(k\cos\varphi - 1) - \mu\dot{\varphi}}$$

non-dimensional

Assume: <u>No friction</u> $\mu = 0 = b$

$$\ddot{\varphi} = \sin\varphi(k\cos\varphi - 1)$$

1st order system: $x = \varphi$, $y = \dot{\varphi}$

$$\dot{x} = y = f_1$$
$$\dot{y} = \sin x(k\cos x - 1) = f_2$$

- Rest pts: $y = 0$, $\sin x(k\cos x - 1) = 0$.

<u>Case</u> I: $k < 1$ $\left(k = \frac{\text{centrip force}}{\text{grav}} < 1\right)$

Rest pts $y = 0$, $x = n\pi$

$$Df(x,0) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0, & 1 \\ \cos x(k\cos x - 1) - k\sin^2 x, & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\cos x + k\cos 2x, & 0 \end{pmatrix}$$

At $x = n\pi$, $\cos 2n\pi = 1$ so

$$Df(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\cos n\pi + k & 0 \end{pmatrix}$$

- $n = 0$ or n even:

$$Df(0,0) = \begin{pmatrix} 0 & 1 \\ -1+k & 0 \end{pmatrix} \qquad (k<1)$$

$\lambda^2 - tr\,\lambda + \Delta = 0$ $\rightarrow$ $\lambda = \pm\sqrt{k-1}$

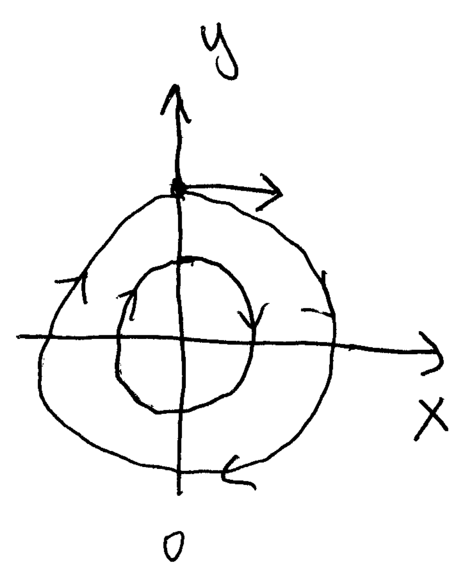
$\lambda^2 - (k-1) = 0$

$$\boxed{\lambda = \pm i\sqrt{1-k}}$$

$\Rightarrow$ center. For direction check linearized equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1+k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$$



- $n = \pi$ or n odd:

$$Df(0,\pi) = \begin{pmatrix} 0 & 1 \\ 1+k & 0 \end{pmatrix}$$

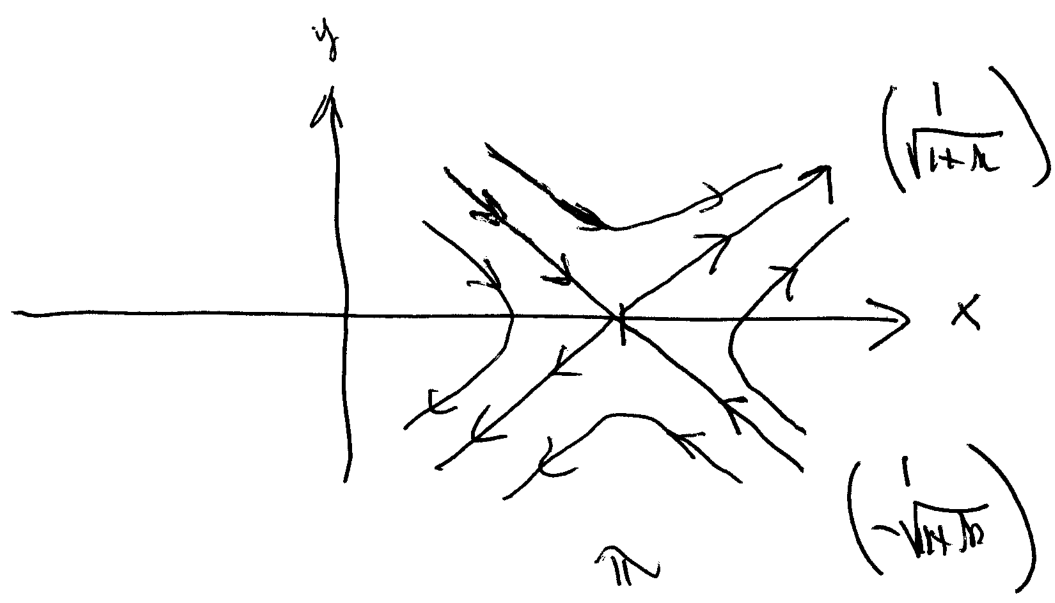
$\lambda^2 - \operatorname{tr}\lambda + \Delta = 0$ $\qquad \lambda_\pm = \pm\sqrt{1+k}$ real

$\lambda^2 - (1+k) = 0$ $\qquad$ <u>saddle</u>:

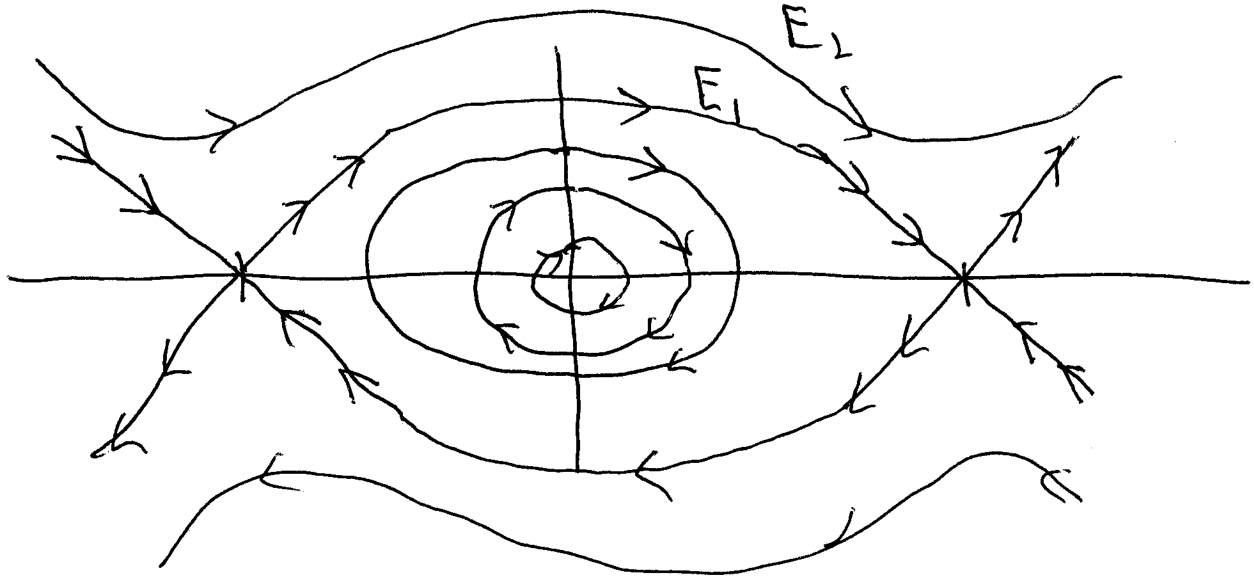
<u>e-vectors</u>: $\begin{pmatrix} \mp\sqrt{1+k} & 1 \\ 1+k & \mp\sqrt{1+k} \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = 0$

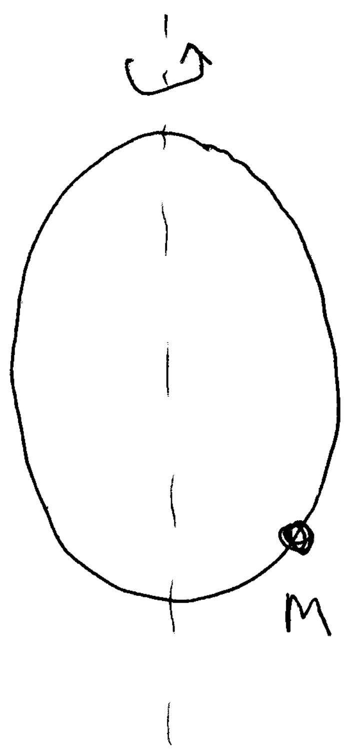
$\mp\sqrt{1+k} + r = 0 \qquad r = \pm\sqrt{1+k} = \lambda_\pm$

$\left(\pm\sqrt{1+k}, \begin{pmatrix} 1 \\ \pm\sqrt{1+k} \end{pmatrix}\right)$ e-pairs



Phase portrait: $k<1$





- critical orbit is bead going from top back to top with neg or pos velocity
- $\ddot{\varphi} = -U'(\varphi)$

  $E = \frac{1}{2}|\dot{\underset{\sim}{x}}|^2 + U(\varphi)$

$E < E_1 \Rightarrow$ periodic orbit

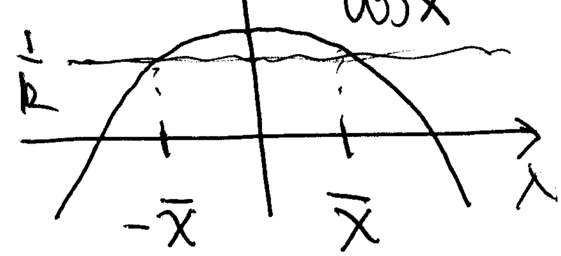
$E > E_1 \Rightarrow$ mass goes around hoop ✓

Case II : $k > 1$ $\left(k = \dfrac{\text{cent. force}}{\text{grav.}} > 1\right)$

Rest pts: $\sin x\,(k\cos x - 1) = 0$, $y = 0$

$x = n\pi$, $y = 0$

$x = \pm\bar{x}$, $y = 0$ $\quad \cos\bar{x} = \dfrac{1}{k} < 1$



(and $\pm\bar{x} + 2n\pi$)

$$Df(x,0) = \begin{pmatrix} 0 & 1 \\ -\cos x + k\cos 2x & 0 \end{pmatrix} \quad (k>1)$$

$x = n\pi \Rightarrow Df(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\cos n\pi + k & 0 \end{pmatrix}$

- n even $\quad Df(0,0) = \begin{pmatrix} 0 & 1 \\ k-1 & 0 \end{pmatrix} \quad (k>1)$

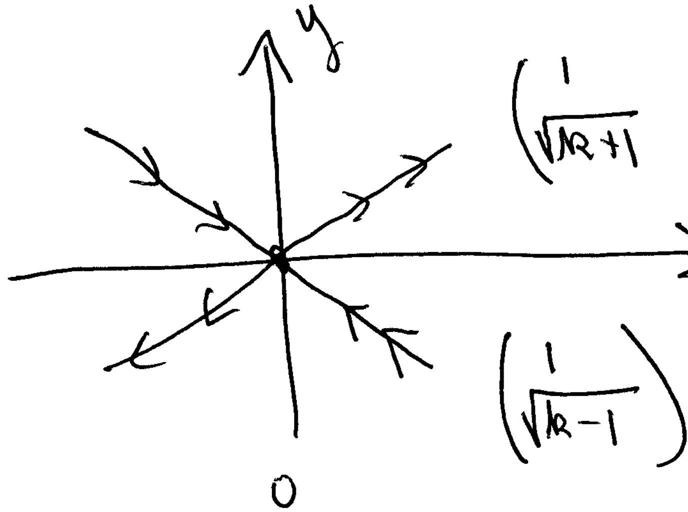
$\lambda^2 - \text{tr}\,\lambda + \det = 0$

$\lambda^2 - (k-1) = 0$

$\lambda = \pm\sqrt{k-1}$ $\rightarrow$ real when $k > 1$ so it switches to a saddle when $k > 1$

e-vectors: $\begin{pmatrix} \mp\sqrt{k-1} & 1 \\ k=1 & \mp\sqrt{k-1} \end{pmatrix}\begin{pmatrix} 1 \\ r \end{pmatrix} = 0$

$r = \pm\sqrt{k-1} \Rightarrow \left(\pm\sqrt{k-1}, \begin{pmatrix} 1 \\ \pm\sqrt{k-1} \end{pmatrix} \right)$ epairs



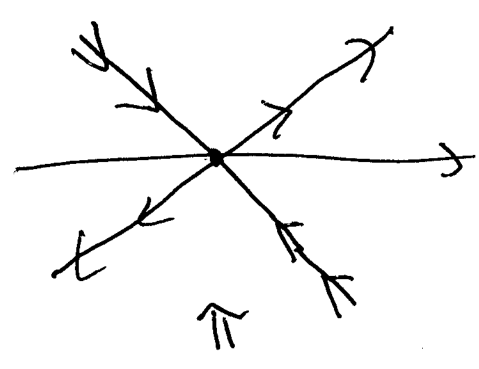
(bottom of hoop unstable)

- n odd: $Df(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1+k & 0 \end{pmatrix}$

$\lambda^2 - tr\lambda + det = 0$ $\quad \lambda = \pm\sqrt{1+k}$ real

$\lambda^2 - (1+k) = 0$ $\quad$ saddle

epairs: $\left(\pm\sqrt{1+k}, \begin{pmatrix} 1 \\ \pm\sqrt{1+k} \end{pmatrix} \right)$



$x = \pm\bar{x}, \; y = 0$

$\cos(\bar{x}) = \frac{1}{k} < 1 \qquad Df(\pm\bar{x}, 0) = \begin{pmatrix} 0 & 1 \\ -\cos\bar{x} + k\cos 2\bar{x} & 0 \end{pmatrix}$

$$-\cos\bar{x} + k\cos 2\bar{x} = -\frac{1}{k} + k(\cos^2 x - \sin^2 x)$$

$$= -\frac{1}{k} + k(\cos^2 x - 1 + \cos^2 x)$$

$$= -\frac{1}{k} + k\left(2\left(\frac{1}{k^2}\right) - 1\right)$$

$$= -\frac{1}{k} + \frac{2}{k} - 1 = \frac{1}{k} - 1 < 0$$

$Df(\pm\bar{x}, 0) = \begin{pmatrix} 0 & 1 \\ -1 + \frac{1}{k} & 0 \end{pmatrix} \qquad \lambda^2 - \mathrm{tr}\,\lambda + \det = 0$

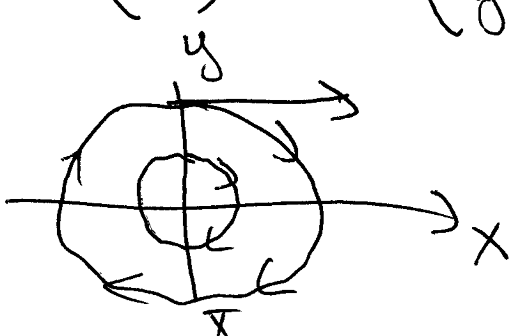
$\lambda^2 - \left(-1 + \frac{1}{k}\right) = 0$

Direction: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 + \frac{1}{k} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

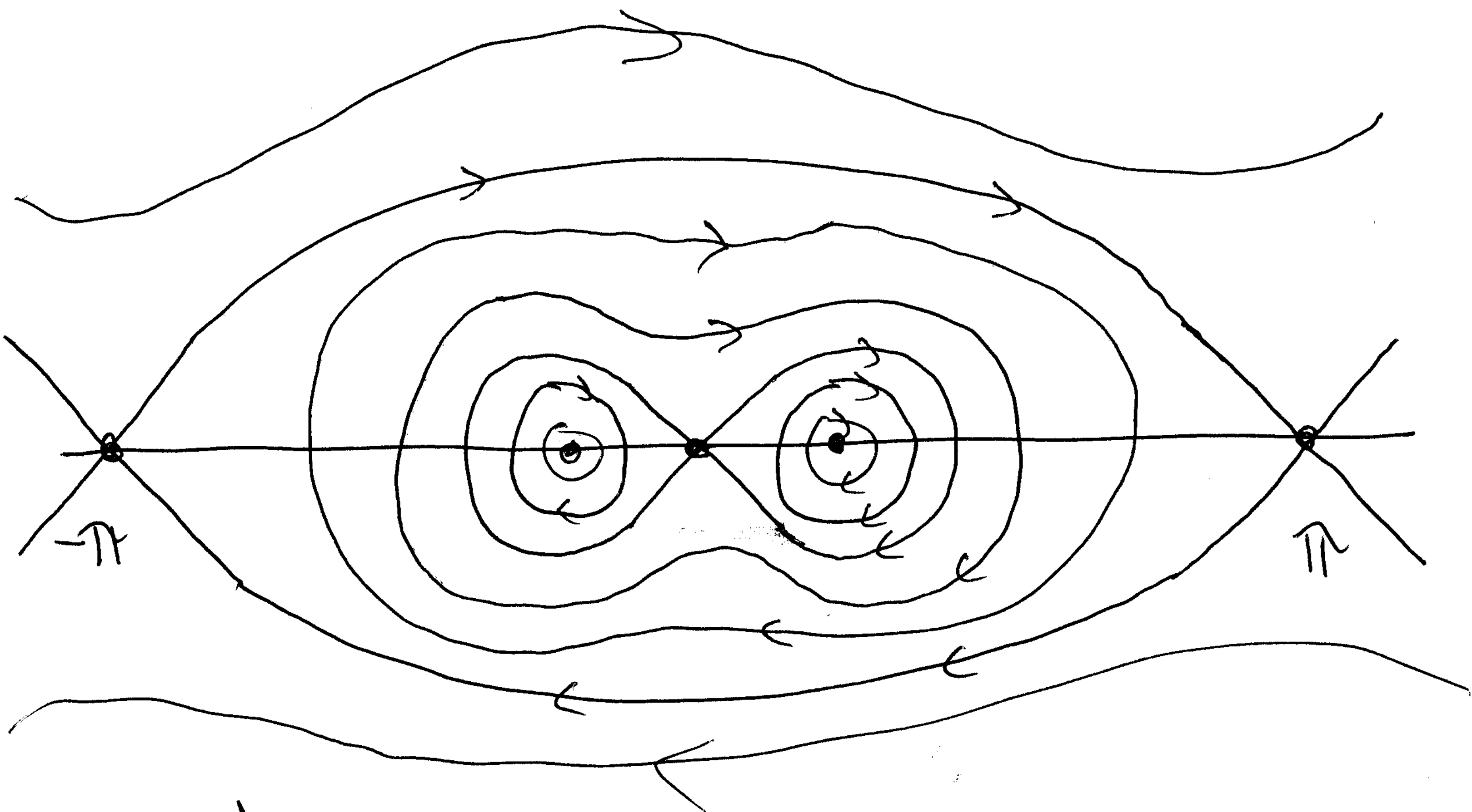
$\lambda_{\pm} = \pm i\sqrt{1 - \frac{1}{k}}$

center

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Phase Portrait



Conclude: "The qualitative features of the phase diagram change at critical values of dimensionless constants"

Here: $k = \frac{r\omega^2}{g} = \frac{mr\omega^2}{mg} = \frac{\text{"centripidal force"}}{\text{grav force}}$

is the dimensionless constant &

$k=1$ is the critical value

Energy: $\ddot{\varphi} = \sin\varphi\,(k\cos\varphi - 1) = -U'(\varphi)$

$$\int k\sin\varphi\cos\varphi - \sin\varphi \, d\varphi$$

$u = \sin\varphi$

$du = \cos\varphi \, d\varphi$

$$k\sin^2\varphi - \cos\varphi + \text{const} = -U(\varphi)$$

$$U(\varphi) = \cos\varphi - k\sin^2\varphi + K$$

$$E = \frac{1}{2}\dot{\varphi}^2 + k\sin^2\varphi - \cos\varphi + K$$