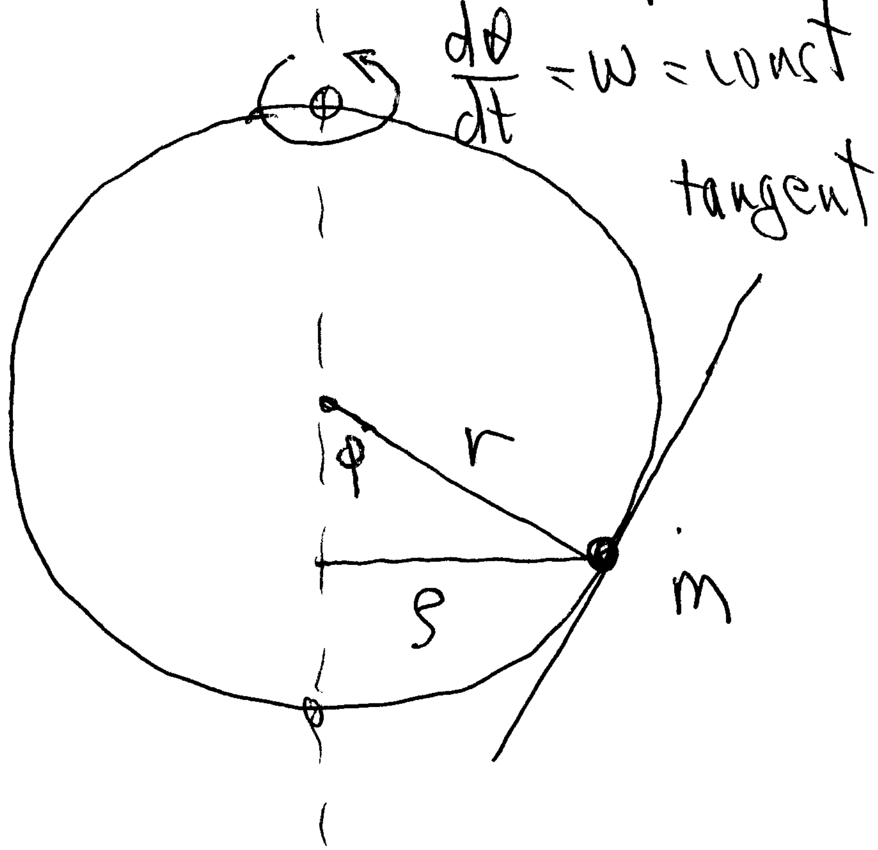
Wed 1-25-02 | 119A - § 3.6 |

E) Overdamped Bead on Rotating Hoop

- We wish to derive the ODE'S describing a mass m (bead) constrained to a fixed rotating hoop.

Picture:



3 forces: gravity, friction, "centrifical force"

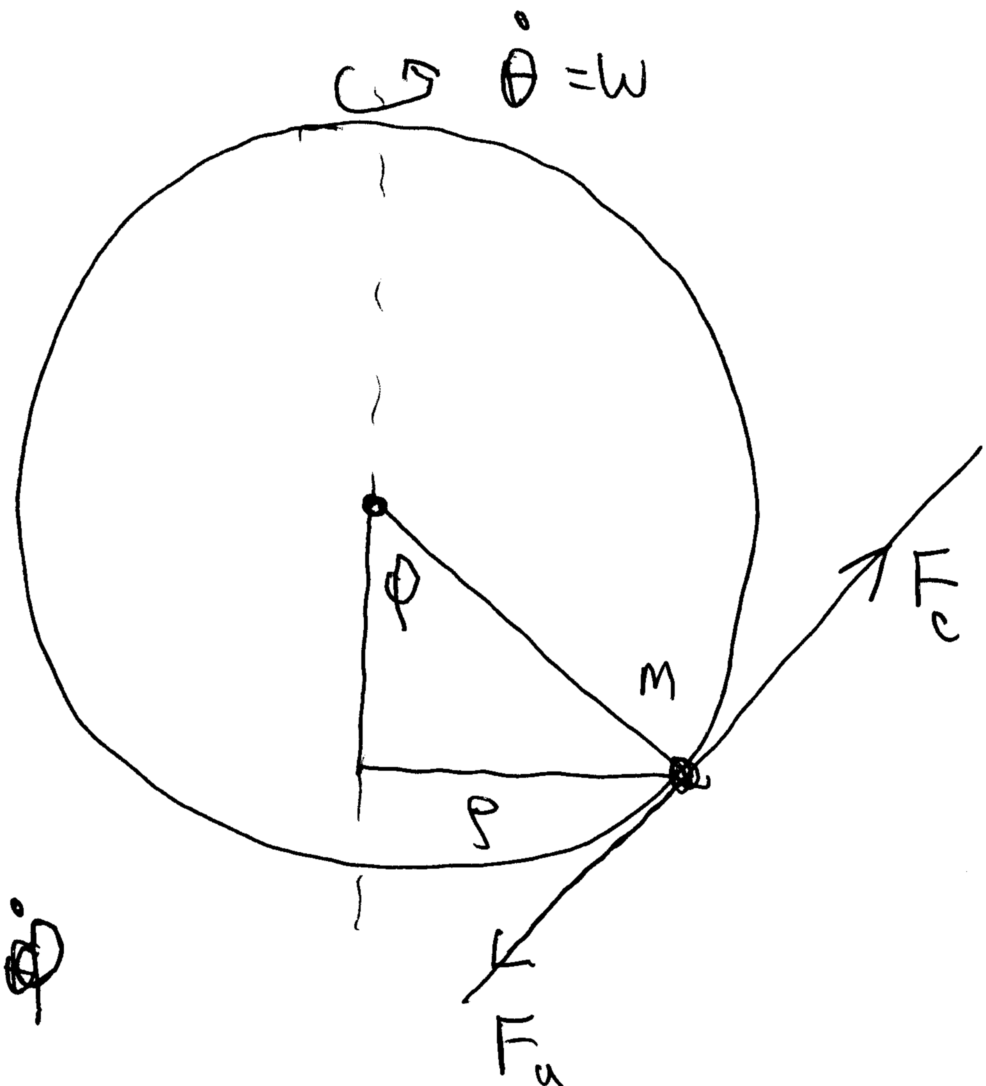
" gravity pulls bead down, spinning pushes it out, and friction slows it down "

- Since the mass can only move tangent to hoop – and $ds = r\,d\varphi$ = arclength

$r\frac{d\varphi}{dt}$ = velocity of mass $= r\dot{\varphi} = \frac{ds}{dt}$

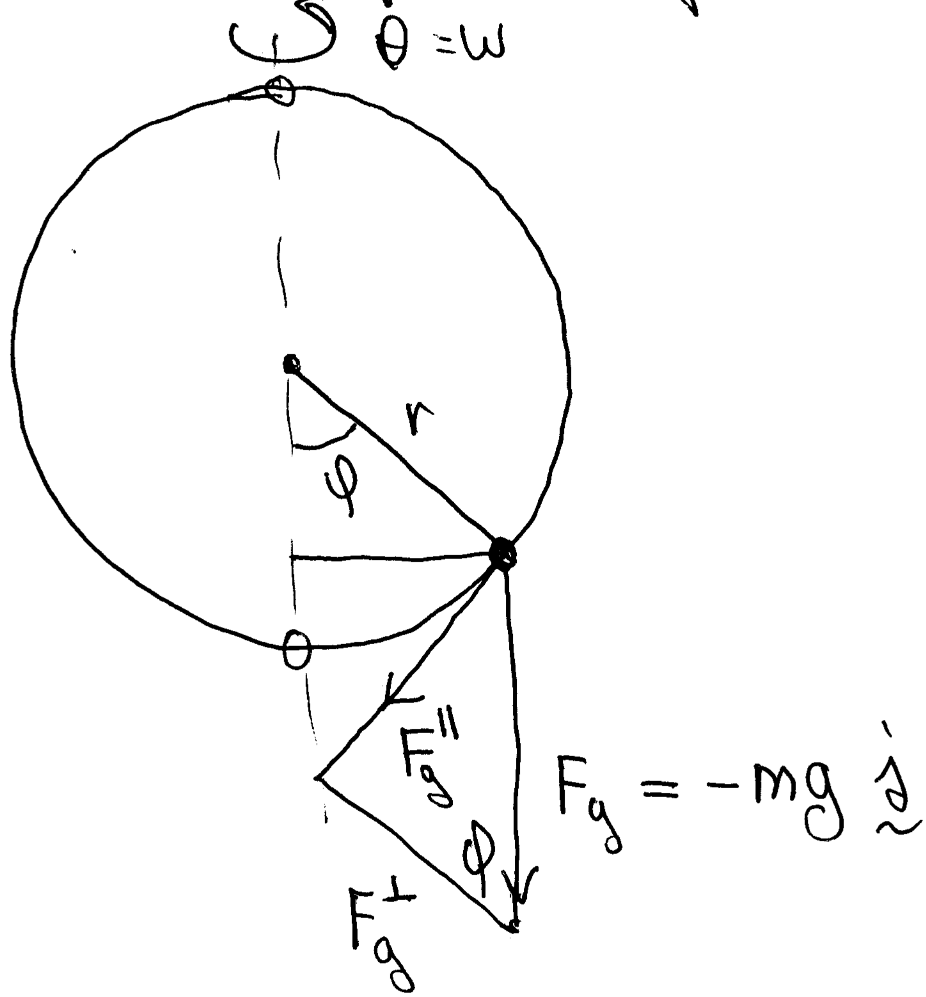
$r\ddot{\varphi}$ = acceleration (tangent to hoop) $= \frac{d^2s}{dt^2}$

We balance forces tangent to hoop –



$F_f = -\underbrace{k}_{b} r\dot{\varphi}$

- Consider force of gravity –

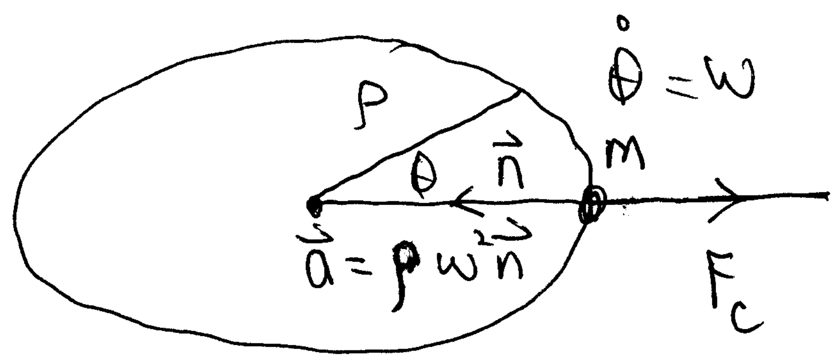


F_g creates m×(acceleration) $mr\ddot{\varphi}$ tangent to hoop by magnitude of the component $F_g^{\parallel}$ of F_g tangent to hoop $= -mg \sin\varphi$

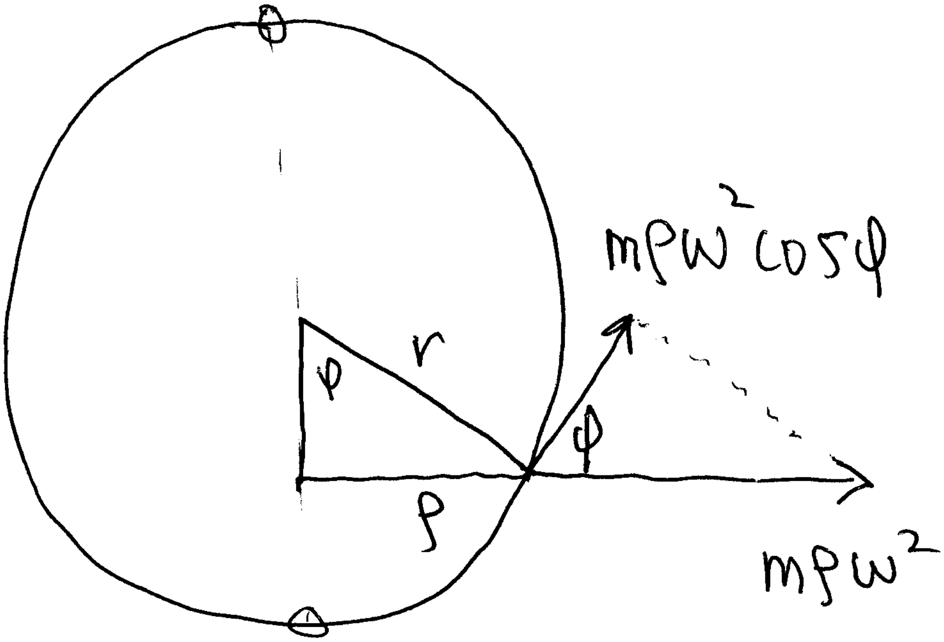
<u>Note</u>: because acceleration downward creates negative acceleration in $r\dot{\varphi}$

• Consider "centripetal force"

Recall: a mass moving at constant angular velocity $\omega = \frac{d\theta}{dt} = \dot{\theta}$ accelerates toward center by $\vec{a} = \rho \omega^2 \vec{n}$



Conclude: the spinning hoop creates a horizontal force outward of magnitude $m \rho \omega^2$



This contributes an acceleration $r\ddot{\varphi}$ tangent to hoop of magnitude

$$+\, m\rho\omega^2\cos\varphi = mr\omega^2\sin\varphi\cos\varphi$$

$\rho = r\sin\varphi$

Equation –

$$\boxed{mr\ddot{\varphi} = -b\dot{\varphi} - mg\sin\varphi + mr\omega^2\sin\varphi\cos\varphi}$$

$\left(m\frac{d^2s}{dt^2}\right)$ $\quad F_f \quad F_g^{\parallel} \quad F_c^{\parallel}$

c Overdamped system – "acceleration small relative to friction"

$$\cancel{mr\ddot{\varphi}} = -b\dot{\varphi} - mg\sin\varphi + mr\omega^2\sin\varphi\cos\varphi$$

≈ 0

$$b\dot{\varphi} = -mg\sin\varphi + mr\omega^2\sin\varphi\cos\varphi$$

$$b\dot{\varphi} = mg\sin\varphi\left(\frac{r\omega^2}{g}\cos\varphi - 1\right)$$

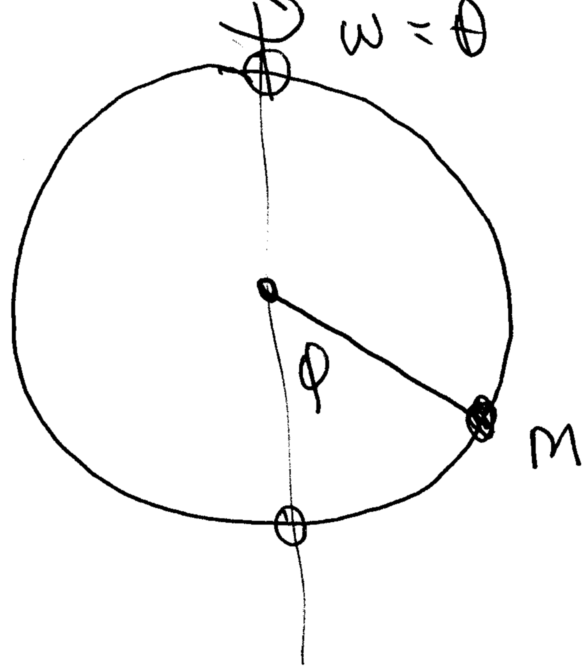
$$\boxed{\dot{\varphi} = \frac{mg}{b}\sin\varphi\left(\frac{r\omega^2}{g}\cos\varphi - 1\right)} = f(\varphi)$$

- Draw phase portrait (autonomous 1st order nonlinear system)

Find rest pts :

$$0 = f(\varphi) \Leftrightarrow \sin\varphi\left(\frac{r\omega^2}{g}\cos\varphi - 1\right) = 0$$

$$\boxed{\begin{array}{c}\sin\varphi = 0 \\ \frac{r\omega^2}{g}\cos\varphi = 1\end{array}}$$ or

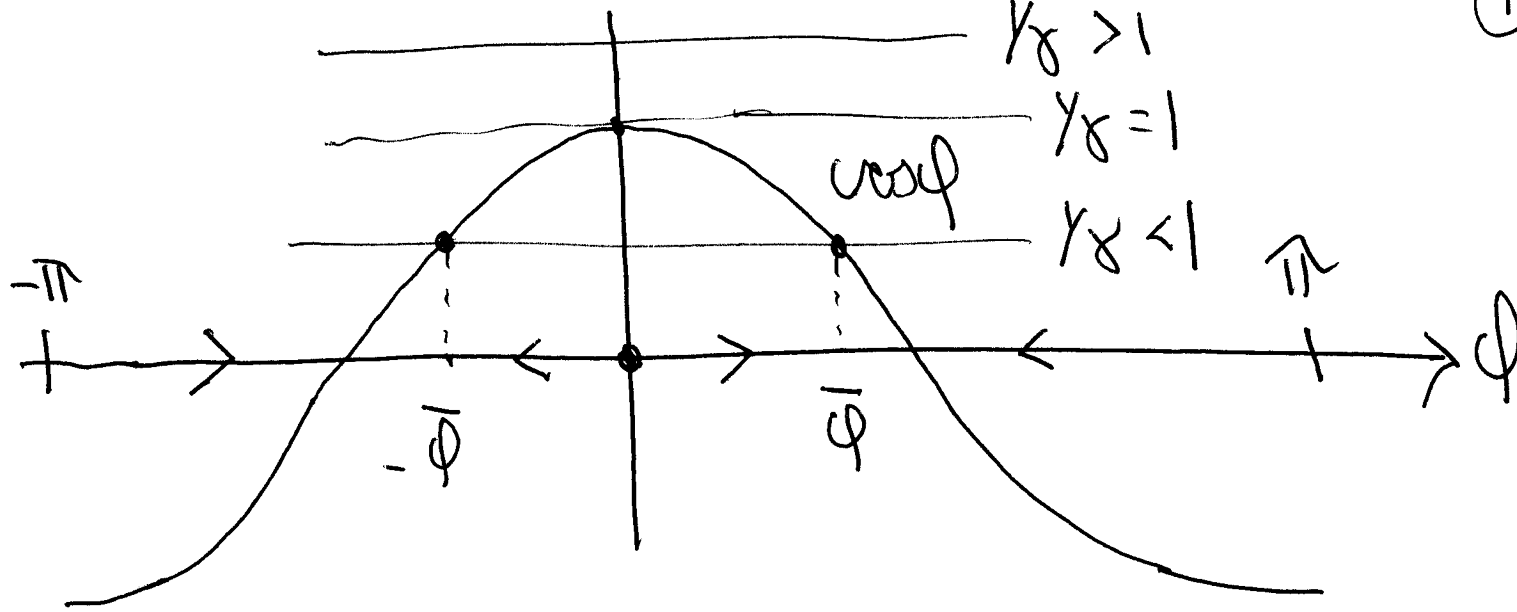


$-\pi \le \varphi \le \pi$

- $\sin\varphi = 0 \Rightarrow \varphi = 0$
  always a rest pt

- $\gamma = \frac{r\omega^2}{g}$ $\cos\varphi = \frac{1}{\gamma}$

  only rest points when $\frac{1}{\gamma} > 1$



- Two new rest points emerge from $\bar{\varphi} = 0$ when γ passes thru $\gamma = 1$.
- γ is a <u>parameter</u> in the equation that creates a "bifurcation" in the phase diagram of solutions —

"Two stable rest points emerge from rest pt $\varphi = 0$ when $\gamma = 1$"

<u>Check</u>: $$f'(\bar{\varphi}) = \frac{d}{d\varphi}\left\{\frac{mg}{b}\sin\varphi(\gamma\cos\varphi - 1)\right\}\Bigg|_{\varphi=\bar{\varphi}}$$

$$= \left(\frac{mg}{b}\cos\bar{\varphi}\right)\cdot 0$$

$$+ \frac{mg}{b}\sin\bar{\varphi}\,(-\gamma\sin\bar{\varphi})$$

$$= -\frac{mg}{b}\gamma\sin^2\bar{\varphi} < 0 \Rightarrow \bar{\varphi} \text{ stable}$$

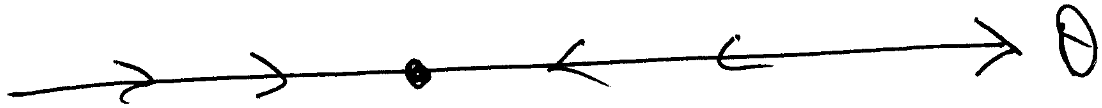
$$f'(-\bar{\varphi}) = -\frac{mg}{b}\gamma\sin^2(-\bar{\varphi}) < 0 \Rightarrow -\bar{\varphi} \text{ stable}$$

$$f'(0) = \frac{d}{d\varphi}\left\{\frac{mg}{b}\sin\varphi(\gamma\cos\varphi - 1)\right\}\Bigg|_{\varphi=0}$$

$$= \frac{mg}{b}\cos(0)(\gamma\cos(0) - 1)$$

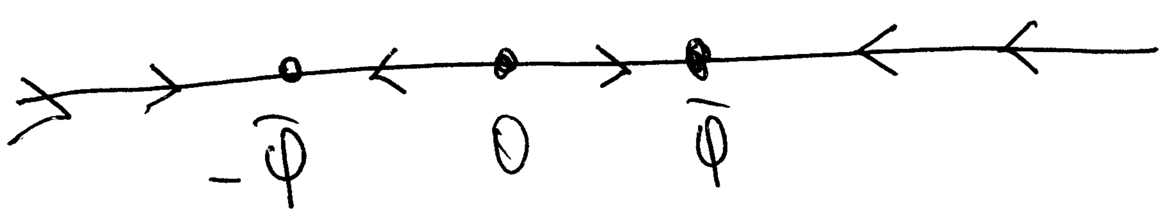
$$= \frac{mg}{b}(\gamma - 1) = \begin{cases} < 0 & \gamma < 1 \\ > 0 & \gamma > 1 \end{cases}$$

Phase Portrait for $\gamma < 1$



(one stable rest pt at $\varphi = 0$)

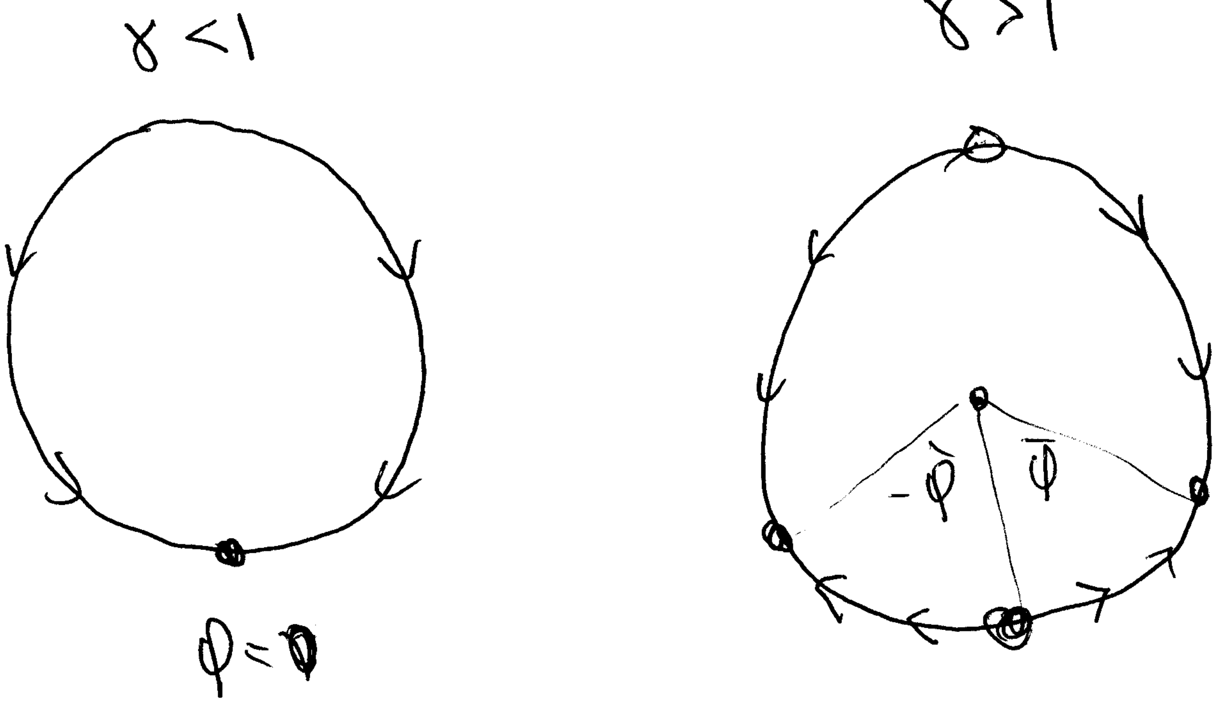
Phase portrait for $\gamma > 1$



(two stable rest points emerge)

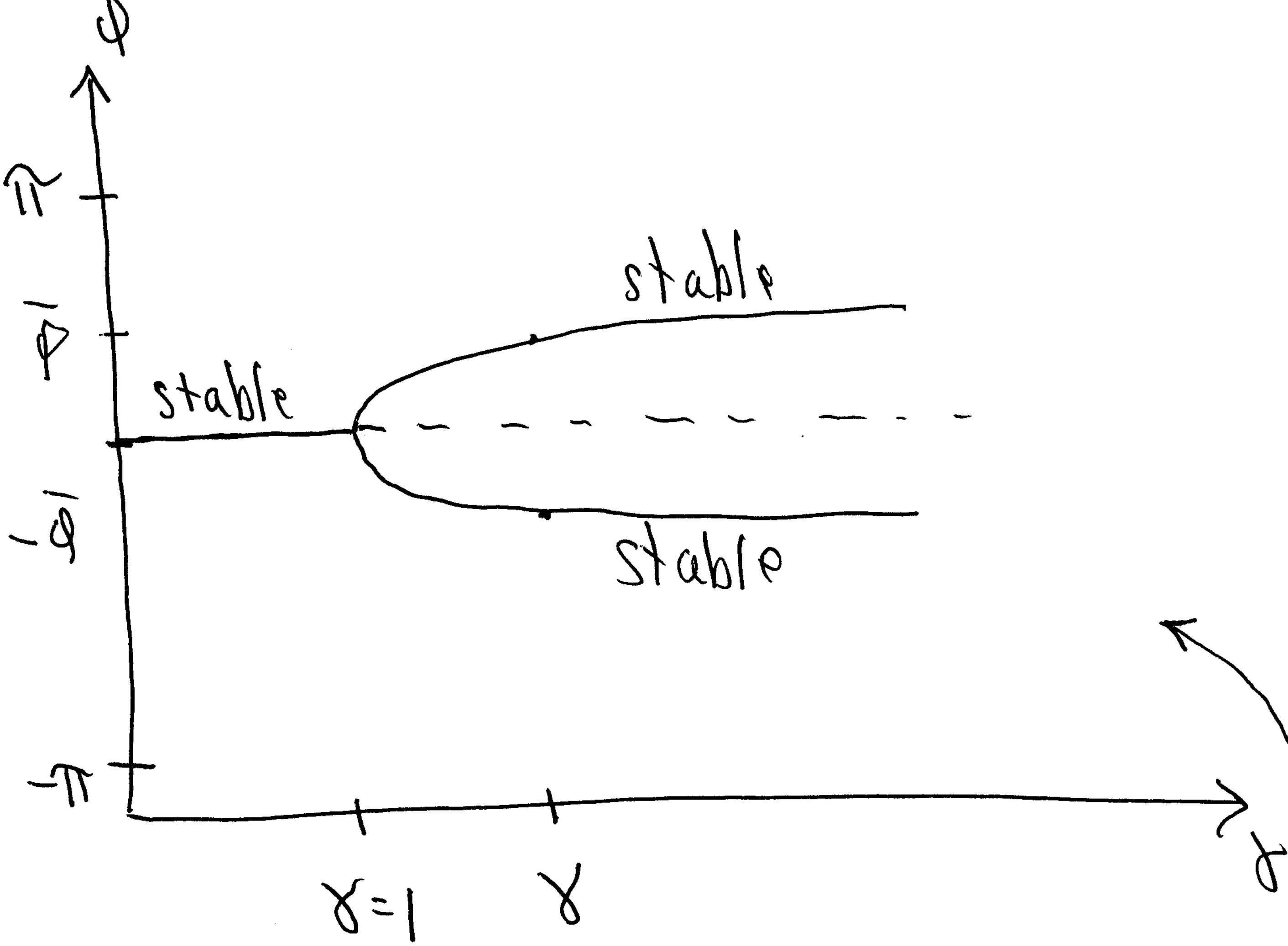
- "Solutions tend to nearest downstream rest point" $\Rightarrow$ "symmetry breaking"

Picture:



"Bead find point of balance betw F_c & F_g"

Bifurcation Diagram –



Q: Three basic types of bifurcation

- saddle node / transcritical / Pitchfork*

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Exam Thru § 3.5 (stop at Dimensional Anal)