Fri Jan 13

§2.1–2.4 Phase Portrait for Autonomous Scalar

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Classification of ODE's –

- An ODE is linear if it is a sum of linear terms –

linear term: $g(t)x^{(n)}$ or just $g(t)$ known

General 1st ord linear scalar ODE

$$a(t)\dot{x} + b(t)x + c(t) = 0$$

General 2nd order –

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x + d(t) = 0$$

- Defn: an ODE is constant coefficient linear if all $g(t)$'s are constant,

Ex $\dot{x} + kx = c$ (1st order linear c.c. eqn)

$a\ddot{x} + b\dot{x} + cx + d = 0$ (2nd order linear cc eqn)

1st order const coeff system of eqn's:

$$\underset{n\times 1}{\dot{\underset{\sim}{x}}} = \underset{n\times n}{A}\ \underset{n\times 1}{\underset{\sim}{x}} + \underset{n\times 1}{\underset{\sim}{b}}$$ "n eqn's in n-unknow

$\underset{\sim}{x}(t) = (x_1(t), \ldots, x_n(t)) \Rightarrow$ n distinct unknown functions

- A linear system is homogeneous if the $g(t)$ term is zero

Eg $\quad a(t)\dot{x} + b(t)x = 0$; $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ etc

(For homogeneous linear equations, superposition holds – $x_1(t)$ & $x_2(t)$ soln's, $c_1\underset{\sim}{x}_1 + c_2\underset{\sim}{x}_2$ also a soln. $\Rightarrow$ the solution space is a vector space...)

- For nonlinear equations – the closest thing to a homogeneous const coeff system is an autonomous system –

__Defn__: a nonlinear ODE is __autonomous__ if all terms depend on t __only__ thru the unknown function $x(t)$

__Eg__: $\dot{x} = f(x)$ some nonlinear fn f

$\dot{x} = \sin$, $\ddot{x} = \sin x$, $\ddot{x} + \dot{x}^2 + x\dot{x} = x^2$

(we almost always assume you can solve for highest order derivative)

$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ 1st order autonomous system.

Unifying Framework –

Every ODE can be written as a 1st order system:

$$\dot{\underset{\sim}{x}} = f(\underline{x}, t)$$

"pf" Consider $\dddot{x} = \ddot{x}x + \dot{x}^2 + tx + t^2$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \end{aligned}\right\} \quad \dot{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \begin{pmatrix} x_2 \\ x_3 \\ +x_3x_1 + x_2^2 + tx_1 + t^2 \end{pmatrix}$$

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, t)$$

Autonomous: $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$

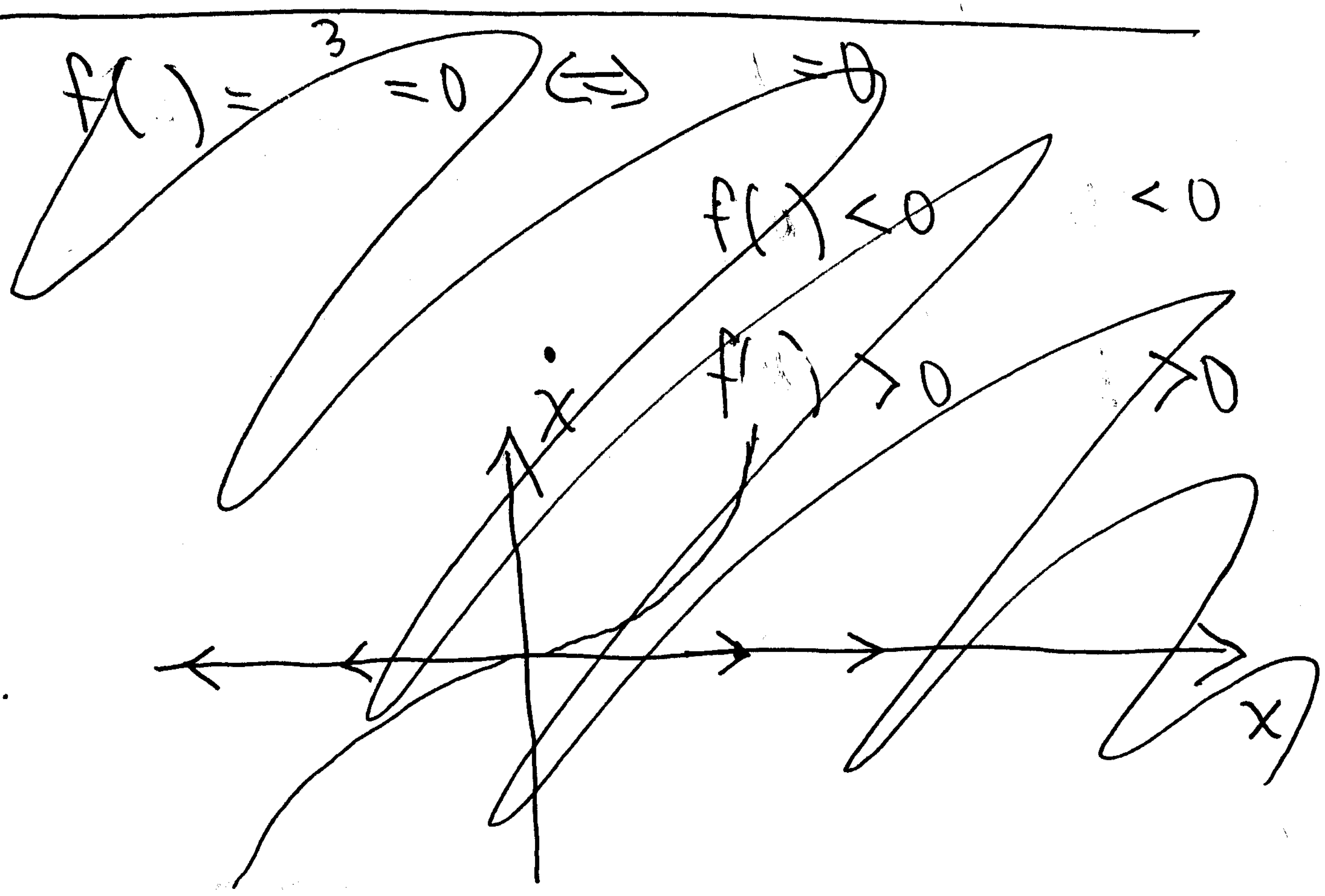
(You have to be able to solve for highest order deriv)

Chapter 2 concerns 1st order autonomous scalar equations.

General: $\dot{x} = f(x)$

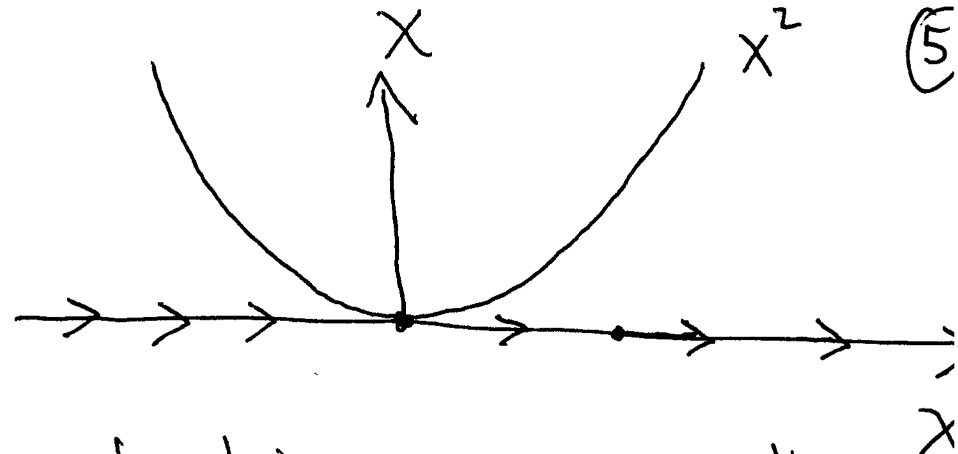
Ex: Describe solutions of $\dot{x} = x^2$

Soln: find where $f(x) = 0$ & draw the phase portrait



$\dot{x} = x^2$

$x(0) = x(0)$



Conclude: solutions starting @ $x_0 > 0$ will have $x(t) \to +\infty$; solutions starting with $x_0 < 0$ will have $x(t) \to 0$.

Idea: The phase portrait tells us what happens to $x(t)$ without giving exact dependence on t.

Check: $\displaystyle\int_{x_0}^{x} \frac{dx}{x^2} = \int_0^t dt \Rightarrow -x^{-1}\Big]_{x_0}^{x} = t$

$$-\frac{1}{x(t)} + \frac{1}{x_0} = t \qquad \boxed{x(t) = \frac{1}{\frac{1}{x_0} - t}}$$

Conclude: if $x_0 > 0$, $x(t) \to \infty$ as $t \to \frac{1}{x_0}$

if $x_0 < 0$, $x(t) = -\dfrac{1}{\left|\frac{1}{x_0}\right| + t} \longrightarrow 0$ as $t \to \infty$ ✓

From the phase portrait we can see what happens t $x(t)$ as $t \to \infty$ w/o having to integrate the equation!

$\Rightarrow$ NICE!

Defn: $\left.\begin{array}{l}\dot{x} = f(x) \\ x(0) = x_0\end{array}\right\}$ initial value problem for autonomous nonlinear 1st order scalar ODE

Defn. $\bar{x}$ is a <u>rest point</u> if $f(\bar{x}) = 0$

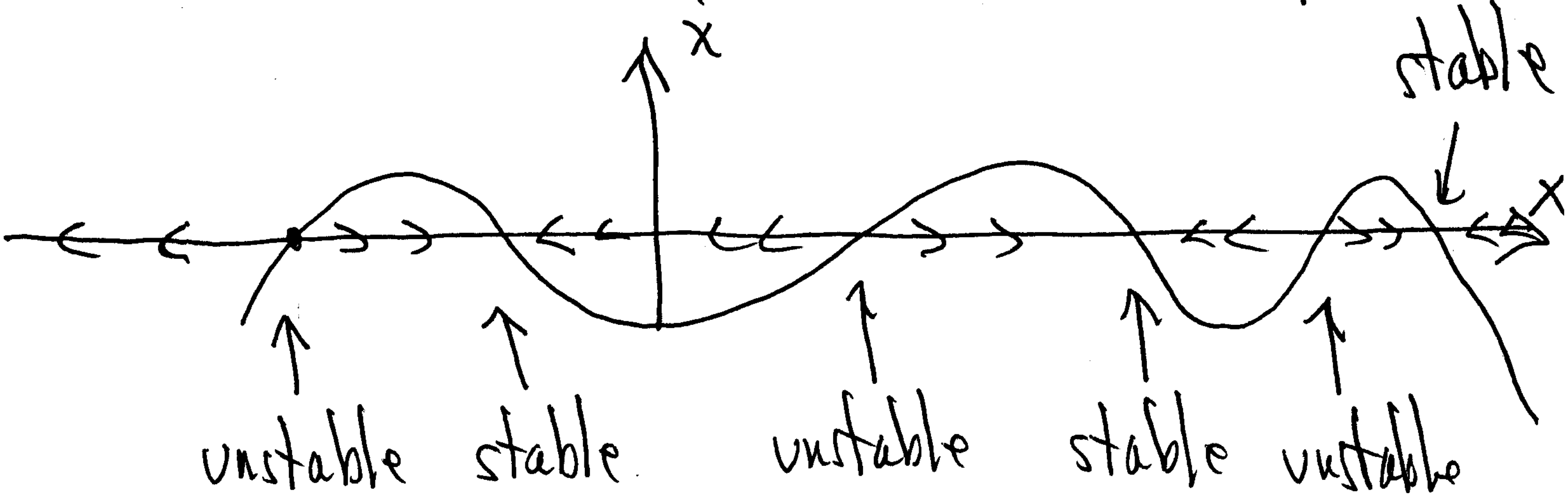
[5] The phase portrait method — $\dot{x} = f(x)$

(1) find the rest pts $f(x_i) = 0$

$\cdots \bar{x}_{-2} < \bar{x}_{-1} < \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \cdots$

(2) $f(x)$ changes sign at rest pts $\Rightarrow$ $f(x) > 0$ or $f(x) < 0$ betw consec rest pts

(3) Draw phase portrait from graph of f



(4) Conclude: solution tends monotonically to nearest downstream rest pt or else it goes to infinity.

This justifies the claim that soln's of ODE's tend to settle down to steady state solutions

__Ex__: Population dynamics
The Economy
Picture is embedded in our Culture!

__Defn__: A rest point $\bar{x}$ is nondegenerate if $f'(\bar{x}) \neq 0$.

__Defn__: A rest point is __stable__ if under small perturbation, the soln returns to the rest pt.

- Stability analysis – From picture, rest pt is <u>stable</u> if $f'(\bar{x}) < 0$, <u>unstable</u> if $f'(\bar{x}) > 0$. Let's check:

- Assume $f(\bar{x}) = 0$, $f'(\bar{x}) \neq 0$.

- Taylor expand f about $\bar{x}$:

$$f(x) = f(\bar{x}) + \underbrace{f'(\bar{x})}_{k}(x-\bar{x}) + \text{Error}$$

$$|\text{Error}| \leq \text{Const}\,(x-\bar{x})^2 \ll 1$$

when $x \approx \bar{x}$.

~~Thus: near~~ $\cancel{\bar{x}}$, ~~$x(t)$ will approximately satisfy~~ $\cancel{\dot{x} = f'(\bar{x})(x-\bar{x})}$

$\cancel{\Leftrightarrow \quad \dot{\overline{(x-\bar{x})}} = f'(\bar{x})}$

• Thus: near $\bar{x}$, $x(t)$ will approximately satisfy

$$\dot{x} = k(x-\bar{x})$$

$$\Leftrightarrow \quad \underbrace{\dot{\overline{x-\bar{x}}}}_{y} = k\underbrace{(x-\bar{x})}_{y} \Leftrightarrow \dot{y} = ky$$

$$\dot{y} = y_0 e^{kt}$$

$$\boxed{x(t)-\bar{x} = (x_0-\bar{x})e^{kt}}$$

<u>Conclude</u>: if $f'(\bar{x}) = k < 0$, then $x(t)$ will move <u>back</u> to $\bar{x}$ at the exponential rate $(x_0-\bar{x})e^{kt}$ (when $x_0 \approx \bar{x}$) $\Rightarrow$ <u>stable</u>

If $f'(\bar{x}) = k > 0$, then $x(t)$ will move away from $\bar{x}$ at the exponential rate $(x_0-\bar{x})e^{kt}$ ✓