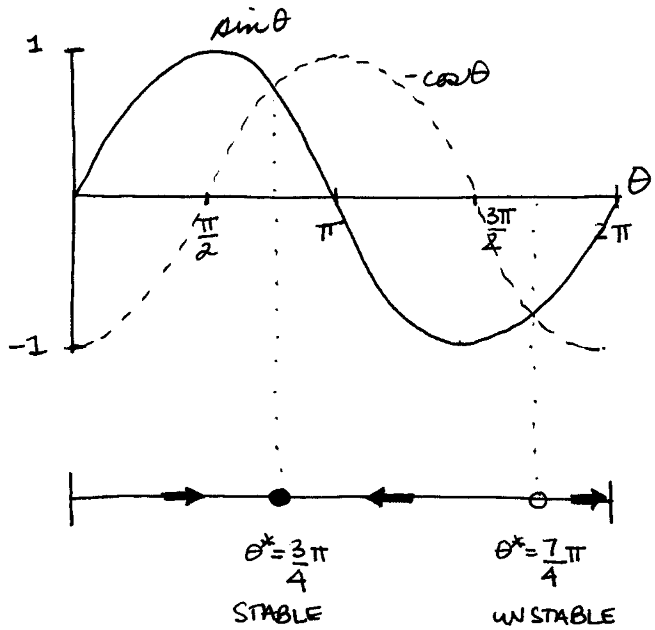
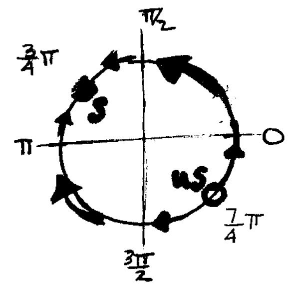
4.1.5

$$\dot{\theta} = \sin\theta + \cos\theta = f(\theta)$$
$$= \sin\theta - (-\cos\theta)$$





F.P.s

$$f(\theta) = 0 \qquad \sin\theta^* = -\cos\theta^*$$
$$\theta^* = \frac{3\pi}{4}, \; \frac{7\pi}{4}$$

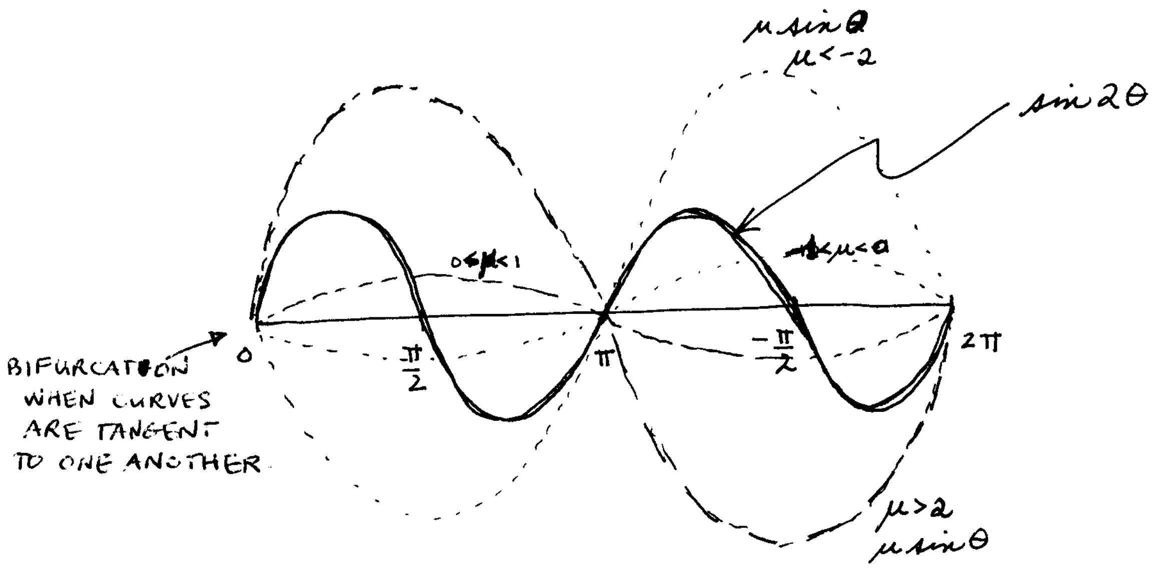
STABILITY

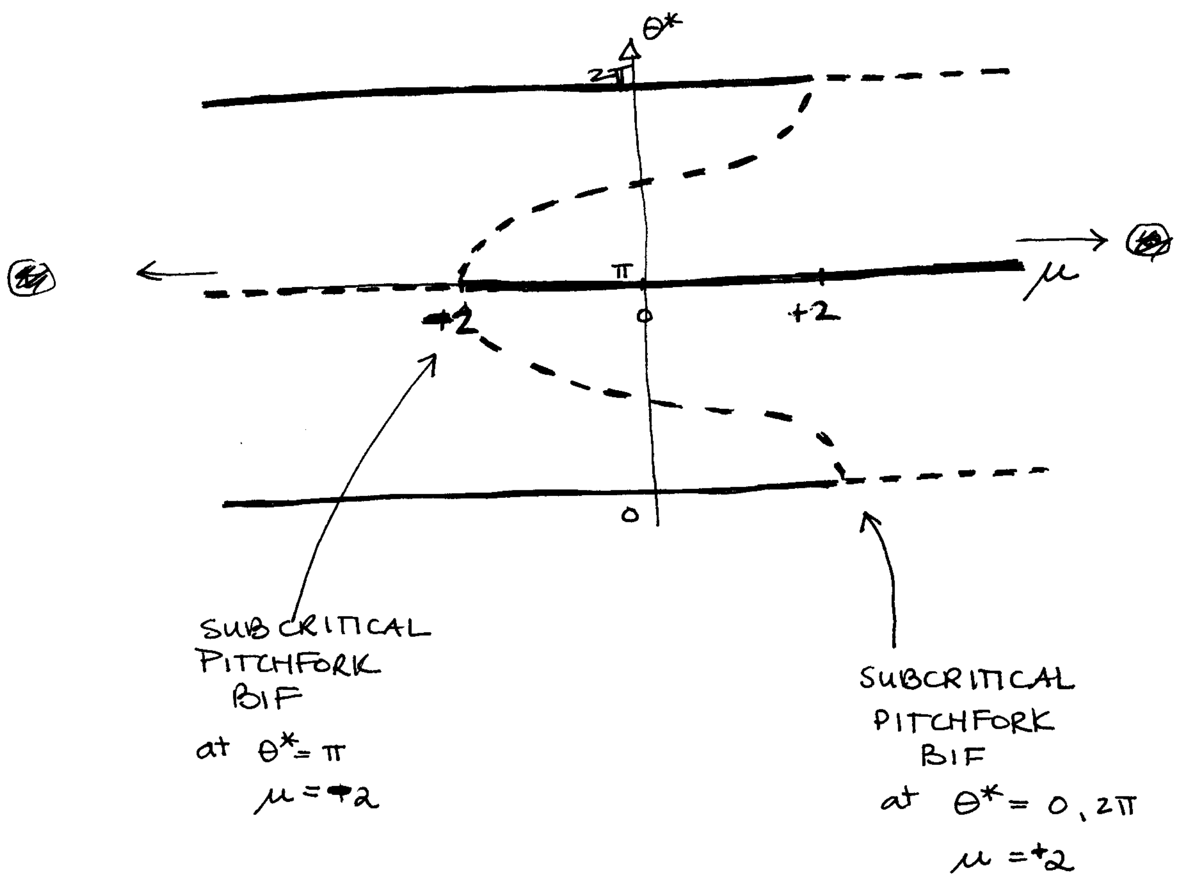
$$f'(\theta) = \cos\theta - \sin\theta$$

$\theta^* = \frac{3\pi}{4} \qquad f'\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} - \sin\frac{3\pi}{4} < 0 \Rightarrow \quad \theta^* = \frac{3\pi}{4}$ is STABLE.

$\theta^* = \frac{7\pi}{4} \qquad f'\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} - \sin\frac{7\pi}{4} > 0 \Rightarrow \quad \theta^* = \frac{7\pi}{4}$ is UNSTABLE.

4.3.3. $\dot{\theta} = \mu \sin\theta - \sin 2\theta = f(\theta;\mu)$





ie. BIF. PTS WHEN $f'(\theta^*)=0$

$f(\theta)=0$ includes $\theta^* = 0, \pi$

$f'(0) = \mu\cos 0 - 2\cos 0 = \mu - 2 = 0 \Rightarrow$ BIF OF $\theta^*=0$ AT $\mu = 2$

$f'(\pi) = \mu\cos\pi - 2\cos 2\pi = -\mu - 2 = 0 \Rightarrow$ BIF OF $\theta^* = \pi$ AT $\mu = -2$

- at $\mu = 2$, $\theta^* = 0$
  $= \mu_{CRIT}$

$$f(\theta^*=0, \mu=2) = 0 \qquad \frac{\partial f}{\partial \mu}(\theta^*=0, \mu=2) = 0$$

$$\frac{\partial f}{\partial \theta}(\theta^*=0, \mu=2) = 0 \qquad \frac{\partial^2 f}{\partial \mu \partial \theta}(\theta^*=0, \mu=2) = 1$$

$$\frac{1}{2}\frac{\partial^2 f}{\partial \theta^2}(\theta^*=0, \mu=2) = 0$$

$$\frac{1}{6}\frac{\partial^3 f}{\partial \theta^3}(\theta^*=0, \mu=2) = \left.\frac{-\mu+8}{6}\right|_{\mu=2} = \frac{6}{6} = 1$$

$$\Rightarrow \quad \dot{\theta} \simeq (\mu-2)\theta + \theta^3 = r\theta + \theta^3$$

which is the normal form for a SUBCRITICAL PITCHFORK BIFURCATION (at $\mu = 2$, $\theta^* = 0$).
$r = 0$

- similarly for $\mu = -2$, $\theta^* = \pi$

$$\Rightarrow \quad \dot{\theta} \simeq -(\mu+2)(\theta-\pi) + (\theta-\pi)^3$$

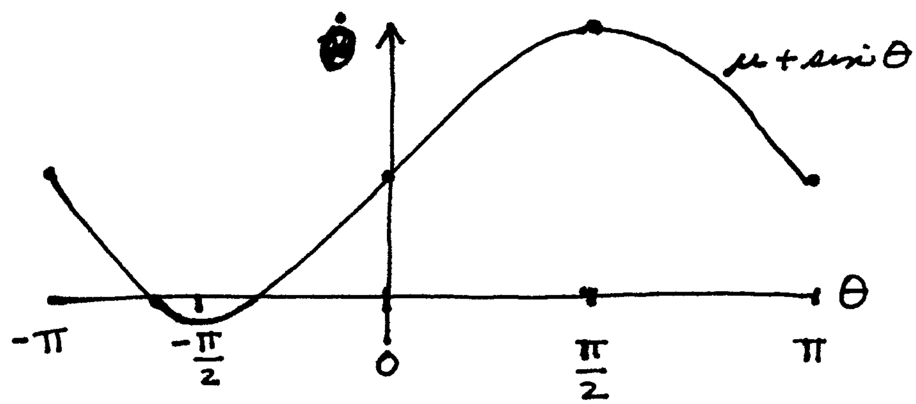
$y = \theta - \pi$
$r = \cancel{\mu}(\mu+2)$ $\Rightarrow$ $\dot{y} = ry + y^3$

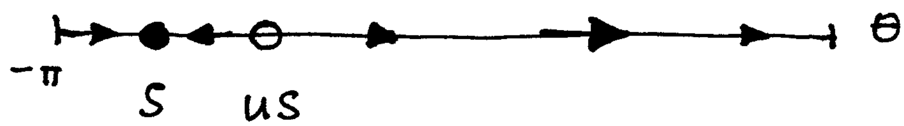
which is the normal for for a SUBCRITICAL PITCHFORK BIF.

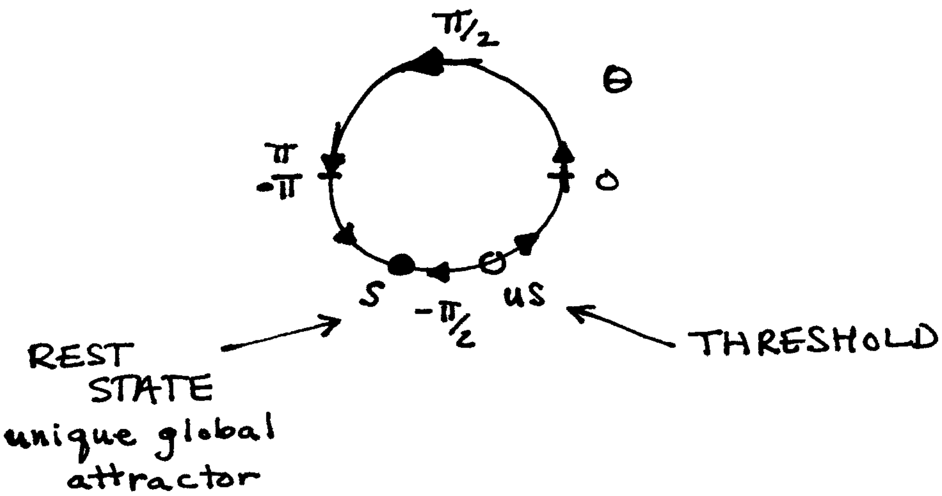
at $y=0, r=0$ i.e. $\mu = -2$, $\theta^* = \pi$

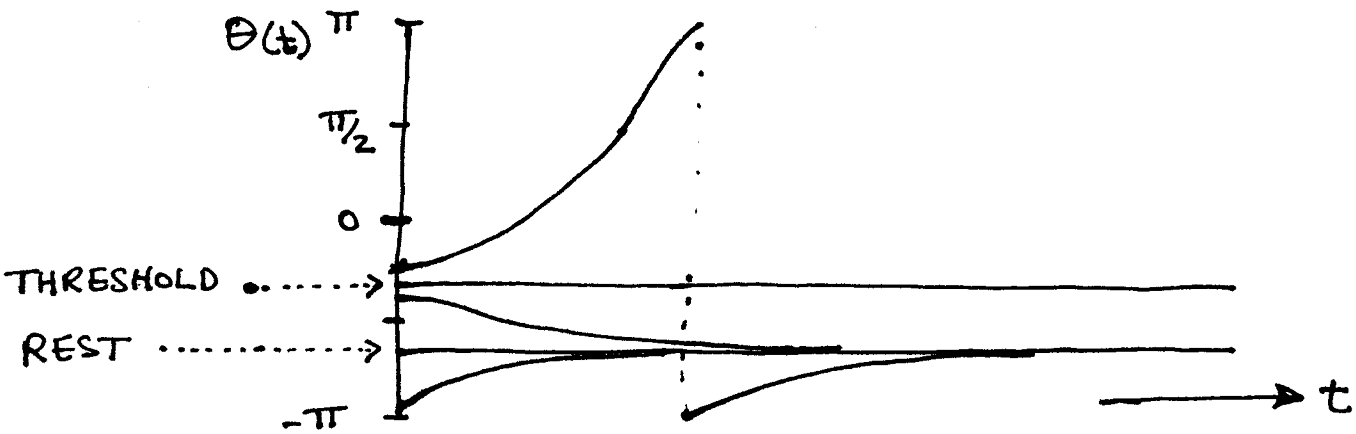
4.5.3 $\dot{\theta} = \mu + \sin\theta$, $\theta \in (-\pi, \pi]$

μ slightly less than 1



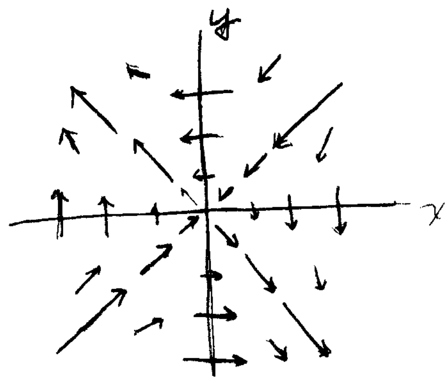






5.1.9 $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

A)



B) note that $x\dot{x} - y\dot{y} = 0$

i.e. $x(-y) - y(-x) = 0$

$$x \frac{dx}{dt} = y \frac{dy}{dt}$$

$$\int_0^t \tilde{x} \frac{d\tilde{x}}{dt}\, dt = \int_0^t \tilde{y} \frac{d\tilde{y}}{dt}\, dt$$

$$\int_{x(0)}^{x(t)} \tilde{x}\, d\tilde{x} = \int_{y(0)}^{y(t)} \tilde{y}\, d\tilde{y}$$

$$\frac{x(t)^2}{2} - \frac{x(0)^2}{2} = \frac{y(t)^2}{2} - \frac{y(0)^2}{2}$$

$$\Rightarrow \quad x(t)^2 - y(t)^2 = C$$

where $C = x(0)^2 - y(0)^2$

⇒ find eigenvalues

$$\det \begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 0 \quad \Rightarrow \quad \lambda_+ = 1 , \ \lambda_- = -1$$

⇒ (0,0) is A SADDLE POINT.

$\lambda_+ = 1$ $\quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ v_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad -1 - v_+ = 0$

$v_+ = -1$

$\lambda_- = -1$ $\quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ v_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 1 - v_+ = 0$

$v_- = 1$

⇒ eigenpairs $\quad \lambda_+ = 1 , \ v_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_- = -1 , \ v_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- UNSTABLE MANIFOLD OF SADDLE PT IS ALONG $\bar{v}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

  ie. $y = -x$

- STABLE MANIFOLD OF SADDLE PT is ALONG $\bar{v}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

  ie. $y = x$

D) Let $u = x + y$, $v = x - y$ $\Rightarrow$ $x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$

$$\frac{du}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -y - x = -u$$

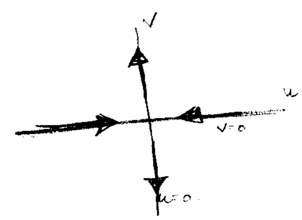
$$\frac{dv}{dt} = \frac{dx}{dt} - \frac{dy}{dt} = -y + x = v$$

$$\Rightarrow \begin{cases} \dfrac{du}{dt} = -u \\[2ex] \dfrac{dv}{dt} = v \end{cases} \longrightarrow \begin{aligned} u(t) &= u_0 e^{-t} \\ v(t) &= v_0 e^{t} \end{aligned}$$

where $u(0) = u_0$, $v(0) = v_0$

E) UNSTABLE MANIFOLD $\Rightarrow$ $u = 0$

STABLE MANIFOLD $\Rightarrow$ $v = 0$



F) $$x(t) = \frac{u(t) + v(t)}{2} = \frac{u_0}{2} e^{-t} + \frac{v_0}{2} e^{t}$$

$$= \frac{x_0 + y_0}{2} e^{-t} + \frac{x_0 - y_0}{2} e^{t}$$

$$y(t) = \frac{u(t) - v(t)}{2} = \frac{x_0 + y_0}{2} e^{-t} - \frac{x_0 - y_0}{2} e^{t}$$

5.2.1

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \bar{x}' = A\bar{x}$$

a) eigenvalues and eigenvectors

$$\det \begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0$$

$$\lambda_+ = 3 \; ; \; \lambda_- = 2$$

$$\lambda_+ = 3 \Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ v_+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_+ = 1$$

$$\lambda_- = 2 \Rightarrow \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ v_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_- = 2$$

$$\Rightarrow \left\{ \lambda_+ = 3, \; \bar{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \; \left\{ \lambda_- = 2, \; \bar{v}_- = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

b) general sol'n of $\bar{x}' = A\bar{x}$:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

c) $\lambda_+ = 3, \lambda_- = 2 \Rightarrow (0,0)$ IS AN UNSTABLE NODE.

d) $\bar{X}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$c_2 = 1 \quad , \quad c_1 = 2$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 2e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

5.2.2 $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$

a) $$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\bar{x}' = A \quad \bar{x}$$

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda - 1)^2 + 1$$

$$= \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i$$

$$\begin{bmatrix} 1 - (1 \pm i) & -1 \\ 1 & 1 - (1 \pm i) \end{bmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = 0$$

$$\begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = 0 \quad \Rightarrow v = \mp i$$

eigenpairs

$$\lambda_1 = 1 + i, \quad \bar{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \longleftarrow \text{OR } \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 - i, \quad \bar{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \longleftarrow \text{OR } \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\bar{x}(t) = c_1 e^{(1+i)t}\begin{pmatrix}1\\-i\end{pmatrix} + c_2 e^{(1-i)t}\begin{pmatrix}1\\i\end{pmatrix}$$

where $c_1, c_2 \in \mathbb{C}$

$$\bar{x}(0) = \begin{pmatrix}x_0\\y_0\end{pmatrix} = c_1\begin{pmatrix}1\\-i\end{pmatrix} + c_2\begin{pmatrix}1\\i\end{pmatrix}$$

$$\Rightarrow \quad c_1 + c_2 = x_0 \Rightarrow \begin{array}{l} Im(c_1) + Im(c_2) = 0 \\ Re(c_1) + Re(c_2) = x_0 \end{array}$$

$$-ic_1 + ic_2 = y_0 \Rightarrow \begin{array}{l} -Re(c_1) + Re(c_2) = 0 \\ Im(c_1) - Im(c_2) = y_0 \end{array}$$

$$c_1 = \frac{x_0}{2} + i\frac{y_0}{2}, \quad c_2 = \frac{x_0}{2} - i\frac{y_0}{2}$$

$$\bar{x}(t) = \frac{e^t}{2}(\cos t + i\sin t)\begin{pmatrix}x_0 + iy_0\\y_0 - ix_0\end{pmatrix}$$

$$+ \frac{e^t}{2}(\cos t - i\sin t)\begin{pmatrix}x_0 - iy_0\\y_0 + ix_0\end{pmatrix}$$

$$= \frac{e^t}{2}\begin{pmatrix}x_0\cos t - y_0\sin t + ix_0\sin t + iy_0\cos t\\y_0\cos t + x_0\sin t - ix_0\cos t + iy_0\sin t\end{pmatrix}$$

$$+ \frac{e^t}{2}\begin{pmatrix}x_0\cos t - y_0\sin t - ix_0\sin t - iy_0\cos t\\y_0\cos t + x_0\sin t + ix_0\cos t - iy_0\sin t\end{pmatrix}$$

$$\bar{x}(t) = e^t\begin{pmatrix}x_0\cos t - y_0\sin t\\y_0\cos t + x_0\sin t\end{pmatrix}$$

5.2.4

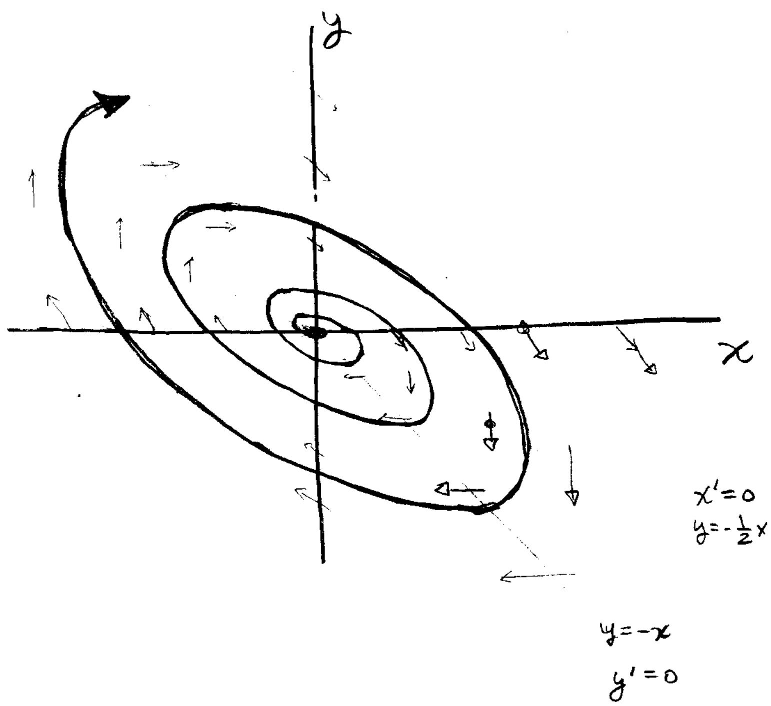
$$\begin{cases} x' = 5x + 10y \\ y' = -x - y \end{cases}$$

- eigenvalues: $\det \begin{bmatrix} 5-\lambda & 10 \\ -1 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 5 = 0$

$$\lambda = 2 \pm i$$

$\Rightarrow$ $(0,0)$ is an UNSTABLE SPIRAL.

* note: eigenvectors will be complex.



5.2.8
$$\begin{cases} x' = -3x + 4y \\ y' = -2x + 3y \end{cases}$$

eigenvalues:

$$\det \begin{bmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{bmatrix} = \lambda^2 - 1 = 0$$

$$\lambda_{1,2} = \pm 1$$

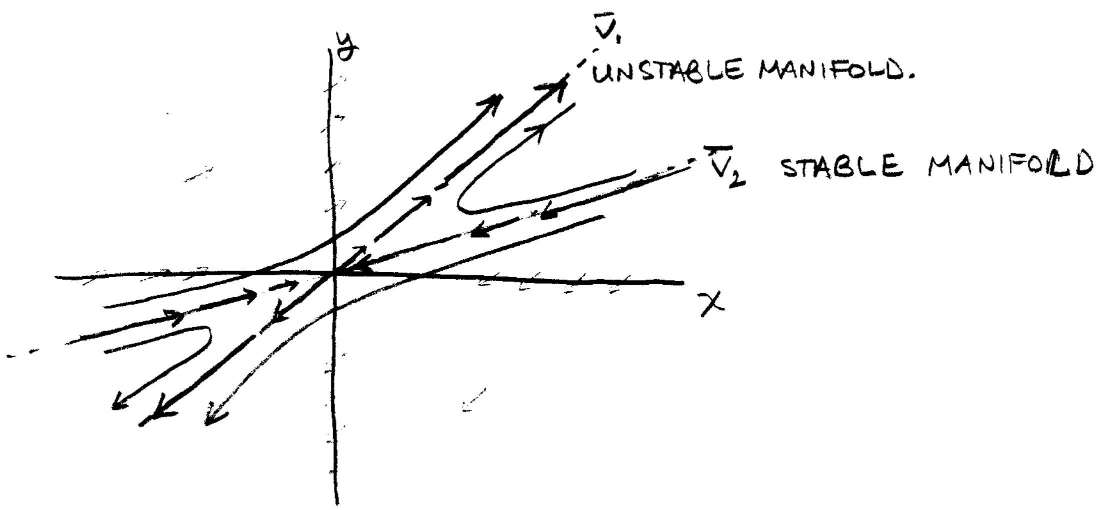
eigenvector:

$$\lambda_1 = 1 \qquad \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ v_1 \end{pmatrix} = \bar{0} \Rightarrow v_1 = 1$$

$$\lambda_2 = -1 \qquad \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} = \bar{0} \Rightarrow v_2 = \frac{1}{2}$$

$$\Rightarrow \left\{ \lambda_1 = 1 , \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \left\{ \lambda_2 = -1 , \bar{v}_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \right\}$$

(0,0) is a SADDLE POINT



5.2.13 $\quad m\ddot{x} + b\dot{x} + kx = 0 \qquad b>0,\ m,k>0$

(a)
$$\dot{x} = v$$
$$\dot{v} = -\frac{k}{m}x - \frac{b}{m}v$$
$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \Rightarrow \bar{x}' = A\bar{x}$$

(b) calculate eigenvalues of A

$$\det \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} - \lambda \end{bmatrix} = \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

because $b, k, m > 0$, $\quad Re(\lambda_{1,2}) < 0$

ie. $(0,0)$ is always STABLE

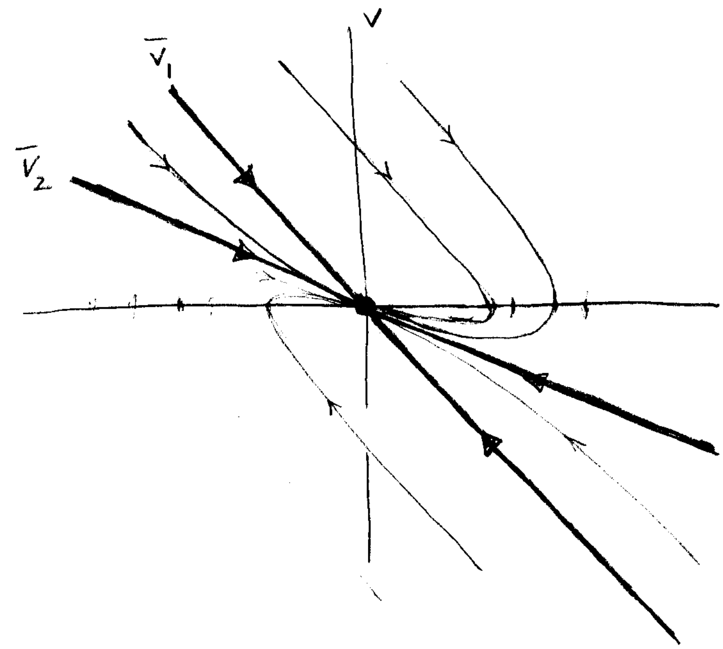
(i) if $\left(\frac{b}{2m}\right)^2 > \frac{k}{m}$, then

$$\lambda_1 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} < \lambda_2 = -\frac{b}{m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} < 0$$

$\Rightarrow$ $(0,0)$ is a STABLE NODE.

note that the eigenvectors of A are

$$\bar{v}_{1,2} = \begin{pmatrix} 1 \\ \lambda_{1,2} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}} \end{pmatrix}$$



$$\bar{v}_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

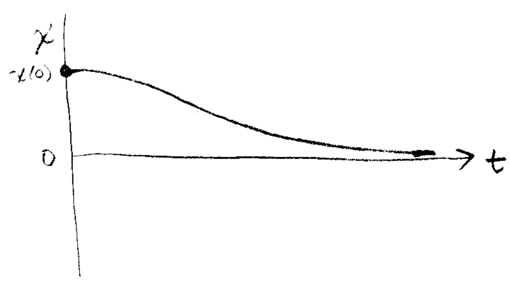
$$\lambda_2 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

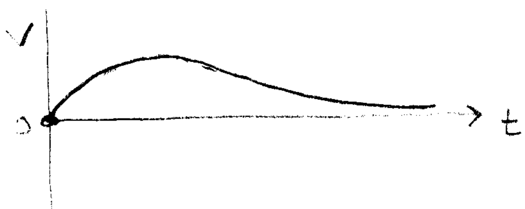
$$\bar{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$$

$$\lambda_1 = \underbrace{-\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}}_{}$$

eigenvalue with larger magnitude i.e. faster decay.

assuming $v(0) = 0$
$x(0) > 0$





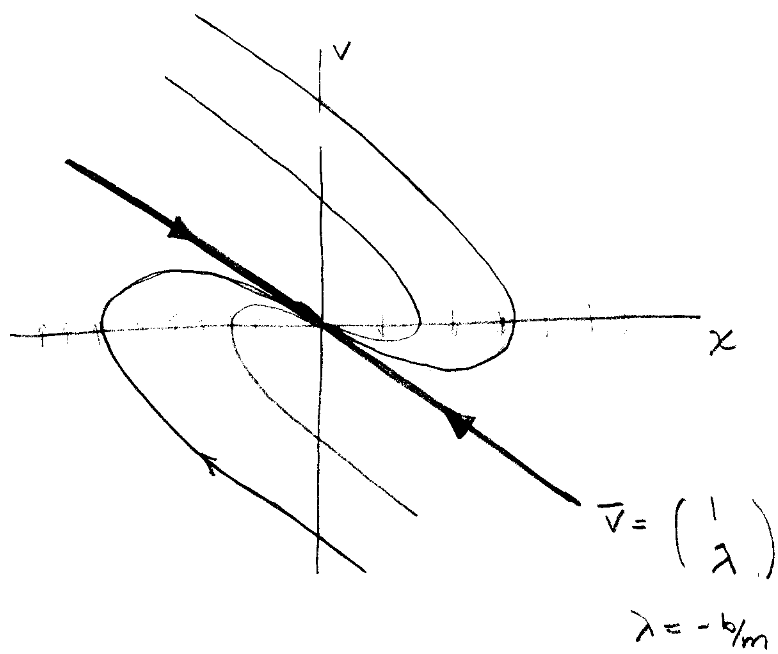
"OVERDAMPED"

iii. if $\left(\frac{b}{2m}\right)^2 = \frac{k}{m}$, then $\lambda_1 = \lambda_2 = \lambda = -\frac{b}{m} < 0$

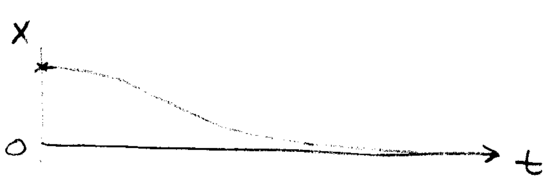
with only one eigenvector $\bar{v} = \begin{pmatrix} 1 \\ -\frac{b}{m} \end{pmatrix}$

$\Rightarrow$ $(0, 0)$ is a STABLE DEGENERATE NODE.

aside $\left\{ \right.$ note that as $\left(\frac{b}{2m}\right)^2 - \frac{k}{m}$ approaches 0 from above (going from case (i) to case (iii)), the eigenvectors $\bar{v}_1$ and $\bar{v}_2$ are squeezed together until they coalesce when $\left(\frac{b}{2m}\right)^2 - \frac{k}{m} = 0$.



$x(0) > 0$
$v(0) = 0$



"CRITICALLY DAMPED"

(iii) if $\left(\frac{b}{2m}\right)^2 < \frac{k}{m}$, then $\lambda_{1,2} \in \mathbb{C}$

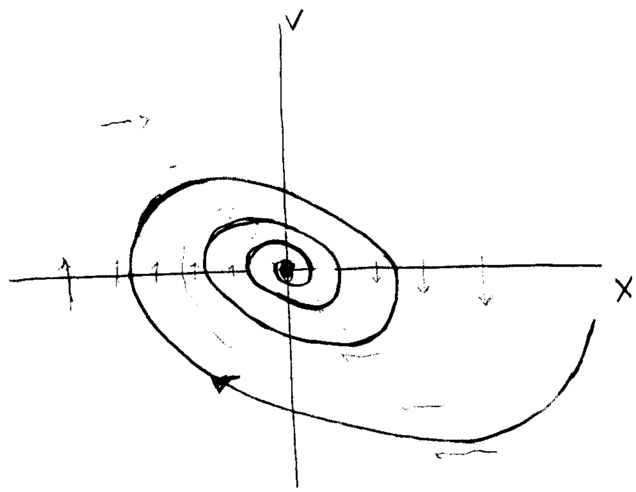
$$\lambda_{1,2} = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

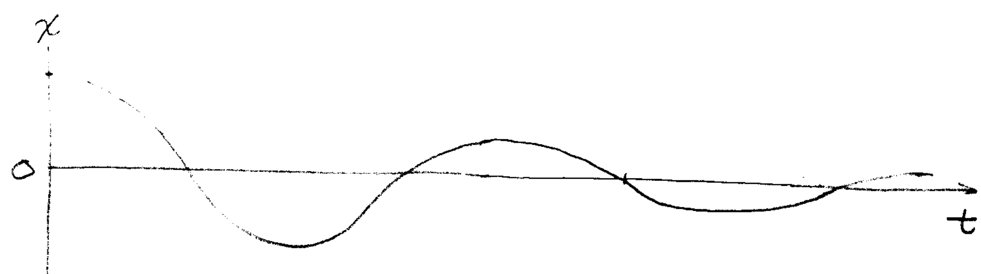
$$= \alpha \pm i\,\omega$$

(0,0) is a STABLE FOCUS.

$$\alpha = Re(\lambda_{1,2}) = -\frac{b}{2m}$$

$$\omega = Im(\lambda_{1,2}) = \pm\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$





damped oscillations

"UNDERDAMPED"