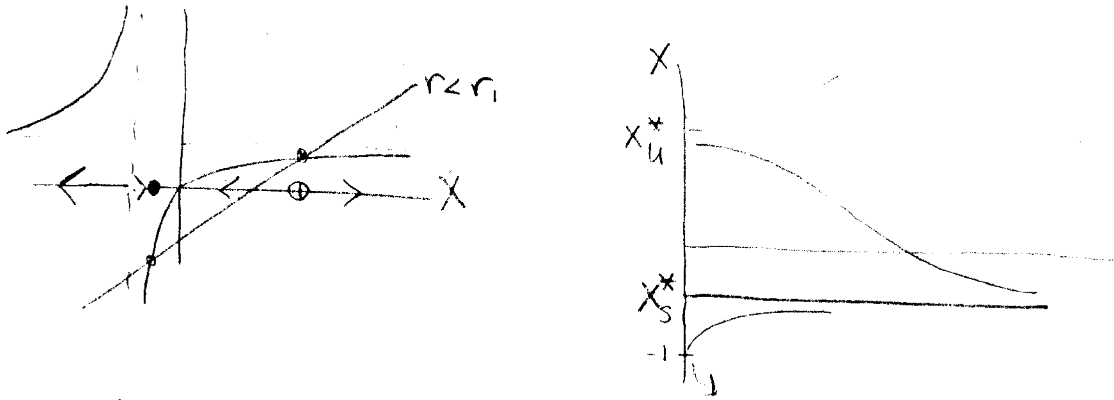
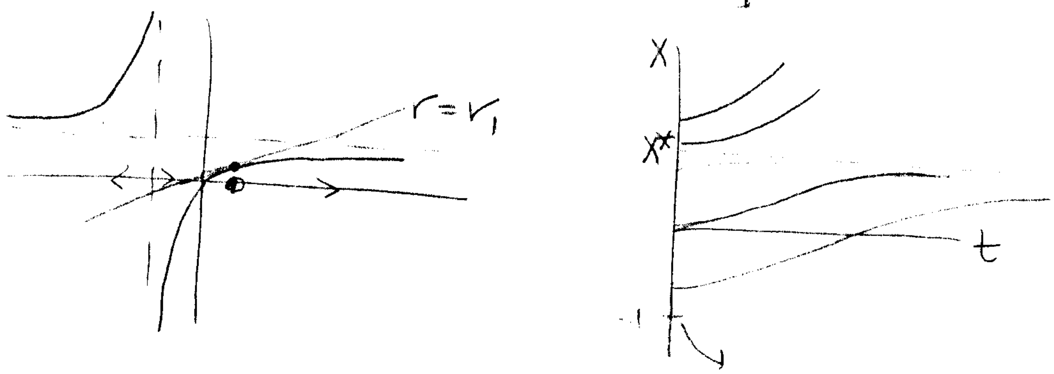
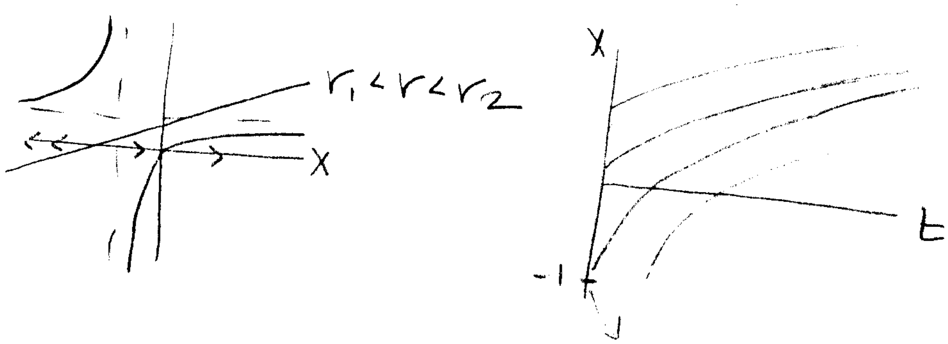
3.1.4 $\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$



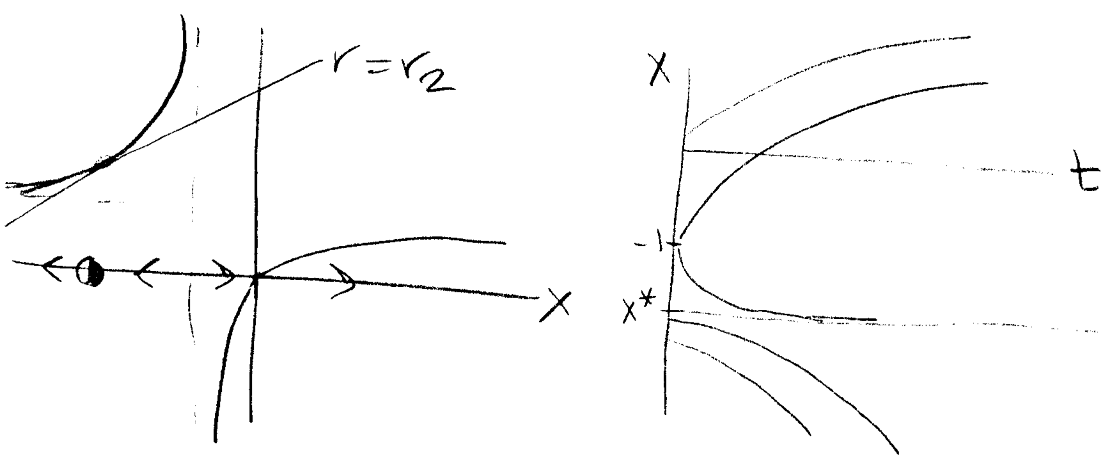
2 fixed pts.



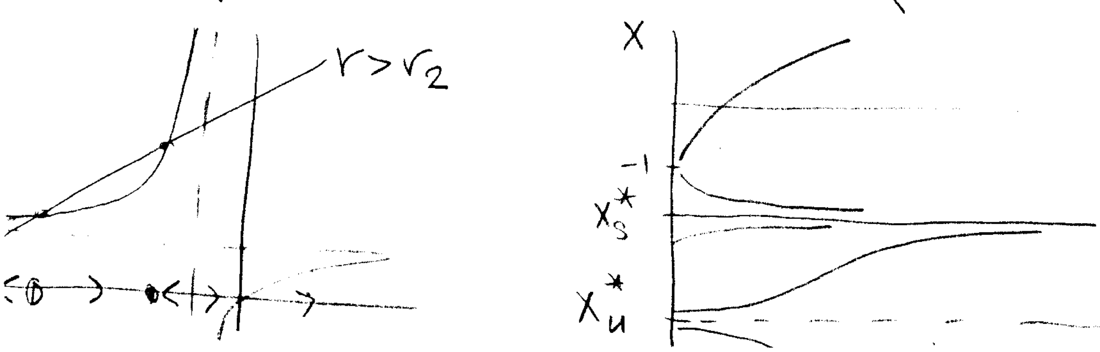
one ½-stable f.p.



no f.p.s

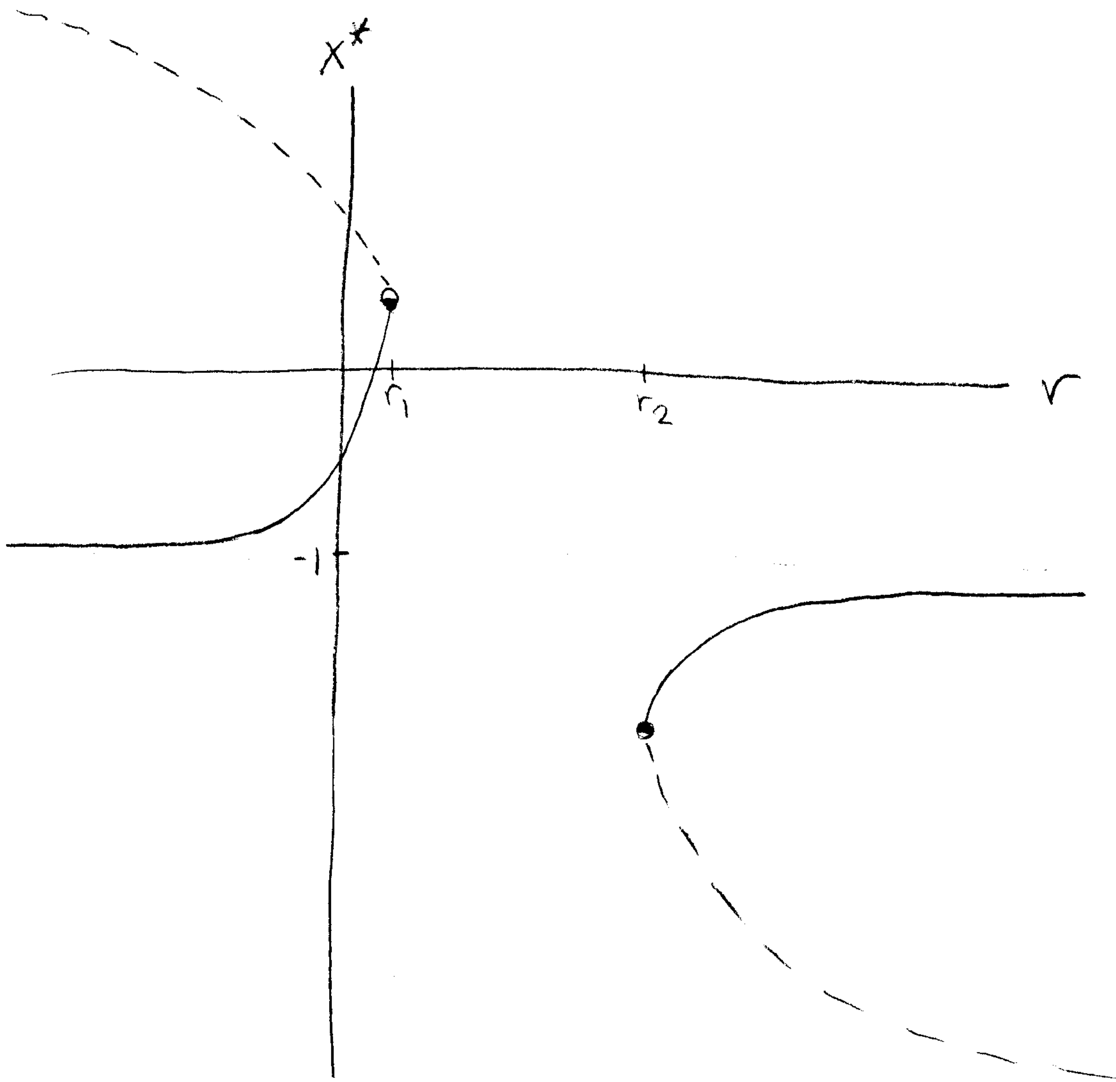


one ½-stable f.p.



2 fixed pts.

We know saddlenode bifurcations occur when $r + \frac{1}{2}x = \frac{x}{1+x}$.
There are 2 solutions to this equation

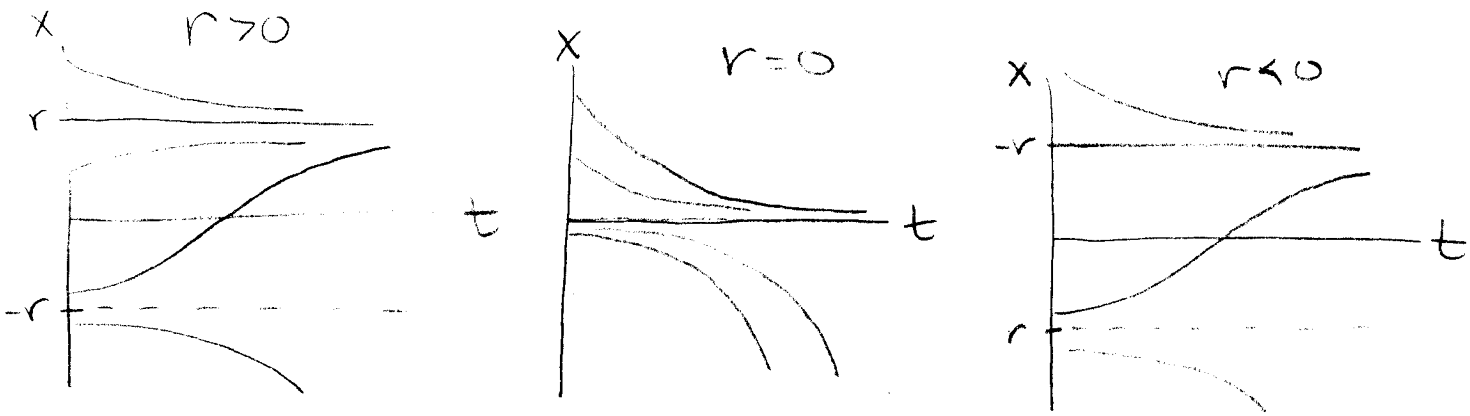


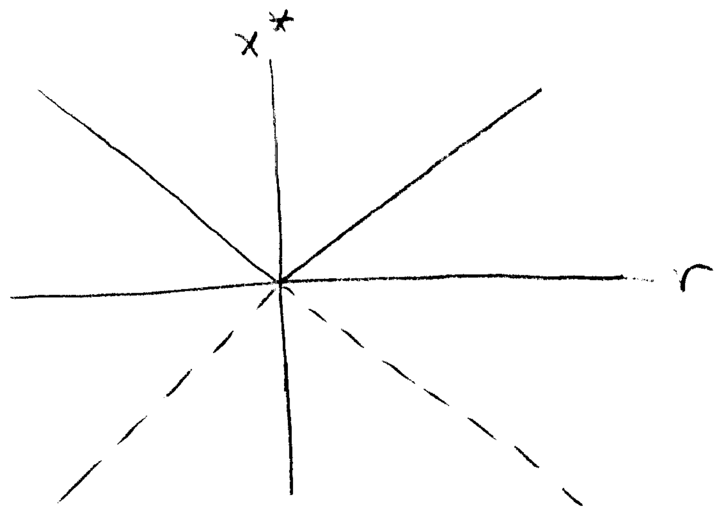
3.1.5

a) $\dot{x} = r^2 - x^2$ fixed points: $x^* = \pm r$

$x^* = r$ is stable if $r > 0$
(unstable if $r < 0$)

$x^* = -r$ is unstable if $r > 0$
(stable if $r < 0$)

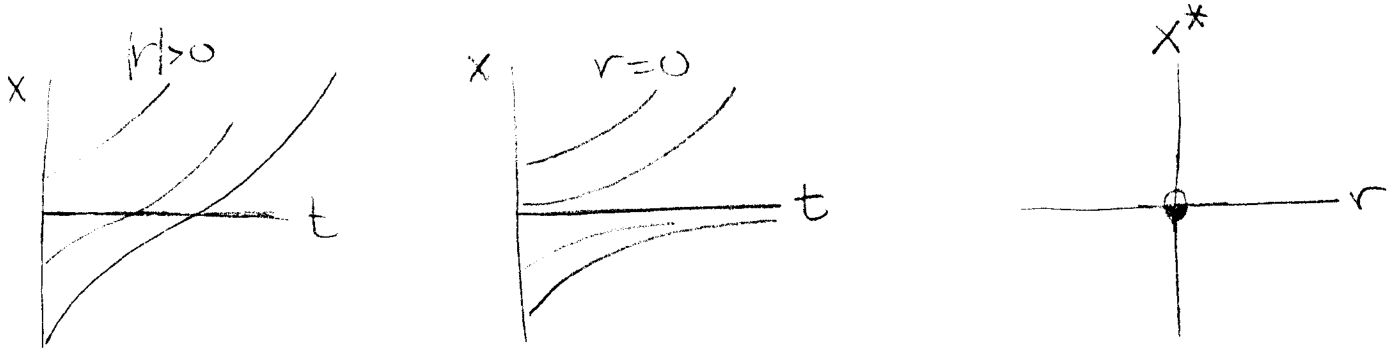




3.1.5

b) $\dot{x} = r^2 + x^2$ the only fixed point is $x^* = 0$ when $r = 0$

Otherwise, flow is always in the positive x direction.



3.2.6 Eliminating the cubic term

$$\dot{X} = RX - X^2 + aX^3 + O(X^4), \quad R \neq 0$$

want x s.t.

$$\dot{x} = Rx - x^2 + O(x^4)$$

Let $x = X + bX^3 + O(X^4)$

(near identity transformation)

a) $x = X + bX^3 + O(X^4)$

if $X = x + cx^3 + O(x^4)$

$$X = (X + bX^3 + O(X^4)) + c(X + bX^3 + O(X^4)) + O(X^4)$$

Note: can absorb $O(X^4)$ terms into one $O(X^4)$

$$X = (X + bX^3) + c(X + bX^3)^3 + O(X^4)$$

$$X = X + bX^3 + c(b^3X^9 + 3X^7b^2 + 3X^5b + X^3) + O(X^4)$$

absorb higher order terms into $O(X^4)$

$$X = X + bX^3 + cX^3 + O(X^4)$$

$$\Rightarrow \quad 0 = bX^3 + cX^3 + O(X^4)$$

$$\Rightarrow \quad c = -b$$

b) $\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)$

$$= (rX - X^2 + aX^3) + 3bX^2(rX - X^2 + aX^3) + O(X^4)$$

$$= rX - X^2 + (a+3br)X^3 + O(X^4)$$

$$= r(x - bx^3 + O(x^4)) - (x - bx^3 + O(x^4))^2 + (a+3br)(x - bx^3 + O(x^4))^3 + O(x^4)$$

$$= rx - rbx^3 - x^2 + (a+3br)x^3 + O(x^4)$$

$$= rx - rbx^3 - x^2 + ax^3 + 3brx^3 + O(x^4)$$

$$= rx - x^2 + ax^3 + 2brx^3 + O(x^4)$$

$$= rx - x^2 + \underbrace{(a+2br)}_{K}x^3 + O(x^4)$$

c) $a + 2br = 0 \Rightarrow b = \dfrac{-a}{2r}$

d) It is not necessary to make the assumption that $r \neq 0$ because if $r = 0$ we have

$$\dot{X} = X^2(1 - aX) + O(X^4)$$

which has a solution.

3.27 (*) $\dot{X} = RX - X^2 + a_n X^n + O(X^{n+1})$

We want to consider the effect of a near identity transformation on this equation. Consider the new variable:

(1) $$x = T(X) = X + b_n X^n + O(X^{n+1})$$

Where $T(X)$ is the transformation function. Let us write the inverse of the transform

(2) $$X = S(x)$$

$\therefore$ (3) $$\dot{X} = \frac{dS}{dx}\dot{x}$$

So (*) becomes:

(4) $$\left(\frac{dS}{dx}\right)\dot{x} = R\,S(x) - [S(x)]^2 + a_n[S(x)]^n + O(S(x)^{n+1})$$

There are different ways to simplify this expression, but the brute force, straight out calculation.

The question is to find $S(x)$, the inverse of the transformation. So we can assume a Taylor series expansion for small X or x:

(5) $X = S(x) = x + \sum_{i=2}^{\infty} c_i X^i$

Notice, I've already assumed that the inverse transformation is also a near identity transformation. Ultimately we will only need the first term in the series - we don't yet know what the exponent or coefficient is so we simplify (5) to

(6) $X = x + c_m X^m + O(X^{m+1})$

Now sub (1) in (6):

$$X = [X + b_n X^n + O(X^{n+1})]$$

$$+ c_m [X + b_n X^n + O(X^{n+1})]^m + O(X^{m+1})$$

Where I have use $O(x^{m+1}) = O(X^{m+1})$ from the near identity transformation. So simplifying we find

$$X = X + b_n X^n + c_m X^m + m b_n c_m X^{m-1} X^n + O(X^{m+1})$$

If we let $m = n$, $c_n = -b_n$ we find

$$X = X + b_n X^n - c_n X^n + O(X^{n+1})$$

∴ (6) becomes

(7) $X = x - b_n x^n + O(x^{n+1}) = S(x)$

$$\frac{dS}{dx} = 1 - n b_n x^{n-1} + O(x^n)$$

∴ $$\left(\frac{dS}{dx}\right)^{-1} = 1 + n b_n x^{n-1} + O(x^n)$$

∴ (4) becomes

(8) $$\dot{x} = \Big\{ R\left[x - b_n x^n + O(x^{n+1})\right]$$

$$- \left[x - b_n x^n + O(x^{n+1})\right]^2$$

$$+ a_n\left[x - b_n x^n + O(x^{n+1})\right]^n\Big\} \cdot \left\{1 + n b_n x^{n-1} + O(x^n)\right\}$$

Remember $O(x^{n+1})$ means all terms with powers $\geq n+1$. Simplify (8)

$$\dot{x} = R\left[x - b_n x^n + n b_n x^n + O(x^{n+1})\right]$$

$$- x^2 + 2 b_n x^{n+1} + O(x^{n+2})$$

$$+ a_n x^n + O(x^{2n-1})$$

We are still free to choose b_n

(9) $$\boxed{R b_n (n-1) + a_n = 0}$$

This choice makes all terms $\propto x^n$ cancel, so

$$(10) \quad \boxed{\dot{x} = Rx - x^2 + \mathcal{O}(x^{n+1})}$$

■

3.3.1

$\dot{n} = GnN - kn \qquad G, k, f > 0$

$\dot{N} = -GnN - fN + p$

a) $\dot{N} \approx 0$

$\Rightarrow 0 \approx -GnN - fN + p$

$\Rightarrow N \approx p/Gn+f$

$\Rightarrow \dot{n} \approx Gn\left(\frac{p}{Gn+f}\right) - kn$

b) $f(n) = \frac{pGn}{Gn+f} - kn \Rightarrow \frac{df}{dn}(n) = \frac{pG(Gn+f) - pG^2 n}{(Gn+f)^2} - k$

$\Rightarrow \frac{df}{dn}(0) = \frac{pGf}{f^2} - k = \frac{pG - fk}{f}$

So, $n^* = 0$ is stable if $\frac{pG - fk}{f} < 0 \Rightarrow p < \frac{fk}{G} = p_c$

and it is unstable when $p > p_c$

c) transcentical bifurcation.

$n^* = \frac{pG - fk}{kG}$ collides with $n^* = 0$ and they switch stability.

d)

3.3.1 (d) The only way to solve this is by <u>first</u> non-dimensionalizing

① $\frac{dn}{dt} = GnN - kn$

② $\frac{dN}{dt} = -GnN - fN + p$

set $N = N_0 \hat{N}$, $n = n_0 \hat{n}$, $t = t_0 \hat{t}$

So ① becomes:

$\frac{n_0}{t_0} \frac{d\hat{n}}{d\hat{t}} = n_0 N_0 G \hat{n} \hat{N} - k \hat{n}$

$\frac{d\hat{n}}{d\hat{t}} = G N_0 t_0 \hat{n} \hat{N} - k t_0 \hat{n}$

define $k t_0 = 1 \Rightarrow \boxed{t_0 = k^{-1}}$ ③

$G N_0 t_0 = 1 \Rightarrow \boxed{N_0 = k/G}$ ④

or (5) $\boxed{\frac{d\hat{n}}{d\hat{t}} = \hat{n} \hat{N} - \hat{n}}$

② becomes:

$\frac{N_0}{t_0} \frac{d\hat{N}}{d\hat{t}} = -G n_0 N_0 \hat{n} \hat{N} - N_0 f \hat{N} + p$

$$\frac{d\hat{N}}{d\hat{t}} = -\left(\frac{G n_0}{k}\right)\hat{n}\hat{N} - \left(\frac{f}{k}\right)\hat{N} + \left(\frac{pG}{k^2}\right)$$

- where I have used ③ and ④
- Now we define

⑥ $$\boxed{\frac{f}{k} \equiv \epsilon}$$

⑦ $$\frac{G n_0}{k} = \frac{1}{\epsilon} \Rightarrow \boxed{n_0 = \frac{k^2}{Gf}}$$

⑧ $$\frac{pG}{k^2} = \frac{\hat{p}}{\epsilon} \therefore \boxed{\hat{p} = \frac{pfG}{k^3}}$$

∴ ⑨ $$\boxed{\frac{d\hat{N}}{d\hat{t}} = \frac{-\hat{n}\hat{N} - \hat{N} + \hat{p}}{\epsilon}}$$

So we can reduce to a first order system when

(a) ϵ is small

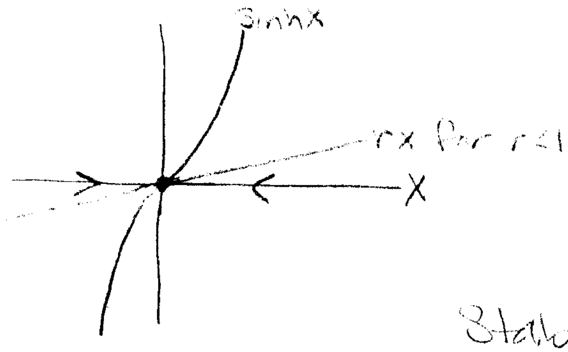
(b) $\hat{p}$ is not large or small (i.e. $\mathcal{O}(1)$)

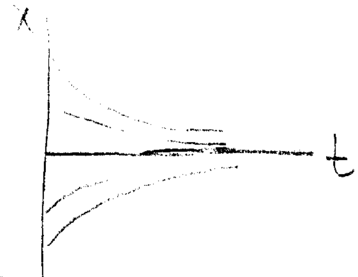
(c) the initial conditions $\hat{n}(0)$, $\hat{N}(0)$ are $\mathcal{O}(1)$

∴ (a) ⇒ $\frac{f}{k}$ is small etc

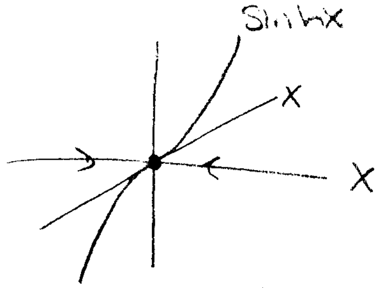
□

3.4.2 $\dot{x} = rx - \sinh x$

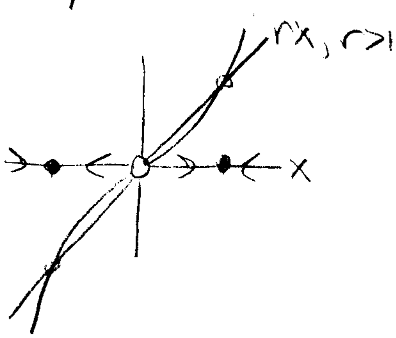


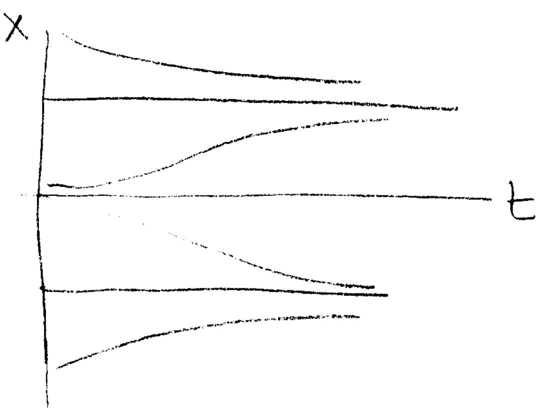


Stable f.p. $x^* = 0$

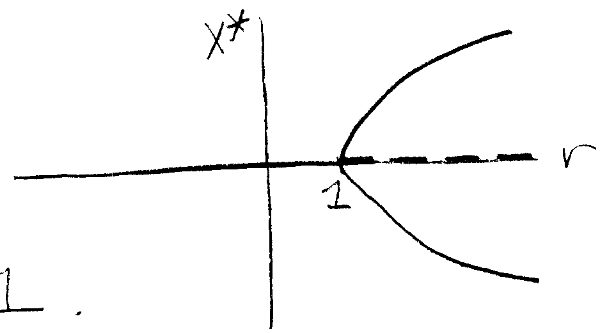


$x^* = 0$ is still stable, but the convergence is slower.





bifurcation diagram:



PFB @ r = 1.

3.4.4 $\dot{x} = x + \frac{rx}{1+x^2}$

Fixed points: $\dot{x} = 0$

$\Rightarrow \quad x\left(1 + \frac{r}{1+x^2}\right) = 0$

$x^* = 0$

$1 + \frac{r}{1+x^2} = 0 \quad \Rightarrow \quad 1 + x^2 + r = 0$

$\Rightarrow \quad x^* = \pm\sqrt{-1-r}$

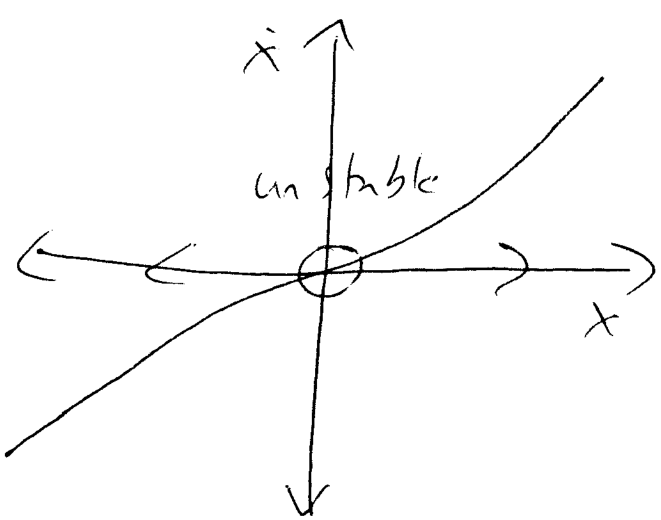
exists for $r < -1$

So,

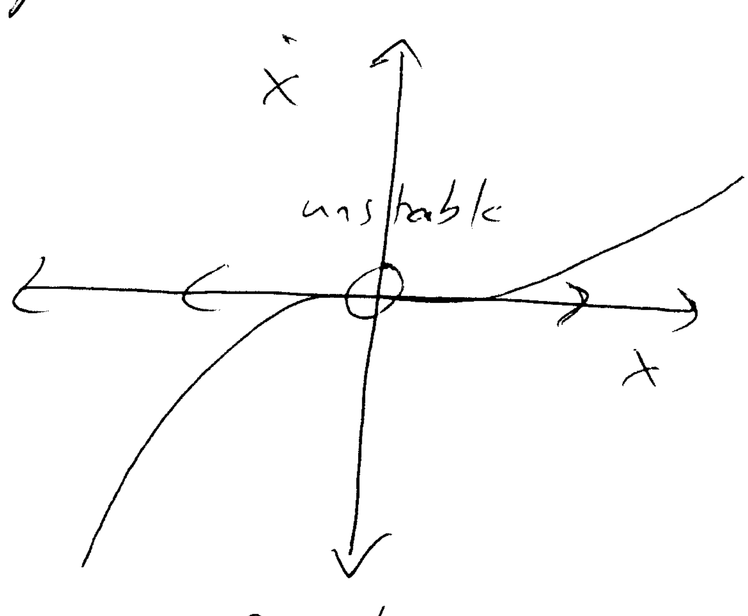
$r > -1$ 1 f.p.

$r = -1$ 1 f.p.

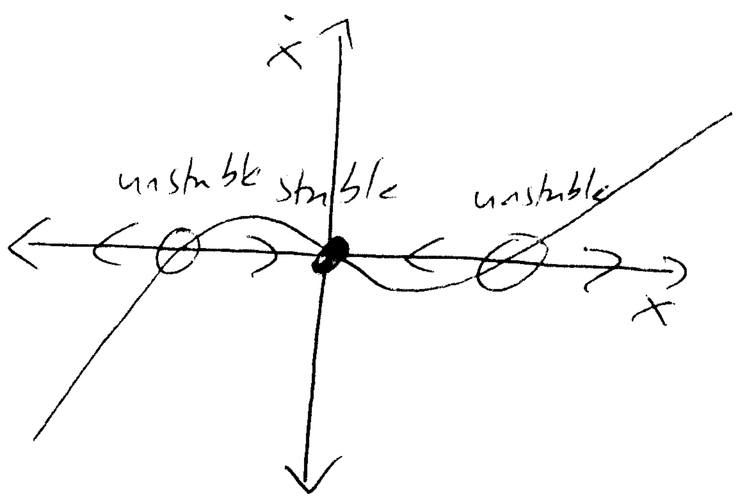
$r < -1$ 3 f.p.



$r > -1$

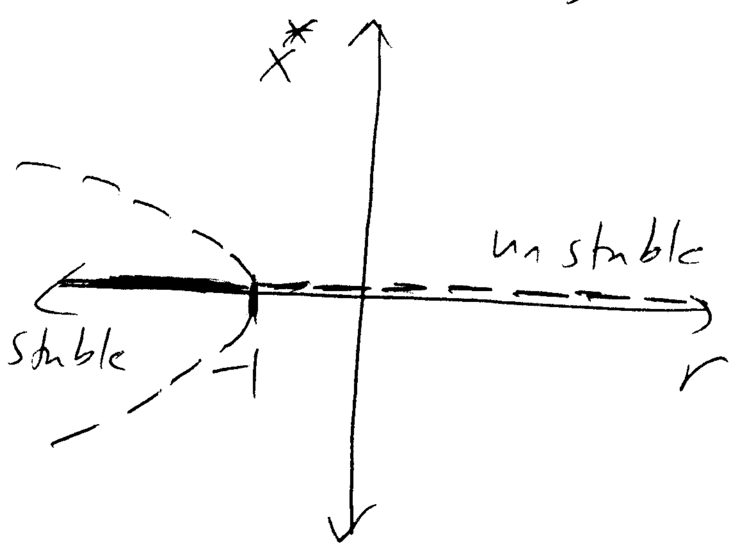


$r = -1$



$r < -1$

Bifurcation diagram



Subcritical pitchfork

3.4.8 $\dot{x} = rx - \frac{x}{1+x^2}$

F.P. $\dot{x} = 0$, $rx - \frac{x}{1+x^2} = 0$

$\Rightarrow x^* = 0$, $r - \frac{1}{1+x^2} = 0$

$\Rightarrow x^* = \pm\sqrt{\frac{1}{r} - 1}$ exists for $0 < r \leq 1$

So,
1 f.p. for $r > 1$
3 f.p. for $0 < r \leq 1$

Stability: $f'(x) = r - \frac{1-x^2}{(1+x^2)^2}$

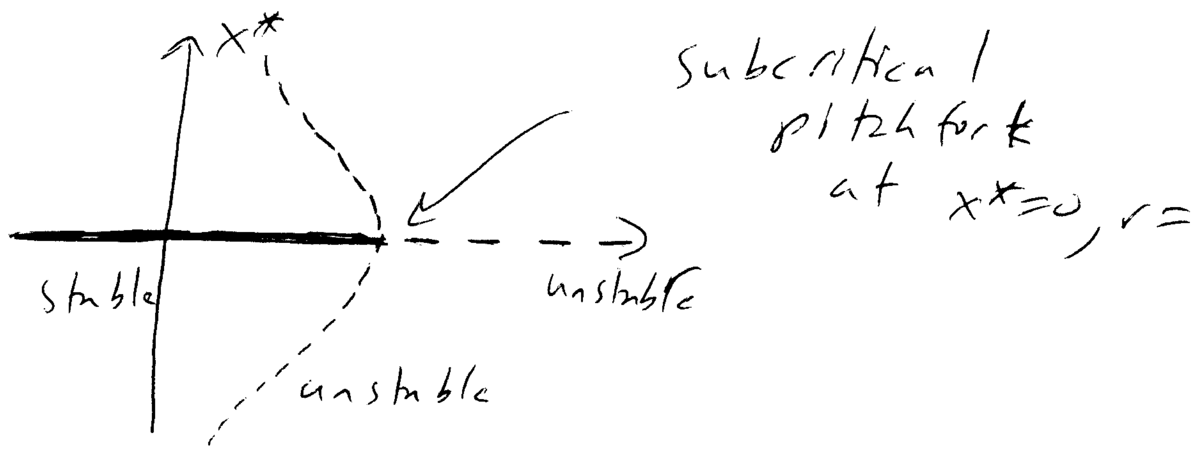
$f'(0) = r - 1 \begin{cases} < 0, & r < 1 \\ > 0, & r > 1 \end{cases}$ $\Rightarrow$ x^* stable for $r < 1$; x^* unstable for $r > 1$

$$f'\left(\pm\sqrt{\tfrac{1}{r}-1}\right) = r - \frac{1-(\frac{1}{r}-1)}{(1+(\frac{1}{r}-1))^2}$$

$$= 2r > 0 \quad \text{for } 0 < r \leq 1$$

$\Rightarrow$ $x^* = \pm\sqrt{\frac{1}{r}-1}$ are unstable when they exist

Bifurcation Diagram



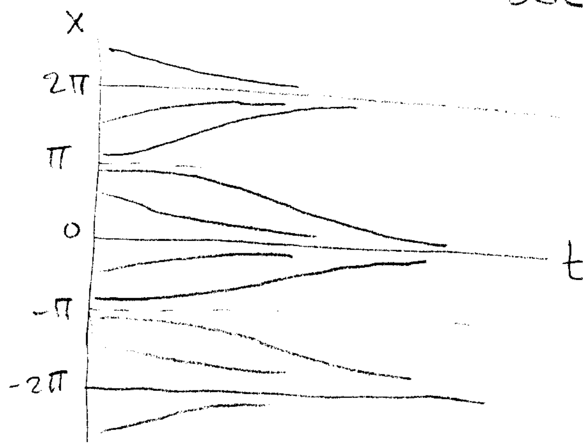
3.4.11 $\dot{x} = rx - \sin x$

a) $r = 0 \Rightarrow \dot{x} = -\sin x$

Fixed points: $x^* = n\pi$ for $n \in \mathbb{Z}$

When n is even $x^* = n\pi$ is stable

" " odd " " unstable



b) $r > 1 \Rightarrow$ the only fixed point is $x^* = 0$

it is unstable.

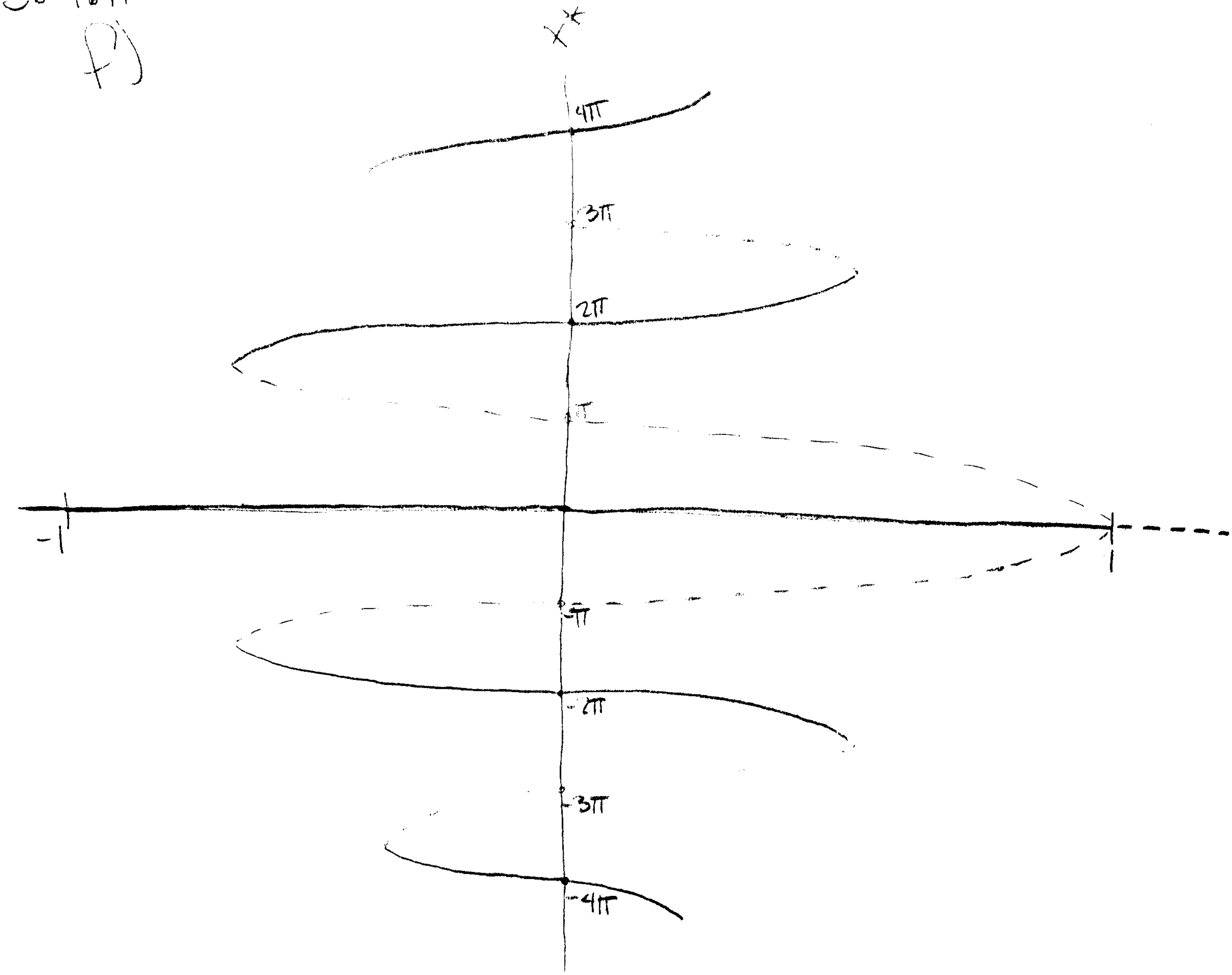
c) there will be a subcritical PFB at $r = 1$ and infinitely many SNB's as r decreases from 1 to 0.

d)

e) As r decreases from 0 to $-\infty$, we get infinitely many SNB's.

3.4.11

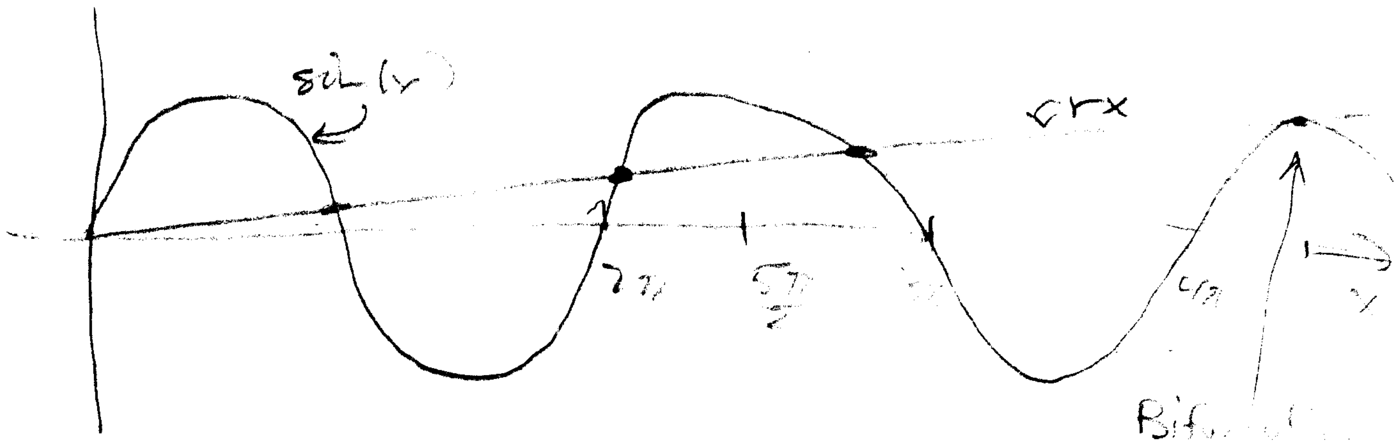
f)



3.4.11 (d)
There are 2 ways to do this problem:

(I) Graphically: $\dot{x} = rx - \sin(x)$

Notice the bifurcations occur when you get a intersection



as $r \to 0$ more and more bifurcations occur. They happen closer and closer to the points $x_* = (2n + \frac{1}{2})\pi$ as $n \to \infty$

$\therefore \quad rx_* = \sin(x_*)$

$$r \approx \frac{1}{\pi(2n + \frac{1}{2})} \quad \text{for large } n \in \mathbb{N}$$

(II) Algebraically
All of these bifurcations have the feature that x_* is a Double root of the equation

$f(x_*) = rx_* - \sin(x_*) = 0$

∴ r_c, x_* satisfy (r_c = critical value of r)

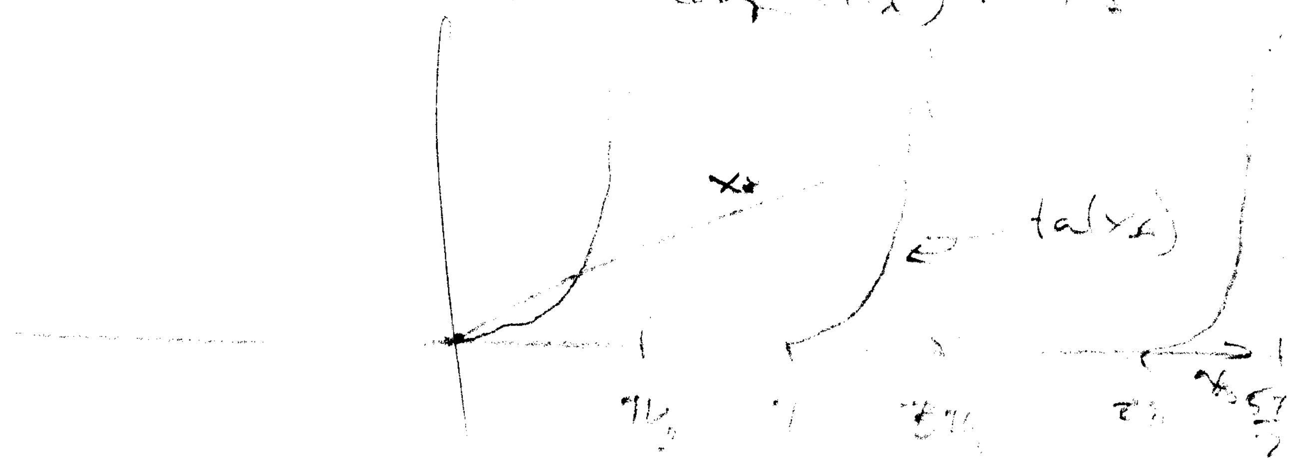
(a) $f(x_*) = 0$

(b) $\left.\dfrac{df}{dx}\right|_{x=x_*} = 0$

∴ (a) ⇒ $r_c = \dfrac{\sin(x_*)}{x_*}$ with $\sin(x_*) \geq 0$

(b) ⇒ $r_c = \cos(x_*)$

∴ divide ⇒ $\tan(x_*) = x_*$



The intersection of these curves gets very close to $x_* = (2n + \frac{1}{2})\pi$ because $3\pi/2$, $7\pi/2$ etc are not admissible ∵ $\sin(x_*) < 0$ there.

∴ $r_c \approx \dfrac{1}{\pi(2n+\frac{1}{2})}$ as $n \to \infty$

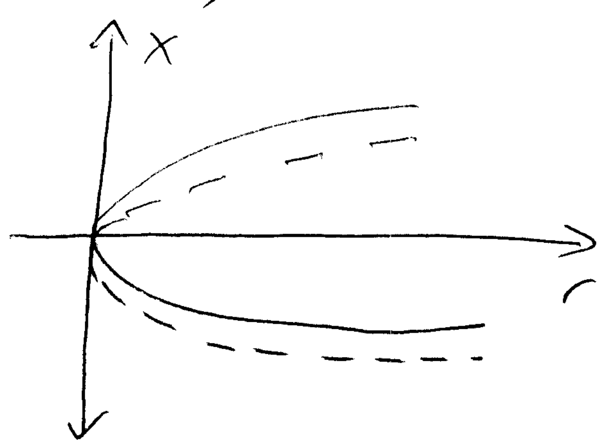
34.12 "Quadfurcation"

$$\dot{x} = (x^2 - \alpha_1 r)(x^2 - \alpha_2 r), \quad \alpha_2 > \alpha_1 > 0$$

fixed points

$$x^* = \pm\sqrt{\alpha_1 r}, \ \pm\sqrt{\alpha_2 r}$$

$\Rightarrow$ $r < 0$, no fixed points
$r > 0$, 4 fixed points



Generalization

(i) $\dot{x} = (x^2 - \alpha_1 r)(x^2 - \alpha_2 r) \cdots (x^2 - \alpha_N r)$

$\alpha_N > \alpha_{N-1} > \cdots > \alpha_1 > 0$

$r < 0$, no fixed points
$r > 0$, $2N$ fixed points.

ii) $\dot{x} = x(x^2 - \alpha_1 r)(x^2 - \alpha_2 r) \cdots (x^2 - \alpha_N r)$

$\alpha_N > \alpha_{N-1} > \cdots > \alpha_1 > 0$

$r < 0$, one fixed pt exists, $x^* = 0$

$r > 0$, $2N+1$ fixed pts. exist.

i.e. pitchfork-like bifurcation.

3.4.14 Subcritical pitchfork

$$\dot{x} = rx + x^3 - x^5$$

a) $\dot{x} = 0$

$$\Rightarrow \quad x(r + x^2 - x^4) = 0$$

$$x^* = 0, \quad r + x^2 - x^4 = 0$$

Quadratic equation

$$x^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1 + 4r}}{2} \quad \text{exists for } r > -1/4$$

$$\Rightarrow \quad x^* = \pm\sqrt{\frac{1 \pm \sqrt{1+4r}}{2}} \quad \text{for } r > -1/4$$

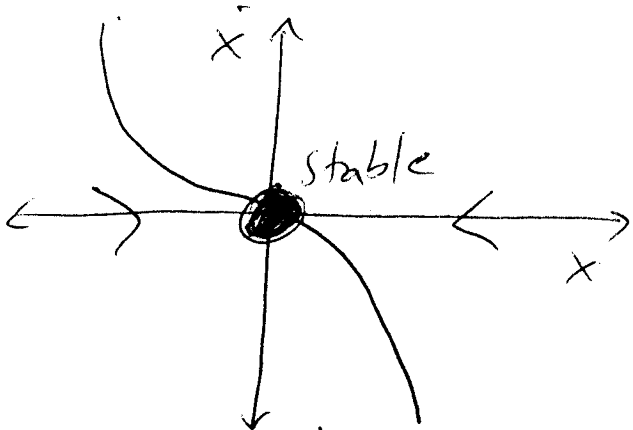
Notice:

1 f.p. for $r < -1/4$

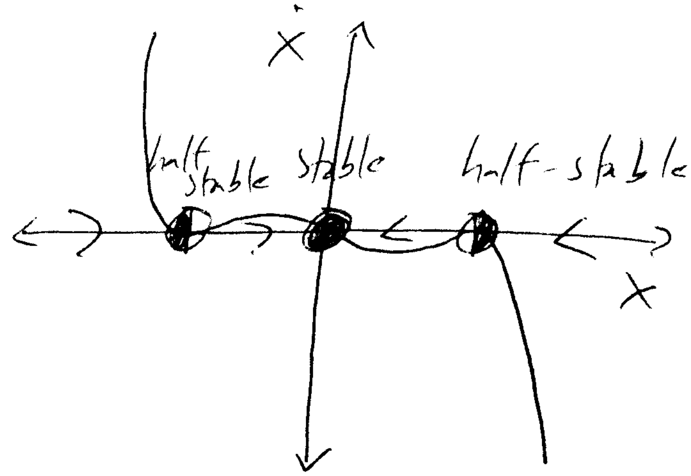
5 f.p. for $-1/4 < r < 0$

3 f.p. for $r > 0$, $r = -1/4$

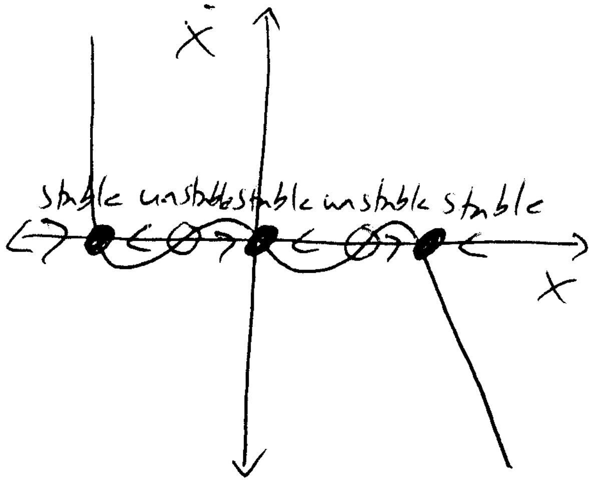
b) Vector fields



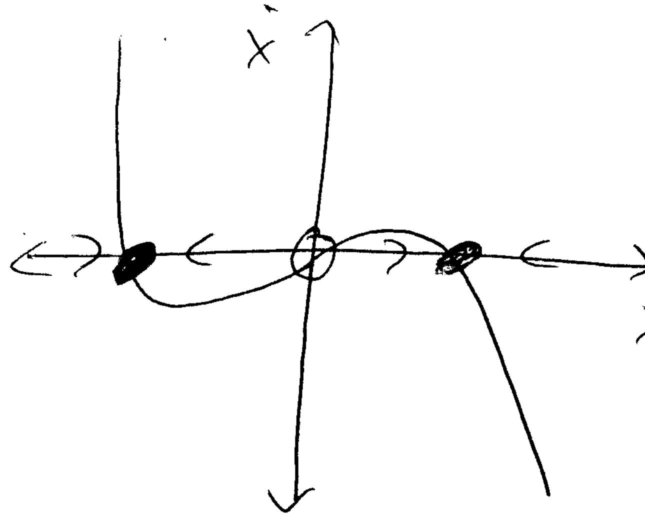
$r < -1/4$



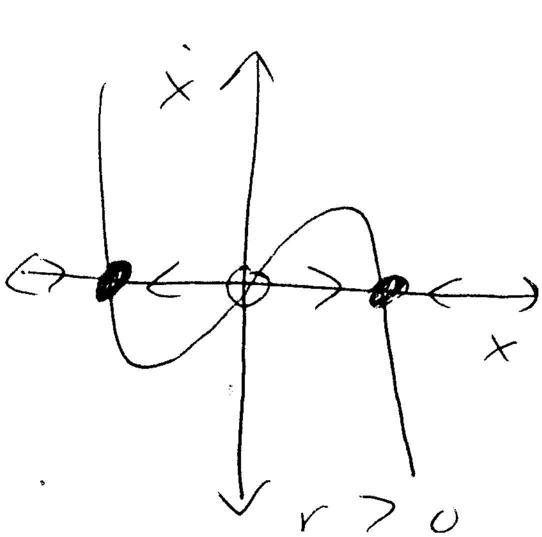
$r = -1/4$



$-1/4 < r < 0$



$r = 0$



$r > 0$

Note that the vector fields for the different values of r match the bifurcation diagram on page 59 (figure 3.4.7) of the text.

c) $r_s = -1/4$.

3.5.7 Nondimensionalizing the logistic equation

$\dot{N} = rN(1 - N/K)$, $N(0) = N_0$

a)

r = growth rate, dimensions are population/sc

K = carrying capacity, dimensions are population size.

N = population size at time t, dimensions are population size.

b) To nondimensionalize time we set

$\tau = rt$, which cancels out seconds

To nondimensionalize N we set

$x = N/K$

Thus, $\frac{dx}{dt} = \frac{dN}{dt}/K$

$$\frac{dN}{dt} = rN(1 - N/K)$$

$$\Rightarrow \frac{1}{r}\frac{dN}{dt} = N(1 - N/K)$$

$$\frac{dN}{d\,rt} = N(1 - N/K)$$

$$\frac{dN}{d\tau} = N(1 - N/K)$$

$$x = N/K, \quad N = Kx$$

$$\frac{dx}{d\tau} K = Kx(1 - (Kx)/K)$$

$$\Rightarrow \frac{dx}{d\tau} = x(1 - x)$$

$$x(0) = N(0)/K = N_0/K$$

c) Different nondimensionalization

$\tau = rt$

$u = \frac{N}{N_0} \Rightarrow u(0) = \frac{N(0)}{N_0} = \frac{N_0}{N_0} = 1$

Thus,

$$\frac{du}{d\tau} = \frac{dN}{d\tau} \Big/ N_0$$

$$\Rightarrow \frac{du}{d\tau} N_0 = \frac{dN}{d\tau}$$

$$N_0 u = N$$

$$\Rightarrow \frac{dN}{dt} = rN(1 - N/K)$$

$$\frac{1}{r}\frac{dN}{dt} = N(1 - N/K)$$

$$\frac{dN}{d\tau} = N(1 - N/K)$$

$$\frac{du}{d\tau} N_0 = uN_0\left(1 - \frac{N_0}{K}u\right)$$

$$\frac{du}{d\tau} = u\left(1 - \frac{N_0}{K} u\right)$$

d) Well, one advantage is that the second nondimensionalization has initial condition $u_0 = 1$ while the first has $x_0 = N_0/K$. $u(0) = 1$ could make it easier to find $u(t)$.