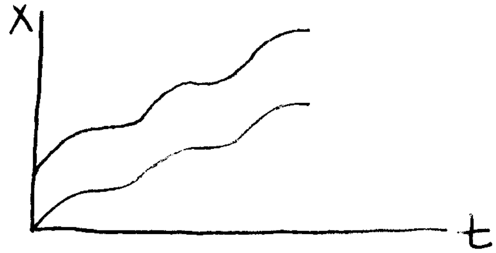
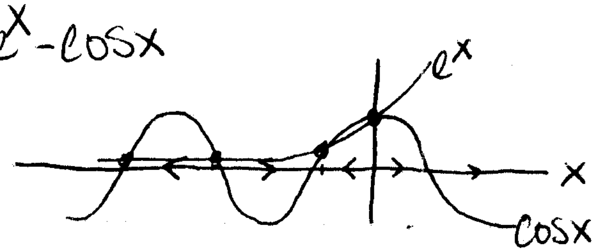
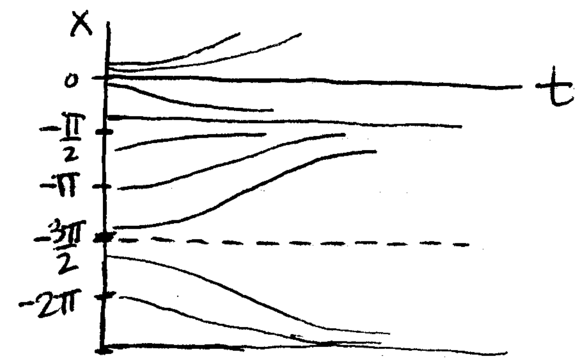
Homework 1 Solutions

2.2.5 $\dot{x} = 1 + \frac{1}{2}\cos x$ this is always positive -- no fixed points.

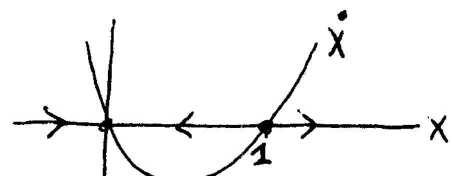


2.2.7 $\dot{x} = e^x - \cos x$





2.2.9 We want



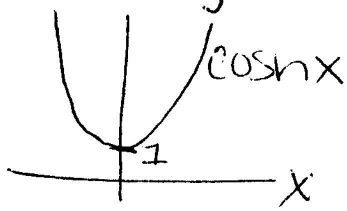
So, something like $f(x) = x(x-1)$ should do it.

2.2.12 We have $I = g(V)$, which we can taylor expand: $I \approx c_1 V + c_3 V^3 + \dots$ (notice g is odd, so no even terms) this system must behave like the linear system near zero, so we get $I \approx V/R + cV^3 + \dots$

If we fix $V = V_0$ and replace $I_R = V/R$ with $I_R = g(V_0)$, then we will get a similar result as in the example. $\dot{Q}$ is still linear with negative slope, but the fixed point will be shifted to the right or left.

2.7.5 $\dot{x} = -\sinh x = -\frac{e^{x}-e^{-x}}{2} = -\frac{dV}{dx}$

$\Rightarrow V(x) = \int \frac{1}{2}(e^{x}-e^{-x})\,dx = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} + C = \cosh x + C$



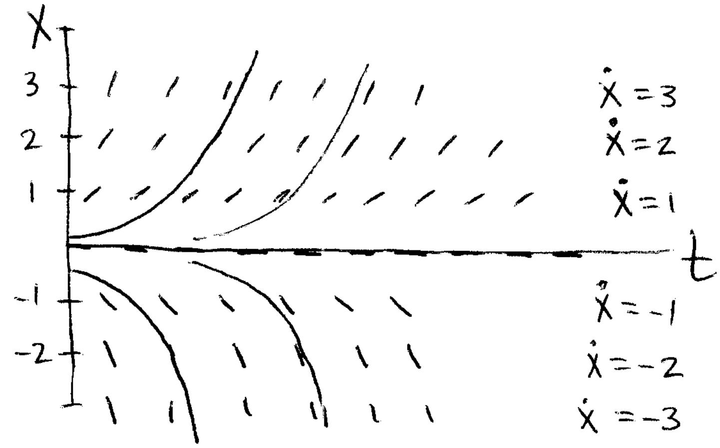
there is a stable fixed point at $x=0$

2.7.7 We know $V(t)$ decreases along trajectories, or we can think of a particle sliding down the walls of a potential well. With this picture in mind, in order for $x(t)$ to oscillate with time, we would have to see the particle slide back up the potential well -- this is impossible.

2.8.1 The slope, $\dot{x}$, is only dependent upon x, NOT time t $\therefore$ this is why the slope is constant along horizontal lines.

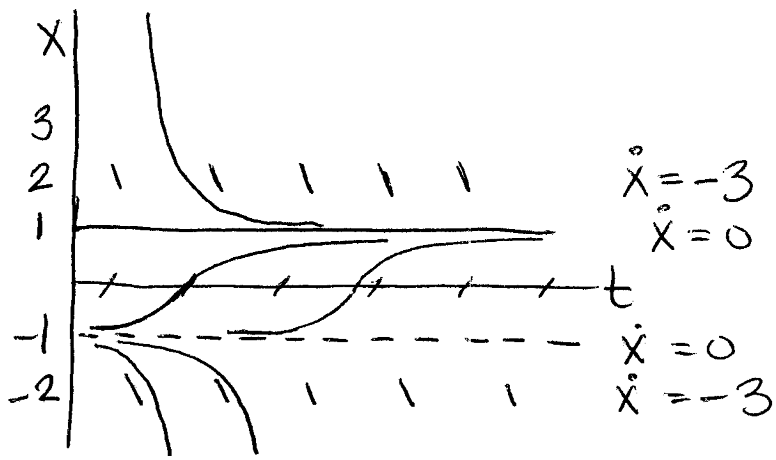
$x = c \Rightarrow \dot{x} = f(c)$ for any constant c.

2.8.2 a) $\dot{x} = x$

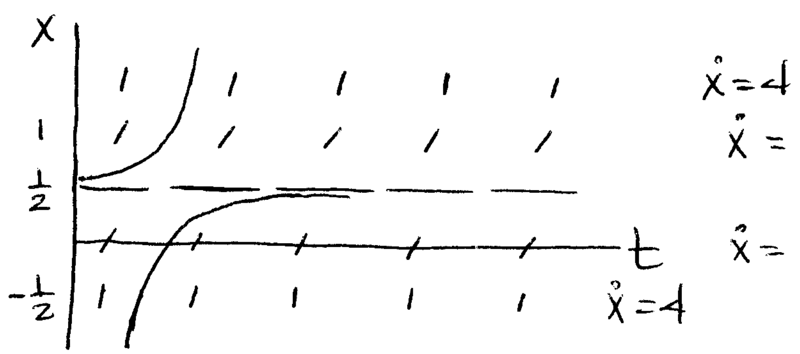


Solutions are of the form $x(t) = x_0 e^{t}$

b) $\dot{x} = 1 - x^2$

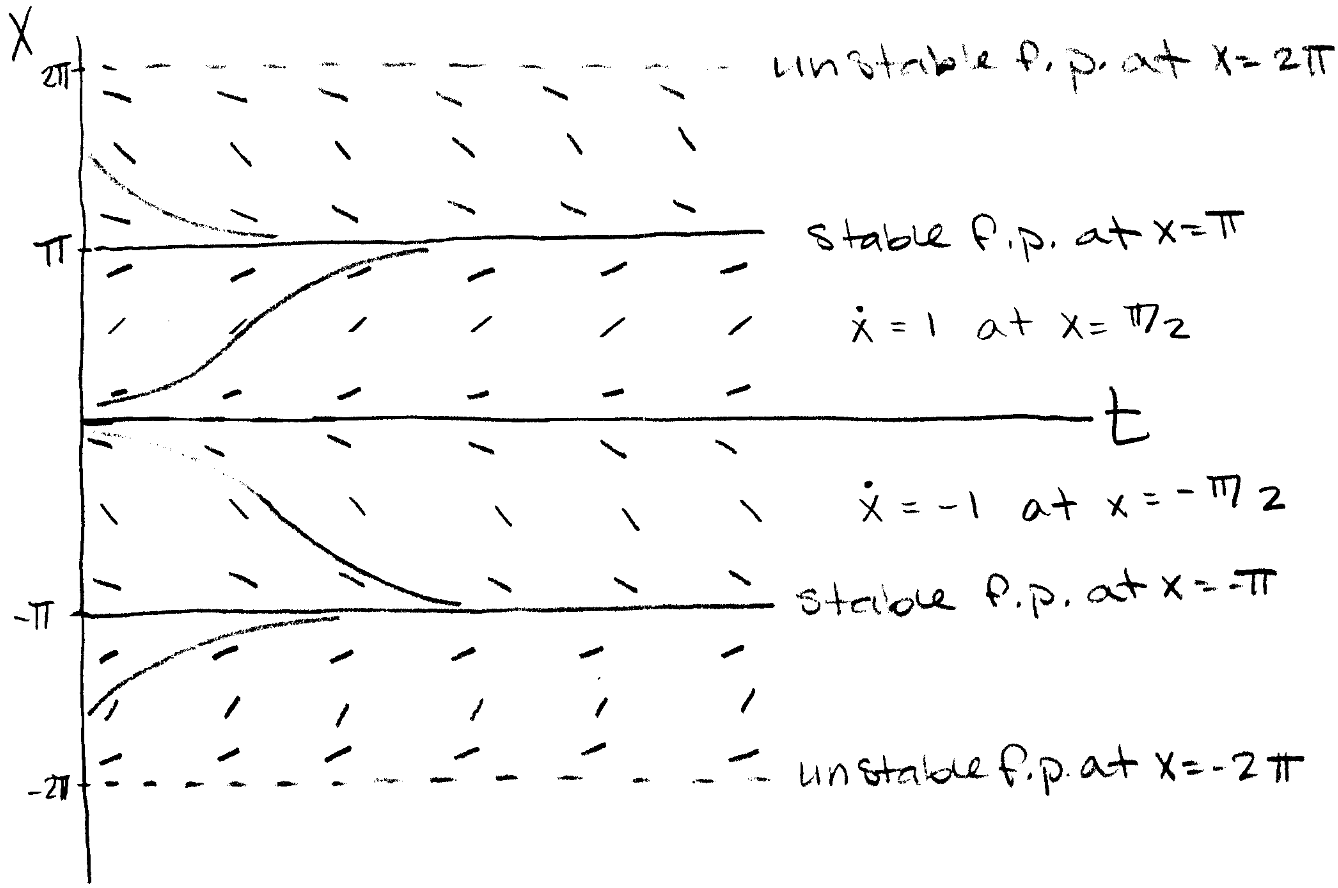


c) $\dot{x} = 1 - 4x(1-x)$



2.8.2

d) $\dot{x} = \sin x$



Prob. 2.8.3

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                        < M A T L A B >
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                   Version 7.3.0.298 (R2006b)
                        August 03, 2006

  To get started, select MATLAB Help or Demos from the Help menu.

>> mat119A_2_8_3

dt =

    0.1000


ans =

    0.3487


dt =

    0.0100


ans =

    0.3660


dt =

   1.0000e-03


ans =

    0.3677


dt =

   1.0000e-04


ans =

    0.3679

>>
```

parts (a) and (b)

actual $X(1) = e^{-1} \approx .3679$

for $\Delta t = 1$, we get

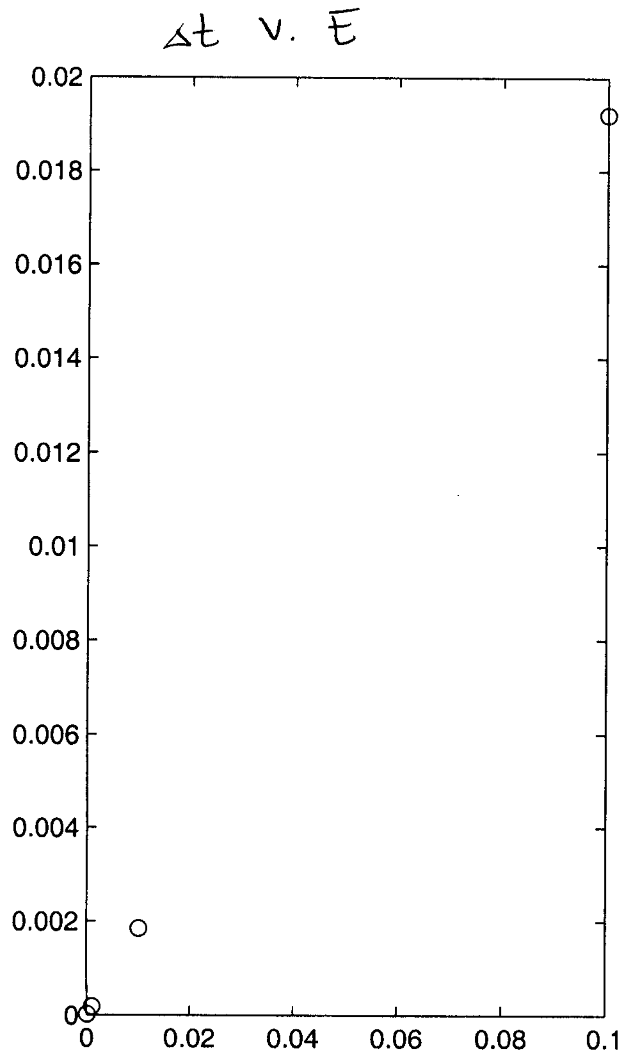
$X_0 = 1$

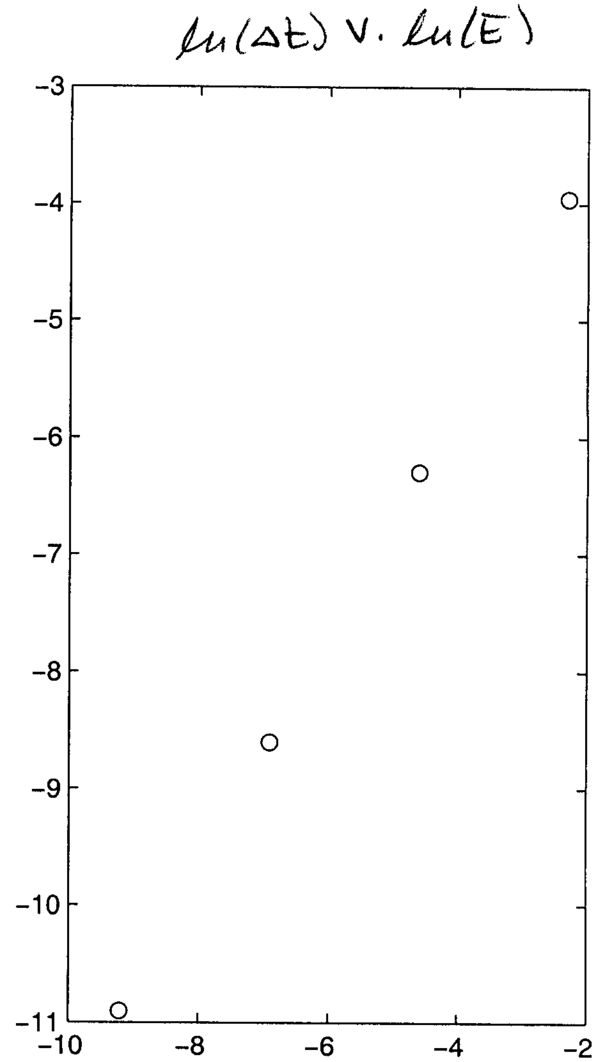
$X_1 = X_0 - \Delta t X_0 = 0$

(bad approximation)

0.3679 ← good approximation

2.8.3 (c)





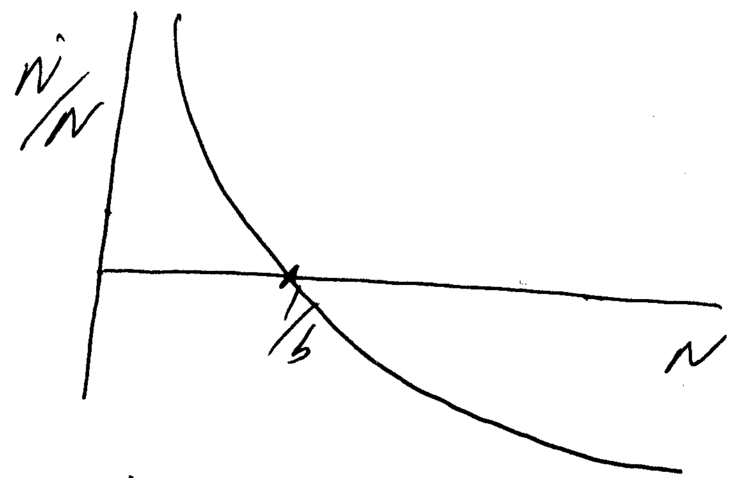
In both plots we see that E decreases as Δt decreases. The log-log plot is useful for very small values because it spreads them out to see their behavior more ~~clearly~~ clearly.

2.3.3 Tumor Growth

$$\dot{N} = -aN\ln(bN), \quad a,b>0$$

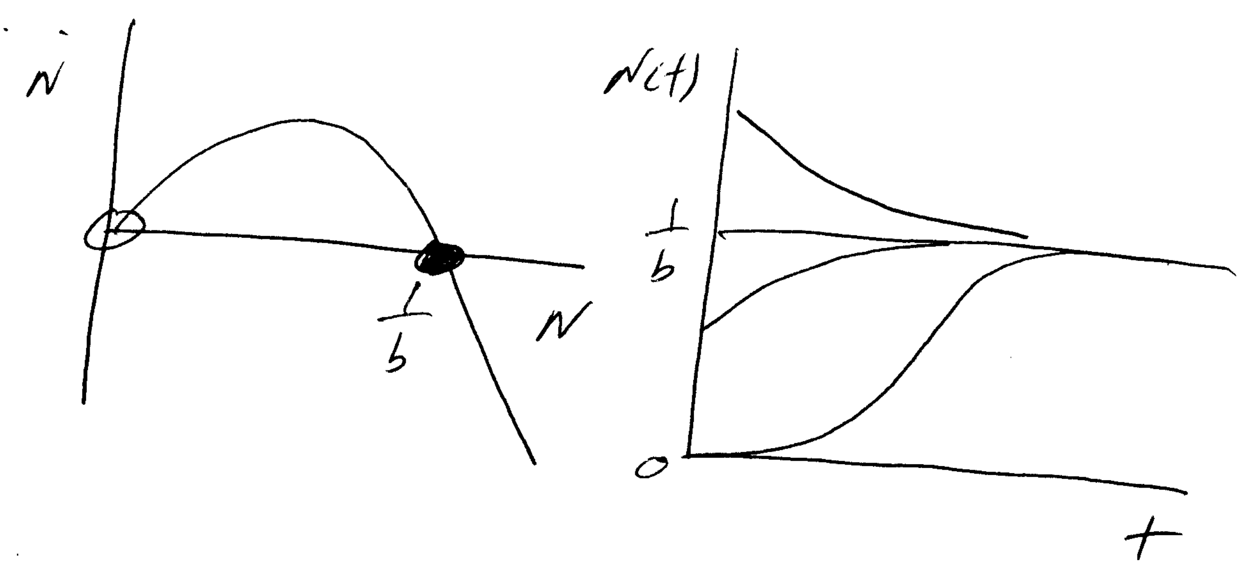
a) $\dot{N}/N = -a\ln(bN)$

$\Rightarrow$



Thus, b controls the maximum population size of the tumor cells and a is a scaling factor that determines how quickly you get to the max.

b)



2.3.4 The Allee effect

$\dot{N} = rN - aN(N-b)^2$

a) $\dot{N}/N = r - a(N-b)^2$

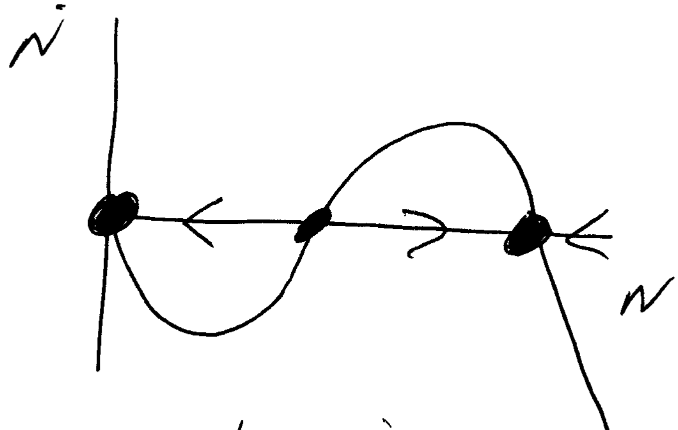
according to the description, we want

$\dot{N} < 0$ when N is small and

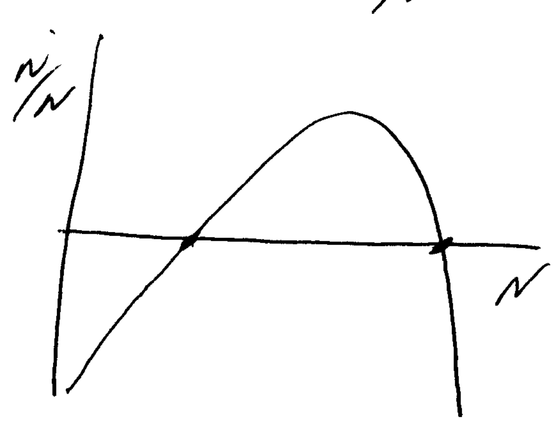
$\dot{N} < 0$ when N is large and

$\dot{N} > 0$ when N is intermediate.

i.e. something like



So we want $\dot{N}/N$ to look like



with $\dot{N}/N < 0$ for $N \leq 0$

So, we choose $a > 0$ in order to make sure that we have a maximum for some intermediate N, and then we set.

$$\dot{N}/N = r - a(N-b)^2 < 0 \quad \text{for } N \leq 0$$

$$\Rightarrow r - a(-b)^2 < 0$$

$$\Rightarrow r < ab^2$$

Thus, our restrictions are

$a > 0$ (but note that if $r < 0$, the polynomial has no real roots, i.e. $\dot{N}/N$ )

$b^2 > \frac{r}{a}$

and r can be any $\mathbb{R}$.

b) $$\dot{N} = rN - aN(N-b)^2 = 0$$

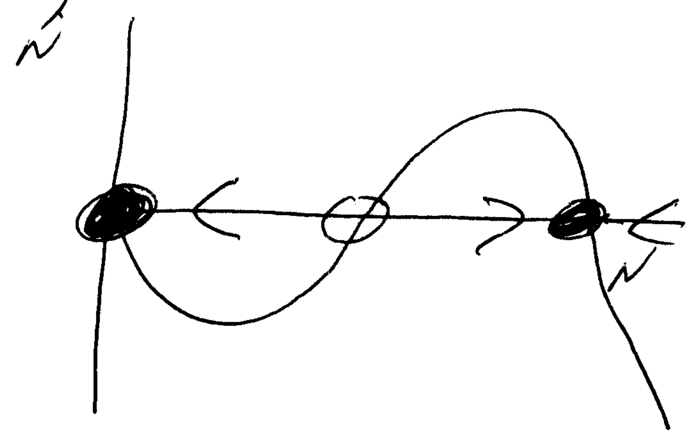
$$r = a(N-b)^2$$

$$(N-b)^2 = \frac{r}{a}$$

$$N - b = \pm\sqrt{\frac{r}{a}}$$

Fixed pts. $N^* = b \pm \sqrt{\frac{r}{a}}\,,\ 0$

Recall,

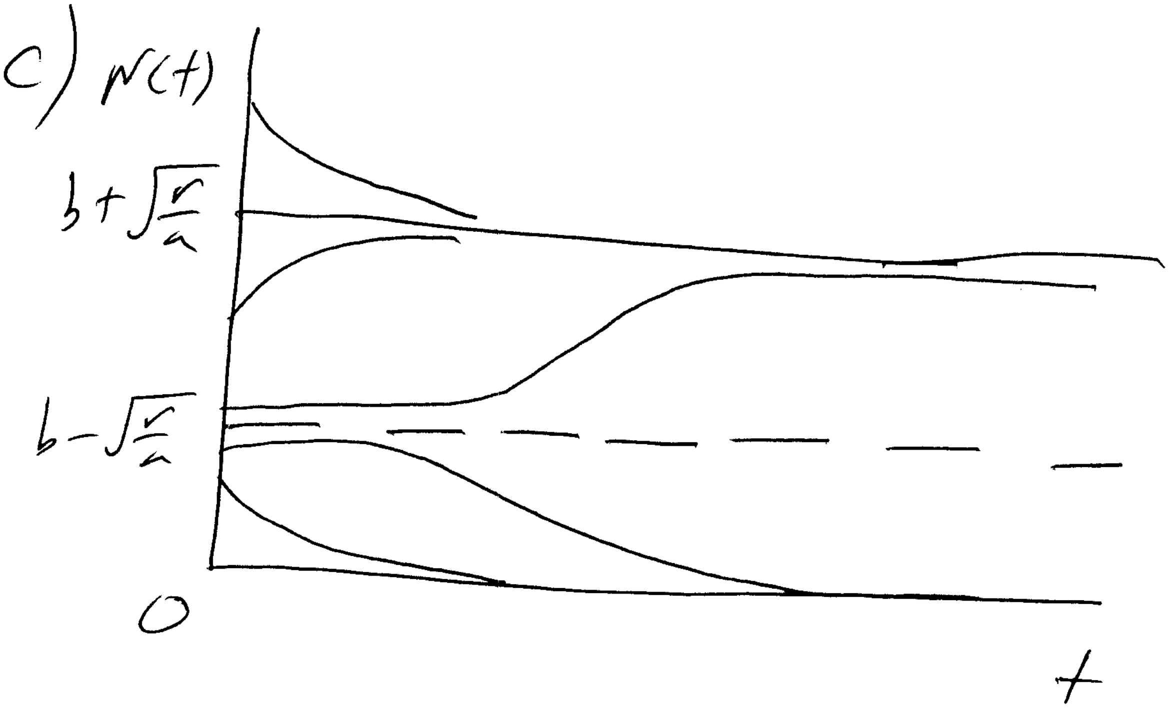


$\Rightarrow$ 0 is stable

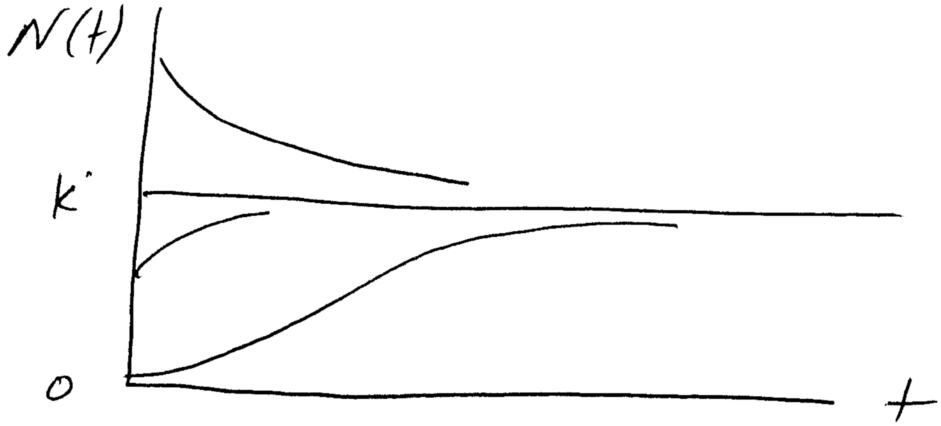
$b - \sqrt{\frac{r}{a}}$ is unstable

$b + \sqrt{\frac{r}{a}}$ is stable

c)



d) Logistic Model



Qualitative Differences:

- Logistic model has 2 f.p. while Allee effect has 3.

2.4.8 $\dot{N} = -aN\ln(bN)$, $a,b > 0$

$f(N) = -aN\ln(bN)$

$f'(N) = -a\ln(bN) - a$

fixed points

$-aN\ln(bW) = 0$

$\Rightarrow \quad N^* = 0, \frac{1}{b}$

$$f'(0) = -a\ln(b\cdot 0) - a$$
$$= -a(-\infty) - a$$
$$= +\infty - a \quad > 0$$

$\Rightarrow$ 0 is unstable

$$f'(\tfrac{1}{b}) = -a\ln(b\cdot\tfrac{1}{b}) - a$$
$$= -a\ln(1) - a$$
$$= -a < 0$$

$\Rightarrow \frac{1}{b}$ is stable.

2.5.2 $\dot{x} = 1 + x^{10}$

Look at $\dot{y} = 1 + y^2$

$$\frac{dy}{dt} = 1 + y^2$$

$$\int \frac{1}{1+y^2}\, dy = \int dt$$

$$\tan^{-1}(y) = t + c$$

$$\Rightarrow y(t) = \tan(t + c)$$

$y(t)$ reaches infinity at $t + c = \pi/2$

Compared to $\dot{x} = 1 + x^{10}$

Note that

$\dot{x} > \dot{y}$ for $x > 1$

$\Rightarrow$ $x(t)$ rises faster than $y(t)$

i.e., $x(t)$ has a steeper slope than $y(t)$, thus $x(t)$ goes to $+\infty$ faster than $y(t)$.

Next, we need to show that $x(t)$ goes to 1 in finite time for $x<1$.
Well, we know that $\dot{x} = 1 + x^{10}$

$\Rightarrow$ for any $x<1$, $\dot{x} > 0$

$\Rightarrow$ $x(t)$ is increasing

Thus, $x(t)$ will get to 1 in finite time for $x<1$.

2.6.2 (No periodic solutions to $\dot{x} = f(x)$)

$\dot{x} = f(x)$

Spse $x(t)$ is a nontrivial periodic solution, i.e. $x(t) = x(t+T)$ for $T > 0$, and $x(t) \neq x(t+s)$ for all $0 < s < T$.

$$\int_t^{t+T} f(x) \frac{dx}{dt} dt = \int_{x(t)}^{x(t+T)} f(x)\, dx$$

by Chain rule

but,

$$\int_{x(t)}^{x(t+T)} f(x)\, dx = 0$$

since $x(t) = x(t+T)$

and, recall that $\dot{x} = f(x)$

$$\Rightarrow \int_t^{t+T} f(x) \frac{dx}{dt} dt = \int_t^{t+T} (f(x))^2 dt > 0$$

by assumption since $x(t)$ is a nontrivial periodic solution and $T > 0$

Thus, we have a contradiction.

2.6.1 $m\ddot{x} = -kx$ is a system that oscillates in one dimension. However, this is not a paradox since the ODE is second order which tells us that we can rewrite the system as two first order ODE's. Therefore, the phase space consists of the variables x, y therefore, it is two dimensional! This is the space that counts, the phase space. i.e.

$$m\ddot{x} = -kx$$

$$\Rightarrow \ddot{x} = -k/m\, x$$

or

$$\begin{cases} \dot{x} = -k/m\, y \\ \dot{y} = x \end{cases}$$

a system of two first order ODE's

2.8.8

Improved Euler

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t$$

$$x_{n+1} = x_n + \frac{1}{2}\left[f(x_n) + f(\tilde{x}_{n+1})\right]\Delta t$$

First, expand $x(t_1) = x(t_0 + \Delta t)$ as a Taylor series in Δt

$$x(t_1) \approx x(t_0) + \dot{x}(t_0)\Delta t + \frac{\ddot{x}(t_0)\Delta t^2}{2} + O(\Delta t^3)$$

Look at Euler approximation

$$x_1 = x_0 + \frac{1}{2}\left[f(x_0) + f(\tilde{x}_1)\right]\Delta t$$

$$= x_0 + \frac{1}{2}\left[f(x_0) + f(x_0 + f(x_0)\Delta t)\right]\Delta t$$

Expand $f(x_0 + f(x_0)\Delta t)$

$$f(x_0 + f(x_0)\Delta t) \approx f(x_0) + f'(x_0)\left[f(x_0)\Delta t\right] + O(\Delta t^2)$$

Thus, plugging this expansion in we get

$$x_1 = x_0 + \tfrac{1}{2}\left[f(x_0) + \left(f(x_0) + f'(x_0)f(x_0)\Delta t + O(\Delta t^2)\right)\right]\Delta t$$

$$= x_0 + \tfrac{1}{2}f(x_0)\Delta t + \tfrac{1}{2}f(x_0)\Delta t + \tfrac{1}{2}f'(x_0)f(x_0)\Delta t^2 + O(\Delta t^3)$$

$$= x_0 + f(x_0)\Delta t + \tfrac{1}{2}f'(x_0)f(x_0)\Delta t^2 + O(\Delta t^3)$$

Next, recall that

$$x(t_1) \simeq x(t_0) + \dot{x}(t_0)\Delta t + \frac{\ddot{x}(t_0)\Delta t^2}{2} + O(\Delta t^3)$$

but

$$\dot{x}(t_0) = f(x_0)$$

$$\ddot{x}(t_0) = f'(x_0)\cdot\dot{x}(t_0)$$

$$= f'(x_0)f(x_0) \quad \text{by chain rule}$$

$$\Rightarrow |x(t_1) - x_1| = \Big|\Big[x_0 + f(x_0)\Delta t + \frac{f'(x_0)f(x_0)\Delta t^2}{2} + O(\Delta t^3)\Big] - \Big[x_0 + f(x_0)\Delta t + \frac{f'(x_0)f(x_0)\Delta t^2}{2} + O(\Delta t^3)\Big]\Big|$$

$$= O(\Delta t^3)$$