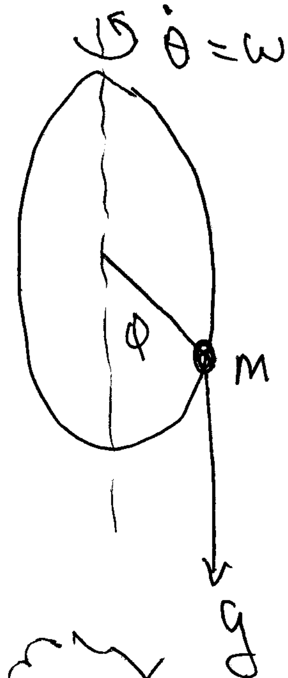
FRI March 16 2012 119A W12 Temple

Nonconservative Systems
Rotating Hoop with Friction

Ex Recall bead with rotating hoop



$$L(\varphi,\dot\varphi) = \frac{1}{2}mr^2\left(\dot\varphi^2 + \omega^2\sin^2\varphi\right) + mgr\cos\theta$$

describes motion when no friction (c.f. Lecture 21) I.e.

Equations are EL: $\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot\varphi} - \dfrac{\partial L}{\partial\varphi} = 0$

$$\Rightarrow \quad mr\ddot\varphi = -mg\sin\varphi + mr\omega^2\sin\varphi\cos\varphi \;\; \cancel{-b\dot\varphi}$$

no friction $\Rightarrow$ energy conserved

Conserved Energy $E = \dot\varphi\dfrac{\partial L}{\partial\dot\varphi} - L$

leads to

$$E = \frac{1}{2}mr^2\{\dot\varphi - \omega^2\sin^2\varphi\} - mgr\cos\varphi$$

• To keep things simple we non-dimensionalize:

Recall nondimensionalized equations –

(1) $$\boxed{\ddot{\varphi} - \sin\varphi\,(k\cos\varphi - 1) = 0}\,, \quad k = \frac{r\omega^2}{g}\,, \quad T = \sqrt{\frac{r}{g}}$$

$$\dot{\ } \equiv \frac{1}{T}\frac{d}{dt}$$

This comes from Lagrangian –

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}k\sin^2\varphi + \cos\varphi$$

I.e. $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\varphi}} - \dfrac{\partial L}{\partial \varphi} = \ddot{\varphi} - k\sin\varphi\cos\varphi + \sin\varphi = 0$ ✓

Incorporating friction gives –

(2) $$\boxed{\ddot{\varphi} - \sin\varphi\,(k\cos\varphi - 1) = -\mu\dot{\varphi}\,,} \quad \mu = \frac{b}{rgT}$$

• <u>Problem</u>: how does the energy that's constant in (1) evolve along solutions of (2)?

- Energy determined by Lagrangian –

$$L(\varphi,\dot\varphi) = \frac{1}{2}\dot\varphi^2 + \frac{1}{2}k\sin^2\varphi + \cos\varphi$$

$$E = \dot\varphi\frac{\partial L}{\partial\dot\varphi} - L = \dot\varphi\,\dot\varphi - \frac{1}{2}\dot\varphi^2 - \frac{1}{2}k\sin^2\varphi - \cos\varphi$$

$$E = \frac{1}{2}\dot\varphi^2 - \frac{1}{2}k\sin^2\varphi - \cos\varphi \quad \left(\begin{array}{c}\text{constant along}\\ \text{soln's of (1)}\end{array}\right)$$

$$\frac{dE}{dt} = \dot\varphi\ddot\varphi - k\sin\varphi\cos\varphi\,\dot\varphi + \sin\varphi\,\dot\varphi$$

$$= \dot\varphi\left\{\ddot\varphi - k\sin\varphi\cos\varphi + \sin\varphi\right\}$$

$$\frac{dE}{dt} = \dot\varphi\left\{\ddot\varphi - \sin\varphi\,(k\cos\varphi - 1)\right\}$$

Along solutions of (1) (no friction) $\frac{dE}{dt} = 0$

Along solutions of (2) (friction)

$$\boxed{\frac{dE}{dt} = \dot\varphi\left\{-\mu\dot\varphi\right\} = -\mu\dot\varphi^2 \leq 0}$$

- <u>Conclude</u>: The energy from the conservation friction free equations is a Liapunov Function for the equations with friction.

<u>Modify Defn</u>: $V(x,\dot{x})$ is a Liapunov function for system $\dot{\underset{\sim}{x}} = f(\underline{x})$ if

$$\frac{dV(\underline{x}(t))}{dt} = \nabla V \cdot \dot{\underset{\sim}{x}}(t) = \nabla V \cdot f(\underline{x}) \leq 0$$

along solutions $\underset{\sim}{x}(t)$ of $\dot{\underline{x}} = f(\underline{x})$.

Its a <u>strict</u> Liapunov function in region R if

$$\nabla V \cdot f(\underline{x}) < 0 \quad \forall\ \underset{\sim}{x} \in R$$

We have: If V strict Liapunov in $\mathbb{R}^2 \setminus \{\bar{\underline{x}}\}$ and V takes a strict minimum at $\bar{\underset{\sim}{x}}$, then all solutions $\underset{\sim}{x}(t) \longrightarrow \bar{\underline{x}}$.

4 For Rotating hoop w/o friction –

$$\ddot{\varphi} - \sin\varphi\,(k\cos\varphi - 1) = 0$$

We could write as 1st order system –

But: $p = \dfrac{\partial L}{\partial \dot{x}} = \dfrac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}k\sin^2\varphi + \cos\varphi\right)$

$$\boxed{p = \dot{\varphi}}$$

$$\therefore\ H(\varphi, p) = E(\varphi, p) = \frac{1}{2}p^2 - \frac{1}{2}k\sin^2\varphi - \cos\varphi$$

$$\dot{\varphi} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial \phi} = k\sin\varphi\cos\varphi - \sin\varphi$$

$$x = \varphi\,,\quad y = \dot{\varphi} = p$$

(1) $\boxed{\begin{aligned}\dot{x} &= y\\ \dot{y} &= k\sin x\cos x - \sin x\end{aligned}}$

$H = \frac{1}{2}y^2 - \frac{1}{2}k\sin^2 x - \cos x$ constant along solutions.

That is: for (1),

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}H_y\\ -H_x\end{pmatrix} = f(\underset{\sim}{x}), \quad \frac{dH}{dt} = \nabla H \cdot \dot{\underline{x}} = \nabla H \cdot f(\underline{x})$$

$$= (H_x, H_y) \cdot (H_y, -H_x) = 0$$

"H is constant along soln's."

• Consider the case with friction —

(2)
$$\dot{x} = y$$
$$\dot{y} = k\sin x\cos x - \sin x - \underbrace{\mu y}_{\text{friction}}$$

$$\begin{pmatrix}\dot{x}\\ y\end{pmatrix} = \begin{pmatrix}H_y\\ -H_x\end{pmatrix} + \begin{pmatrix}0\\ -\mu y\end{pmatrix} = \hat{f}(\underset{\sim}{x})$$

$$\frac{d}{dt} H(\underset{\sim}{x}(t)) = \nabla H \cdot \dot{\underset{\sim}{x}} = \nabla H \cdot \hat{f}(\underset{\sim}{x})$$

$\nabla H = \left(\frac{\partial}{\partial x} H, \frac{\partial}{\partial y} H\right)$

$\frac{\partial H}{\partial y} = y$ ✓

$$= \cancel{\nabla H \cdot (H_y, -H_x)}^{\,0} + \nabla H \cdot (0, -\mu y)$$
$$= (0, y) \cdot (0, -\mu y) = -\mu y^2$$

<u>Conclude</u>: $H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}k\sin^2 x - \cos x$ is a Liapunov function (not strict) on solutions with friction).

<u>Conclude</u>: H decreases along solutions <u>except</u> when they cross the y-axis

"$\Rightarrow$" solution must tend to a rest pt. @ $y=0$

Ie: rest pts – same as $\mu = 0$ case:

$$f(\bar{x}) = 0 \iff \begin{array}{l} \bar{y} = 0 \\ \sin\bar{x}\,(k\cos\bar{x} - 1) - \underbrace{\cancel{\mu\bar{y}}}_{0} = 0 \end{array}$$

Rest pts $(n\pi, 0)$, $(\bar{x}, 0)$ same as $\mu = 0$.

Q: which rest pts have the <u>least</u> <u>energy</u>

i.e. where is $H(x,y)$ minimized?

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}k\sin^2 x - \cos x \geq$$

$$\geq -\frac{1}{2}k\sin^2 x - \cos x$$

$$= -\frac{1}{2}k(1-\cos^2 x) - \cos x$$

$$= -\frac{1}{2}k + \frac{1}{2}k\cos^2 x - \cos x$$

Minimum energy occurs at min of $\frac{1}{2}k\cos^2 x - \cos x$

$$f(x) = \frac{1}{2}k\cos^2 x - \cos x$$

$$0 = f'(x) = -k\cos x\sin x + \sin x$$

$$0 = \sin x(1 - k\cos x)$$

Conclude: Minimum energy is at a rest pt $\bar{y}=0$, $\sin\bar{x}=0$ or $\cos\bar{x}=\frac{1}{k}$

check: $H(n\pi, 0) = -\cos n\pi = \begin{cases} -1 & n \text{ even} \\ +1 & n \text{ odd} \end{cases}$

$$H(\bar{x},0) = -\frac{1}{2}k - \frac{1}{2}k\cos^2\bar{x} - \cos\bar{x}, \quad \bar{x} = \frac{1}{k}$$

$$H(\bar{x},0) = -\frac{k}{2} + \frac{1}{2k} - \frac{1}{k} = -\frac{k}{2} - \frac{1}{2k}$$

$$= -\frac{k^2+1}{2k} \qquad k \geq 1$$

For what k is this minimized?

$$\frac{d}{dk}\left(\frac{k^2+1}{2k}\right) = \frac{1}{2} - \frac{1}{2k^2} = 0 \text{ @ } k=1$$

(MINIMUM)

$$\therefore \quad H(\bar{x},0) = -\frac{k^2+1}{2k} < -1 \text{ when } k>1$$

<u>Conclude</u>: When $k \leq 1$, minimum energy occurs @ $n\pi$, n even. When $k>1$, minimum energy occurs at $\cos\bar{x} = \frac{1}{k}$.

Exam Question

<u>Homework</u>: These are the <u>only</u> stable rest points when $\mu > 0$. (ie $DF(\underset{\sim}{\bar{x}})$ has neg evals <u>only</u> at rest pts of minimum energy)