Mar 14, 12
Math 119A
W12

Liapunov Stability §7.2

①

☒ We have: for systems that conserve energy, centers perturb to nonlinear centers "periodic orbits are preserved" & $\underline{\underline{no}}$ stable nodes

$$m\ddot{x} = -\nabla U(x)$$

$$E = \frac{1}{2}mv^2 + U(x)$$

const. on soln's

More generally –

$$\frac{d}{dt}(p) = -\nabla U(x) \quad p = m\dot{x} = \text{momentum}$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{x}}}_{p}\right) = \frac{\partial L}{\partial x} \qquad L(x,\dot{x}) =$$

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L$$

const on soln's

General $L(x,\dot{x})$

Eg $L(x,\dot{x}) = x\dot{x}^2 - x^3$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}(2x\dot{x}) - \dot{x}^2 + 3x^2$$

$$2x\ddot{x} + \dot{x}^2 + 3x^2 = 0$$

$$E = \dot{x}\cdot 2x\dot{x} - x\dot{x}^2 + x^3 = x\dot{x}^2 + x^3$$

Const along solns –

Rewrite as 1st order Hamiltonian System:

$H(x,p) = E(x, f(x,p))$ $\quad \dot{x} = f(x,p)$

$p = \frac{\partial L}{\partial \dot{x}}(x,\dot{x})$ & solve for $\dot{x}$

(H) $$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial x} \end{aligned}}$$

Eg: $\frac{\partial L}{\partial \dot{x}} = 2x\dot{x} = p$

$\dot{x} = \frac{p}{2x} = f(x,p)$

$H = E(x, f(x,p))$

$= x\left(\frac{p}{2x}\right)^2 + x^3 = \frac{p}{4x} + x^3$

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = \frac{1}{4x} \\ \dot{p} &= -\frac{\partial H}{\partial x} = -\frac{p}{4x^2} + 3x^2 \end{aligned}}$$

H.W. Show H is constant along soln's of (H) (Must be because E is const along equivalent 2nd order system $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$)

Soln: $\frac{d}{dt} H(x(t), p(t)) = H_x \dot{x} + H_p \dot{p}$

$= + H_x H_p - H_p H_x = 0$ ✓

"Lagranges equations are the canonical equations that Conserve Energy"

Note: RHS of Hamilton's equations are almost a gradiant:

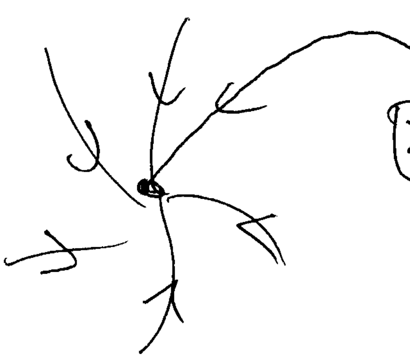
$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\substack{\text{Antisymmetric} \\ \text{matrix } \underset{\approx}{\varepsilon}}} \begin{pmatrix} H_x \\ H_p \end{pmatrix} = \underset{\approx}{\varepsilon} \nabla H$$

- Lagrange / Hamilton's Equations conserve energy $\Leftrightarrow$ no stable nodes & centers perturb to nonlinear centers $\Rightarrow$ rest points are not asymptotically stable

Q: What sorts of nonlinear systems have asymptotically <u>stable</u> solutions?

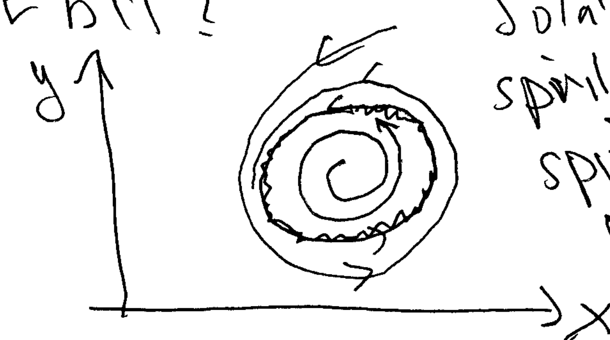
1. Rest Pts

"Global soln's tend to rest points"



2. Stable periodic orbits

"Global solutions tend to one periodic orbit?"



Soln's spiral in or spiral out

1 <u>Gradiant</u> <u>Systems</u>:

$$(G) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla V(\underset{\sim}{x}) = -\begin{pmatrix} \frac{\partial V}{\partial x}(x,y) \\ \frac{\partial V}{\partial y}(x,y) \end{pmatrix}$$

close to Hamilton but no $\underset{\sim}{\varepsilon}$

<u>Lemma 1</u>: The function V <u>decreases</u> along soln's of (G) when $\nabla V \neq 0$.

Pf.
$$\frac{d}{dt} V(\underline{x}(t)) = \frac{d}{dt} V(x(t), y(t))$$

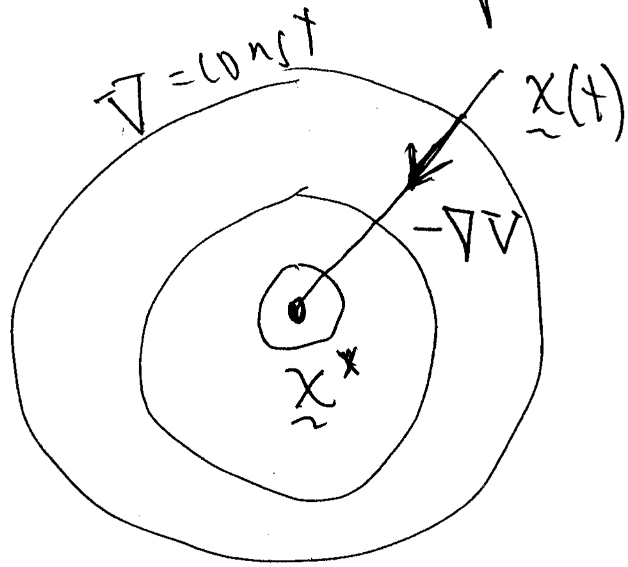
$$= \frac{\partial V}{\partial x}\dot{x}(t) + \frac{\partial V}{\partial y}\dot{y}(t)$$

$$= -\frac{\partial V}{\partial x}\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}\frac{\partial V}{\partial y} = -\|\nabla V\|^2.$$

✓

- Expect: if $V(\underset{\sim}{x})$ has a global minimum $\underset{\sim}{x}^*$, $V(\underset{\sim}{x}) > V(\underset{\sim}{x}^*)$ $\forall \underset{\sim}{x}$, then we expect all trajectories tend to $\underset{\sim}{x}^*$ $\Rightarrow$ $\underset{\sim}{x}^*$ is <u>globally</u> <u>stable</u>

"all trajectories tend to the rest pt."



I.e. $\underset{\sim}{x}(t)$ crosses the level curves of V in normal direction until it reaches minimum at $\underset{\sim}{x}^*$

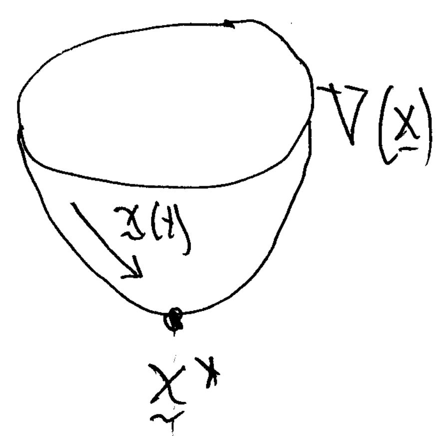
<u>Note</u>: wlog we can assume $V(\underset{\sim}{x}^*) = 0$

I.e. $\hat{V} = V - V(\underset{\sim}{x}^*) \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla \tilde{V}$ also

This is a special case of the <u>Liapunov Stability Theorem</u>:

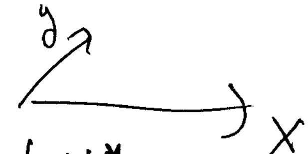
"If V is a function that strictly decreases on solutions of ODE $\dot{\underline{x}} = f(\underline{x})$ except at $\underline{x}^*$ where it takes a minimum value, then all soln's of $\dot{\underline{x}} = f(\underline{x})$ satisfy $\lim_{t\to\infty} \dot{\underline{x}}(t) = \underline{x}^*$ "

<u>Precisely</u>: Assume



(1) $V(\underline{x}) > 0 \;\; \forall \;\; \underline{x} \neq \underline{x}^*$ &

$V(\underline{x}^*) = 0$



(2) $\dot{V} = \nabla V \cdot \dot{\underline{x}}(t) = \nabla V \cdot f(\underline{x}) < 0 \;\; \forall \; \underline{x} \neq \underline{x}^*$

then $\lim_{t\to\infty} \underline{x}(t) = \underline{x}^* \;\; \forall$ soln's $\dot{\underline{x}} = f(\underline{x})$.

- Thm gives conditions under which a rest point is <u>globally asymptotically stable</u>.

- A function $L(\underset{\sim}{x})$ satisfying ① & ② is called a Liapunov function for system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ & rest pt $\underset{\sim}{x}^*$.

- If you want to prove a rest pt is globally asymptotically stable, find $V(\underset{\sim}{x})$ satisfying ① & ②

- <u>Note</u>: simplest case: $\dot{\underset{\sim}{x}} = -\nabla V(\underset{\sim}{x})$

$\Rightarrow V$ a Liapunov fn if it has a global regular minimum.

- Cor. Systems with a Liapunov function have no periodic orbits (i.e. V decreases along soln's $\Rightarrow$ cannot come back to same pt.)

Ex Consider $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x+4y \\ -x-y^3 \end{pmatrix}$

Describe the asymptotic behavior.

Soln. $V(x,y) = x^2 + ay^2$

$$\nabla V \cdot f(\underline{x}) = (2x, 2ay) \cdot (-x+4y, -x+y^3)$$

$$= -2x^2 + 8xy - 2axy - 2ay^{\cancel{3}4}$$

$$= -2x^2 - 2ay^4 + (8-2a)xy \quad \text{choose } a=4$$

$$= -2x^2 - 8y^4 < 0 \quad (x,y) \neq 0 \Rightarrow \underset{\sim}{x}(t) \longrightarrow \underset{\sim}{0} \checkmark$$

LST