FRI Jan 20 2012 // 1:29A W12

Pf Sketch of $\exists!$ Soln by Picard

$\boxed{E}$ Existence & Uniqueness Thm for ODE's

<u>**Thm 1**</u>: $\exists\ \varepsilon > 0$ st $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}, t)$ has a
$\underset{\sim}{x}(0) = \underset{\sim}{x}_0$

unique solution $\underset{\sim}{x}(t)$ defined for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ if $\exists\ \delta, K$ st

$$\| f(\underset{\sim}{x}_2, t) - f(\underset{\sim}{x}_1, t) \| \le K \| \underset{\sim}{x}_2 - \underset{\sim}{x}_1 \| \quad (1)$$

for all $\underset{\sim}{x}_1, \underset{\sim}{x}_2$ within a distance δ from the initial value $\underset{\sim}{x}_0$.

("we say f is Lipschitz cont in $\underset{\sim}{x}$ in a δ-neighborhood of $\underset{\sim}{x}_0$")

- <u>Note</u>: we always assume f is a <u>continuous function</u> $\Leftrightarrow$ no breaks in graph

- This is so important I am going to sketch the proof.

- Recall: $\underset{\sim}{x} = (x, y)$

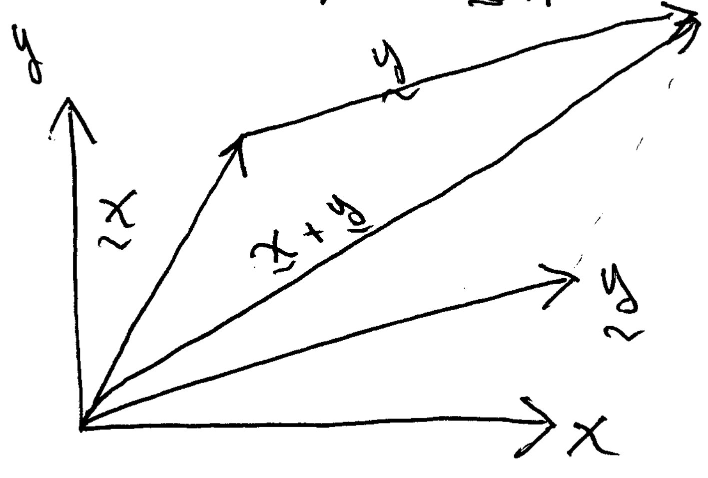
$$\|\underset{\sim}{x}\| = \sqrt{x^2 + y^2}$$

If $\underset{\sim}{x}$ is scalar, $\underset{\sim}{x} = x \in \mathbb{R}$, then

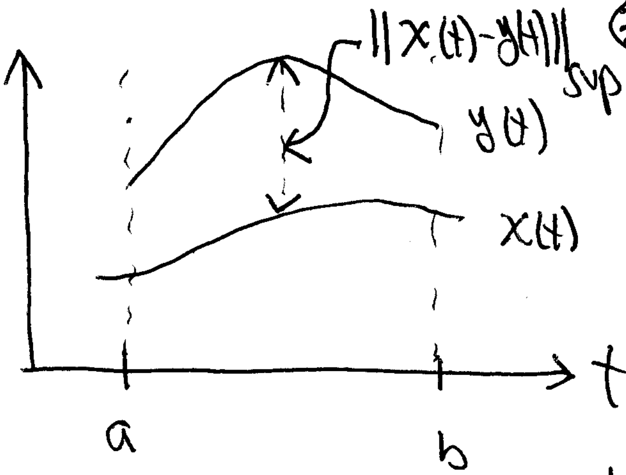
$$\|\underset{\sim}{x}\| = |x|$$

Recall: $\|\underset{\sim}{x}\|$ satisfies Δ-inequality

$$\|\underset{\sim}{x} + \underset{\sim}{y}\| \leq \|\underset{\sim}{x}\| + \|\underset{\sim}{y}\|$$



- Restrict to scalar ODE's $\dot{x} = f(x,t)$ (nxn same proof?)



☐ Let $x(t)$ & $y(t)$ be two continuous functions defined for $a \le t \le b$

<u>Defn</u>: $\|x(t)-y(t)\|_{sup}$ = "Max'al difference betw $x(t)$ & $y(t)$ as t ranges from a to b"

$$= \max_{a \le t \le b} |x(t)-y(t)|$$

≡ "a measure of the distance between functions called <u>supnorm</u>"

- Not hard to show:

$$\|x(t)+y(t)\|_{sup} \le \|x(t)\|_{sup} + \|y(t)\|_{sup}$$

(Δ-inequality)

$$\|cx(t)\|_{sup} = |c|\,\|x(t)\|_{sup}$$

"just like abs value & Euclidean norm for vectors in $\mathbb{R}^3$"

<u>Defn</u>: $C[a,b]$ = set of all continuous functions $x(t)$ defined for $a \le t \le b$

<u>Defn</u>: Let T be a mapping from $C[a,b]$ to $C[a,b]$:

$$T : C[a,b] \longrightarrow C[a,b]$$

$$T[x(t)] = y(t)$$

cont fn on $[a,b]$

cont fn on $[a,b]$

<u>Theorem</u>: (Contraction Mapping Thm) if T is a contraction, then T has a unique fixed point $\hat{x}(t)$: $T(\hat{x}(t)) = \hat{x}(t)$

<u>Defn</u>: $T : C[a,b] \to C[a,b]$ is a <u>contraction</u> if

$$\| T[x(t)] - T[y(t)] \|_{sup} \leq r \, \| x(t) - y(t) \|_{sup}$$

for some $r < 1$

- This is a special case of the Banach Fixed Pt Theorem: "If T is a contraction on a complete metric space, then T has a unique fixed point" <cf. Advanced Calculus>

Moreover, starting with any initial function $x_0(t)$, and setting

$$x_1(t) = T[x_0(t)]$$

$$x_2(t) = T[x_1(t)] \qquad (*)$$

$$\vdots$$

$$x_{n+1}(t) = T[x_n(t)]$$

BFT $\Rightarrow$ it follows that $x_n(t) \longrightarrow \hat{x}(t)$

That is: $\|x_n(t) - \hat{x}(t)\|_{sup} \longrightarrow 0$.

Conclude: if T is a contraction, then (*) is a numerical method for finding the unique fixed point of T (a function)

- Using the Contraction Mapping thm we can prove the local exist. thm:
- Assume $f(x,t)$ is Lipschitz continuous in x: $\exists\, K>0$ st

$$|f(x_2,t)-f(x_1,t)| \le K|x_2-x_1|$$

$\forall\, x_1, x_2 \in \mathbb{R}$.

- We prove $\exists\, \varepsilon>0$ and a function $\hat{x}(t)$ such that

$$\begin{cases} \dot{\hat{x}}(t) = f(\hat{x}(t),t) & \text{for all} \\ \hat{x}(t_0) = x_0 & t_0-\varepsilon \le t \le t_0+\varepsilon \end{cases}$$

Method- Construct a contraction mapping —

- Start with $$\dot{x} = f(x,t) \qquad (A)$$ $$x(t_0) = x_0$$

Integrate:

$$x(t) - x(t_0) = \int_{t_0}^{t} \dot{x}(\xi)\, d\xi = \int_{t_0}^{t} f(x(\xi),\xi)\, d\xi$$

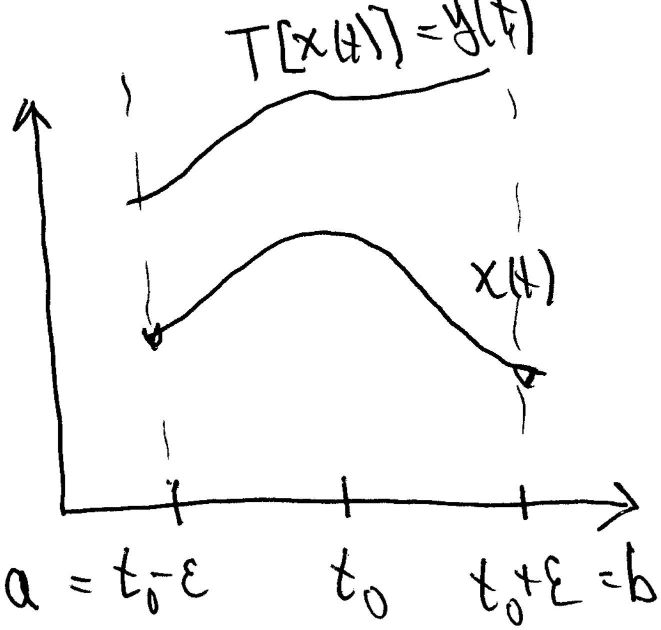
$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\xi),\xi)\, d\xi \qquad (B)$$

It is not difficult to show that if $x(t)$ is a continuous function that solves (B), then it solves (A) as well, & vise-versa.

- Define a contraction mapping T:

$$T[x(t)] = x_0 + \int_{t_0}^{t} f(x(\xi),\xi)\,d\xi \qquad t \in [t_0-\varepsilon, t_0+\varepsilon]$$

T takes functions in $C[a,b]$ to functions in $C[a,b]$.



<u>Claim</u>: T is a contraction for ε sufficiently small $\Rightarrow$ the unique fixed pt is the unique solution!

$$T[x(t)] = x_0 + \int_{t_0}^{t} f(x(\xi),\xi)\,d\xi$$

$$T[y(t)] = x_0 + \int_{t_0}^{t} f(y(\xi),\xi)\,d\xi$$

So

$$\| T[x(t)] - T[y(t)] \|_{sup} = \left\| \int_{t_0}^{t} f(x(\xi),\xi) - f(y(\xi),\xi)\,d\xi \right\|_{sup}$$

$$\leq \left\| \int_{t_0}^{t} | f(x(\xi),\xi) - f(y(\xi),\xi) |\,d\xi \right\|_{sup}$$

$$\leq \left\| \int_{t_0}^{t} K\,|\underbrace{x(\xi) - y(\xi)}_{\text{largest value}}|\,d\xi \right\|_{sup}$$

$$\leq 2\varepsilon K \, \| x(t) - y(t) \|_{sup}$$

Use: $|t-t_0|\leq\epsilon$

Conclude: if $\varepsilon < \frac{1}{2K}$, then T is a contraction $\Rightarrow$ fixed pt $\hat{x}(t)$ is our unique solution.

- Moreover: we have proven that a numerical method converges —

$$x_0(t) = x_0$$

$$x_1(t) = T[x_0] = x_0 + \int_{t_0}^{t} f(x_0, t)\,dt$$

$$\vdots$$

$$x_{n+1}(t) = T[x_n(t)] = x_0 + \int_{t_0}^{t} f(x_n(t), t)\,dt$$

Thm $\Rightarrow$ $x_n(t) \longrightarrow x(t)$ in sense

$$\|x_n(t) - x(t)\|_{sup} \longrightarrow 0 \checkmark$$

this is call Picards Method

<u>Note</u>: Proof assumes Lipshitz const K holds $\forall\, x_1, x_2$. A more careful analysis shows that we only needed

$$|f(x_2,t) - f(x_1,t)| \leq K|x_2 - x_1|$$

$\forall\, x_1, x_2 \in (x_0 - \delta, x_0 + \delta)$ since ~~this~~ approx soln's x_n always take values in this interval —