Mon 1-23-02 MAT119A

Potential Energy & Overdamped System ①

§2.6, 2.7

Energy:

Q: How much energy is stored in a car of mass M moving at velocity v? I.e., how much work would be done on you if you abruptly came to a stop?

Ans: $KE = \frac{1}{2} M v^2$

- How do we know this is correct?

2 main answers —

① Work Done $= \int_a^b F \cdot ds = \int_a^b m a \, ds$

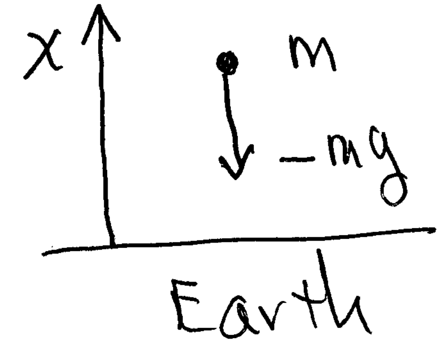
$$= \int_a^b m \frac{dv}{dt} ds = \int_{v_a}^{v_b} m \frac{ds}{dt} dv$$

$$\frac{ds}{dt} = v$$

$$= \int_{v_a}^{v_b} m v \, dv = \frac{1}{2} m v^2 \Big]_{v_a}^{v_b} = \Delta KE$$

So: The work done is the change in Kinetic Energy –

(2) Assume mass m moves in a force field:



Eg $F = ma = -mg$

Rewrite: $ma = -\frac{d}{dx}(mgx)$

"mgx = potential energy" $\boxed{U(x) = mgx}$

More generally: assume $F = -U'(x)$

$$\boxed{F = ma = -U'(x)}$$ ODE that describes the motion

$U(x) \equiv$ "potential"

Claim: $E = \frac{1}{2}mv^2 + U(x)$ is constant along solutions

Pf. $\frac{d}{dt}E(x(t)) = \frac{d}{dt}\left\{\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x(t))\right\}$

$= \frac{1}{2}m \cdot 2 \frac{dx}{dt}\frac{d^2x}{dt^2} + U'(x(t))\frac{dx}{dt}$

$$E(x(t)) = \left[m\frac{d^2x}{dt^2} + U'(x(t))\right]\frac{dx}{dt} = 0$$

$$ma + U' = 0 \text{ by equation!}$$

"The energy is constant in a conservative forcefield where $F = -\frac{d}{dx}U(x)$"

$$E = \frac{1}{2}mv^2 + U(x) = \text{const}$$

"All along the solution KE is stored & released as potential energy"

- Note: same is true when $\underset{\sim}{x}$ is a vector & equation is

$$m\ddot{\underset{\sim}{x}} = -\nabla U(\underset{\sim}{x})$$

$\Rightarrow E = \frac{1}{2}m|\dot{\underset{\sim}{x}}|^2 + U(\underset{\sim}{x})$ is constant along soln's. This describes every conservative force field!

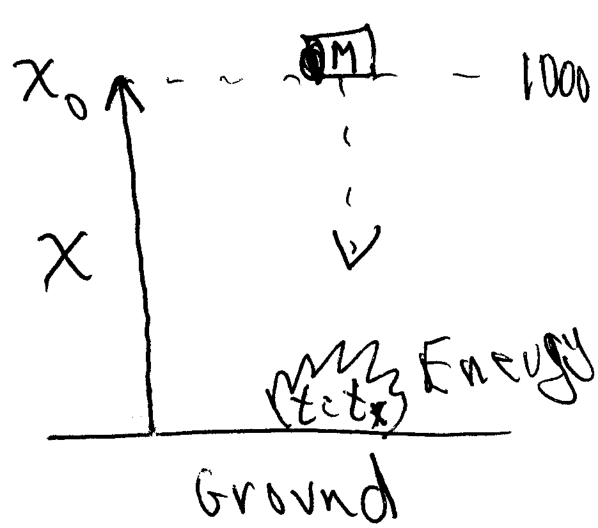
☒ **Note**: only special ODE's are conservative. Eg if $x(t) \in \mathbb{R}$ &

$$m\ddot{x} = -U(x) \quad \text{is the eqn:}$$

Write as 1st order system: $\dot{x} = y$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -U(x) \end{pmatrix}$$ Very special 1st order system

• **Ex**: How much energy is stored in a 100 kg mass sitting 1000 meters above the ground?



$$E = \frac{1}{2}mv_0^2 + mgx_0$$

$t=0 \quad v_0 = 0 \quad E = mgx_0$

$t = t_* \quad x_0 = 0 \quad E = \frac{1}{2}mv_*^2$

Conclude from E = const:

$$\frac{1}{2} m v_*^2 = \text{energy of impact} = m g x_0$$

$$= 100\,kg \times 9.8 \frac{meter}{s^2}\, 1000\, meter$$

$$= 980000 \frac{kg\ meter^2}{s^2} \checkmark$$

Also: $v_* = \sqrt{2 g x_0} = \sqrt{2 \times 9800}\, \frac{meter}{s}$

$$\approx 140 \frac{meters}{s} \checkmark$$

☐ Example from Book – Force with Friction: 2.6-2.7

- Friction is a non conservative force proportional to minus velocity:

Eg $$ma = \text{Force} = \underbrace{-U'(x)}_{\text{conservative part of force}} \underbrace{- k\dot{x}}_{\text{nonconservative force of friction}}$$

Equation:

$$\boxed{m\ddot{x} = -U'(x) - k\dot{x}}$$

<u>Eg</u>: Spring with Friction:

Force of Spring $\propto$ minus displacement

$$F_s = -ax = -\frac{d}{dx}\left(\frac{a}{2}x^2\right)$$

a ↑ spring const

$\frac{a}{2}x^2$: $U(x)$ the potential

Force of Friction:

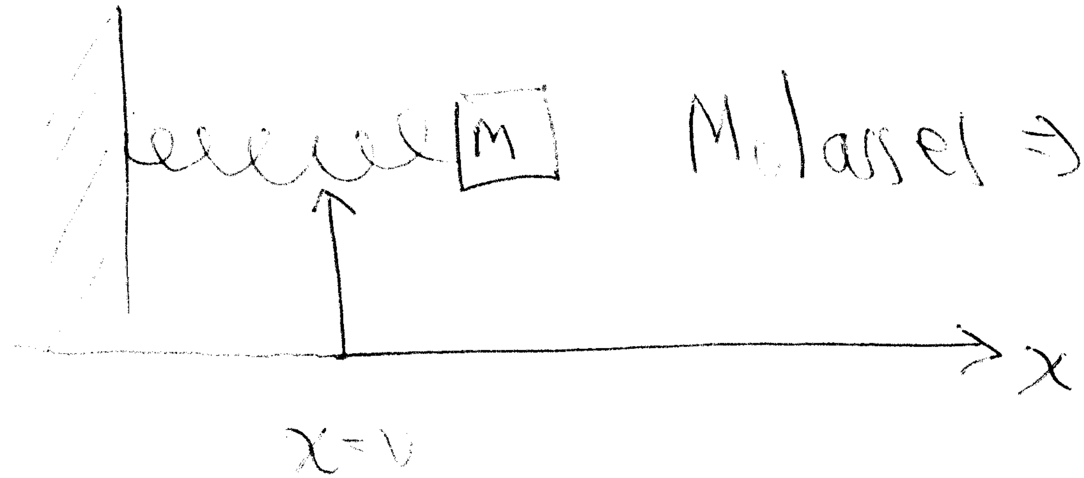
$$F_f = -kv = -k\dot{x}$$

Equation:

$$m\ddot{x} = -\frac{d}{dx}\left(\frac{a}{2}x^2\right) - k\dot{x}$$

$-\frac{d}{dx}\left(\frac{a}{2}x^2\right)$: conservative Force

- Overdamped: "When friction is much larger than acceleration and mass"



Molasses ⇒ $k \gg 1$

Solve for $\dot{x}$:

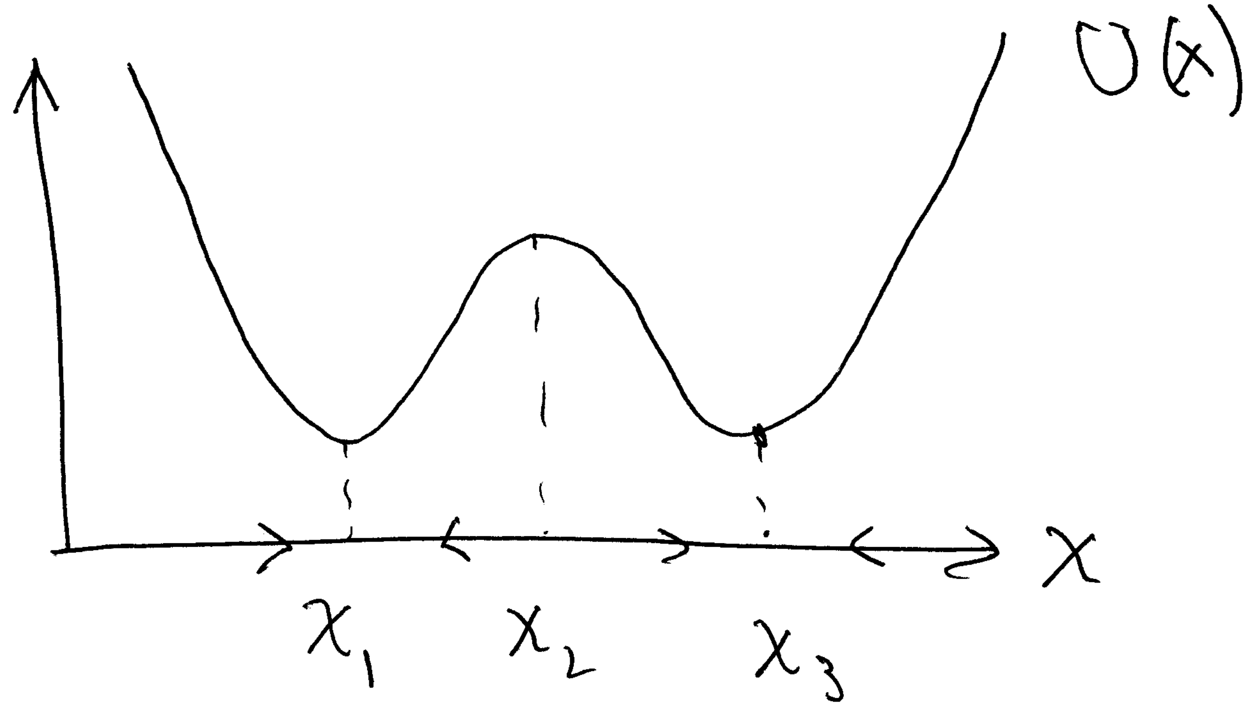
$$\dot{x} = -\frac{d}{dx}\left(\frac{a}{2k}x^2\right) - \frac{m}{k}\ddot{x}$$

when this is really small ≈ overdamped ≈ 0

$$\boxed{\dot{x} = -\frac{d}{dx}U(x)}$$

Equation for overdamped system with potential $U(x)$

- $\dot{x} = -\frac{d}{dx}U(x)$ is a 1st order scalar eqn an interpretation as an overdamped motion in potential.
- Solve by phase portrait –



- rest pts ≡ max/min of $U(x)$
- Solutions head straight for the minima of the potential –