Math 119A
Winter 2012

∃! Theorem – Wed Jan 18 2012

Every ODE can be written as a 1st order system:

$$\dot{\underset{\sim}{x}}(t) = f(\underset{\sim}{x}(t), t) \qquad (*)$$

To solve: "find a function $x(t)$ such that $(*)$ holds at evey t"

<u>Ex</u>: $\ddot{x}(t) + x(t)\,\dot{x}(t)^2 + x(t)\,\dot{x}(t) = \sin t$

suppress dependence of x on t:

$$\ddot{x} + x\dot{x} + x\dot{x}^2 + x\dot{x} = \sin t$$

Solve for highest order deriv:

$$\ddot{x} = -x\dot{x} - x\dot{x}^2 - x\dot{x} + \sin t = g(x, \dot{x}, t)$$

$$x_1 = x \qquad \dot{\underset{\sim}{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ g(x, \dot{x}, t) \end{pmatrix} = f(\underset{\sim}{x}, t)$$
$$x_2 = \dot{x}$$

$$\underset{\sim}{x} = (x_1, x_2)$$

- **Linear**: $\dot{\underset{\sim}{x}}(t) = A(t)\,\underset{\sim}{x}(t) + \underset{\sim}{b}(t)$

$$\underset{(n\times 1)}{\dot{\underset{\sim}{x}}(t)} = \underset{(n\times n)}{A(t)}\,\underset{(n\times 1)}{\underset{\sim}{x}(t)} + \underset{(n\times 1)}{\underset{\sim}{b}(t)}$$

"coeff's of $x_1, \dots, x_n$ are functions of t.

Linear Homo: $\dot{\underset{\sim}{x}}(t) = A(t)\,\underset{\sim}{x}(t)$ $\quad \underset{\sim}{b} = 0$

Linear Homo CC.: $\dot{\underset{\sim}{x}} = A\,\underset{\sim}{x}$

$$\underset{(n\times 1)}{\dot{\underset{\sim}{x}}} = \underset{(n\times n)}{A}\,\underset{(n\times 1)}{\underset{\sim}{x}}$$

$$\underset{n\times 1}{\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix}} = \underset{n\times n}{\begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix}} \underset{n\times 1}{\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}}$$

Nonlinear autonomous: $\dot{\underset{\sim}{x}}(t) = f(\underset{\sim}{x})$

Ex: Harmonic Oscillator —

$$\ddot{x} + kx = 0 \qquad \ddot{x} = -kx$$

$$x = x \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -kx \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$y = \dot{x}$$

$$\underset{2\times 1}{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} x \\ y \end{pmatrix}}$$

"Harmonic oscillator is equivalent to a 2x2 homo c.c. eqn"

Existence & uniqueness Theorem —

<u>Motivation</u>

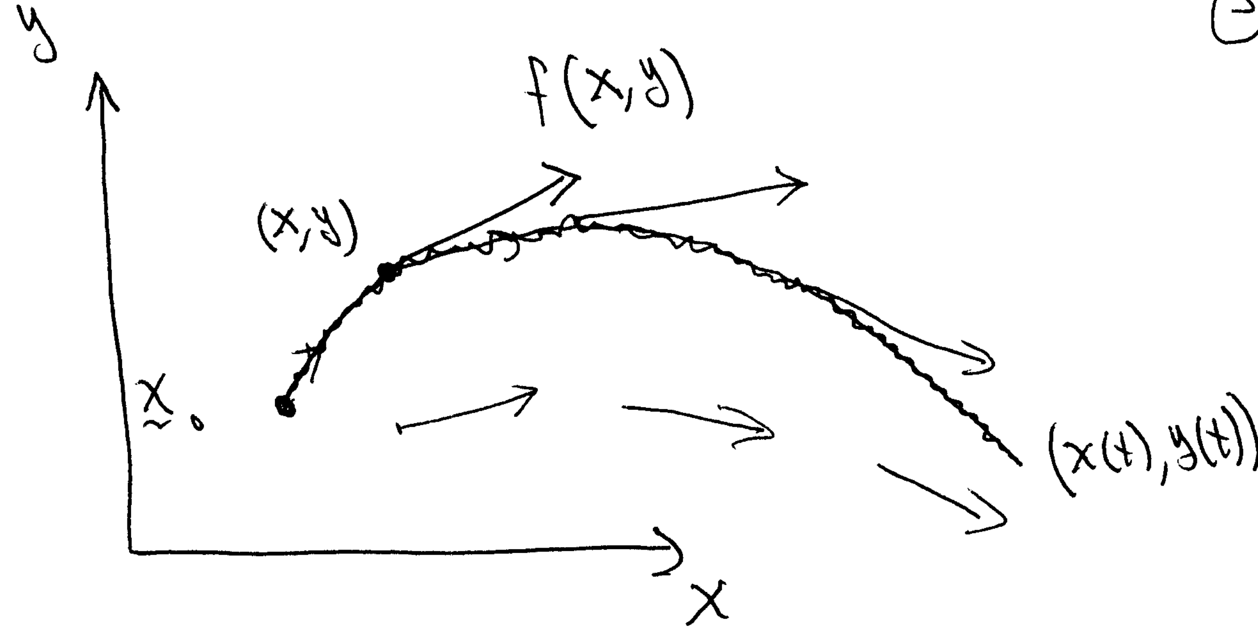
- Its best to visualize a general 1st order ODE $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ as a 2x2 autonomous system:

$\underset{\sim}{x} = (x, y)^T$:

$$\dot{\underset{\sim}{x}}(t) = \begin{pmatrix} \dot{x}(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = f(\underset{\sim}{x})$$

<u>Defn</u>: $f(\underset{\sim}{x})$ is called a <u>vector field</u>

It gives an arrow at each pt (x,y)



- $\dot{\underset{\sim}{x}}(t) = f(\underset{\sim}{x})$: "Find a curve $\underset{\sim}{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ whose tangent vector is $f(x,y)$ at each point"
- If we start at $\underset{\sim}{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ at $t=0$, there should exist a curve whose tangent vector is $f(\underset{\sim}{x}(t))$ at each $\underset{\sim}{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$
- __But__: it might not exist <u>for all</u> t when nonlinear because $\dot{y} = y^2 \Rightarrow y(t) \to \infty$
  $t \to t_*$

☒ Existence & Uniqueness Theorem — Stated in terms of 1st order systems

<u>Theorem 1</u>: Let $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, t)$ be a general $n \times n$ system, $\underset{\sim}{x}(t) = (x_1(t) \cdots x_n(t))^T$.

Assume f is <u>Lipschitz Continuous</u> <u>in $\underset{\sim}{x}$</u> in some interval nbhd containing $\underset{\sim}{x}_0$.

Then a unique solution $\underset{\sim}{x}(t)$ exists in some interval $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ satisfying

$$\dot{\underset{\sim}{x}}(t) = f(\underset{\sim}{x}(t), t) \qquad (1)$$

$$\underset{\sim}{x}(t_0) = \underset{\sim}{x}_0 \qquad (2)$$

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<u>Defn</u>: $f(\underset{\sim}{x}, t)$ is Lipschitz continuous in $\underset{\sim}{x}$ if $\exists$ constant $K > 0$ s.t.

$$\| f(\underset{\sim}{x}_2, t) - f(\underset{\sim}{x}_1, t) \| \le K \| x_2 - x_1 \|$$

for all x_1 & x_2.

$$\| \underset{\sim}{x} \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

<u>Theorem</u> 2: Suppose $f(\underset{\sim}{x}, t)$ is Lipschitz continuous in $\underset{\sim}{x}$ with <u>the same</u> K for all $a \le t \le b$. Then a unique soln $\underset{\sim}{x}(t)$ of (1) & (2) exists $\forall\, t \in [a, b]$.

- <u>Note</u>: Thm's 1 & 2 apply to all ODE's of <u>any</u> order & <u>any size</u> (No such unifying thm exists for PDE's)

You only have to remember these two thm's

Ex ① Prove that $\dot{x} = kx$, $x(0) = x_0$ has a unique soln for all time.

(We know $x(t) = x_0 e^{kt}$: are there others?)

Soln : $f(x,t) = kx = f(x)$

$\Rightarrow$ linear, homogeneous, autonomous, cc

$\|x\| = \sqrt{|x|^2} = |x|$

$$|f(x_2) - f(x_1)| = |kx_2 - kx_1|$$

$$= k|x_2 - x_1|$$

So f is Lipschitz continuous with const $K = k$ for all t (indept of t!)

$\in (-\infty, \infty)$

By Theorem ②: $\dot{x} = kx$, $x(t_0) = x_0$ has a unique

Soln defined $\forall\, t$ ✓

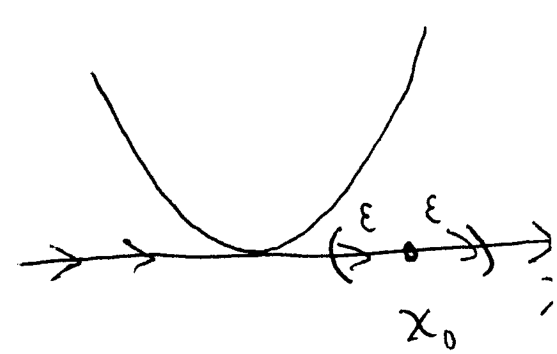
<u>Ex ②</u>: What does Thm ① tell you about solutions of $\dot{x} = x^2$, $x(t_0) = x_0$?

<u>Soln</u>: $f(x) = x^2$

$$|f(x_2) - f(x_1)| = |x_2^2 - x_1^2| = \underbrace{|x_2 + x_1|}_{\text{Not bounded by } K}\,|x_2 - x_1|$$

However: in $(x_0 - \varepsilon, x_0 + \varepsilon)$

$K = 2x_0 + 2\varepsilon > x_1 + x_2$ $\forall$ $x_1, x_2 \in (x_0 - \varepsilon, x_0 + \varepsilon)$.



Conclude: $|f(x_2)-f(x_1)| \leq K|x_2-x_1|$ in some interval around x_0. $\Rightarrow$

Thm① implies $\exists !$ soln $x(t)$ of $\begin{cases} \dot{x}=x^2 \\ x(t_0)=x_0 \end{cases}$

defined in some interval

$$t \in (t_0-\delta, t_0+\delta)$$

We know we can't do much better

because the soln $x(t) = \dfrac{1}{\frac{1}{x_0}-t}$

solves $\dot{x}=x^2$

$x(0)=x_0$

& $x(t) \to \infty$ as $t \to \frac{1}{x_0} \ll 1$ when $x_0 \gg 1$!

Ex: ($\exists !$ fails) $\dot{x} = \sqrt{x}$, $x(0) = 0$ Q: does there exist a unique soln

$$\int_0^x \frac{dx}{\sqrt{x}} = \int_0^t dt \Rightarrow 2x^{1/2}\Big]_0^x = t \qquad 2x^{1/2} = t$$

$x = \left(\frac{t}{2}\right)^2$ check: $x(0) = 0$ ✓

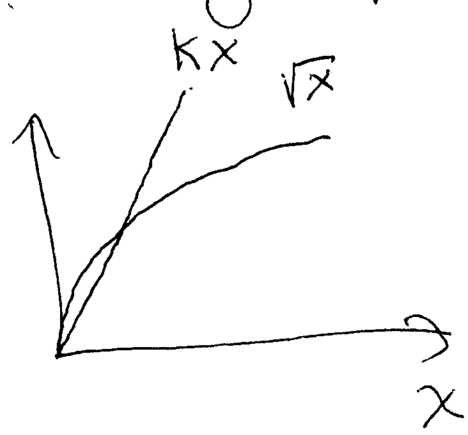
$\dot{x} = t/2 = \sqrt{x}$ ✓

But: $x(t) = 0$ also a soln. What goes wrong?

$$|f(x_2) - f(x_1)| = |\sqrt{x_2} - \sqrt{x_1}| \overset{?}{\leq} K|x_2 - x_1|$$

for x_1, x_2 near $x = 0$? NO! try $x_1 = 0$

$$|\sqrt{x_2}| \leq K|x_2|$$



always $\sqrt{x} > Kx$ for x suff small $\Rightarrow$ not Lipschitz cont!