6.3.9 $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$

a) $\dot{x} = 0$ when $x = y^3/4$
$\dot{y} = 0$ when $x = \frac{y^3 - y}{3}$ $\Big\} \Rightarrow$ f.p. when $\frac{y^3}{4} = \frac{y^3 - y}{3}$

$\Rightarrow 4y = y^3$

$\Rightarrow y = 0, \pm 2 \Rightarrow x = 0, \pm 2$

$A_{(0,0)} = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix} \Rightarrow (-4-\lambda)(-1-\lambda) = 0 \Rightarrow \lambda = -1, -4 \Rightarrow$ stable node

$A_{(2,2)} = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix} \Rightarrow (-4-\lambda)(11-\lambda) + 36 = 0 \Rightarrow \lambda^2 - 7\lambda + 8 = 0$

$\Rightarrow \lambda = -1, 8 \Rightarrow$ saddle

$(-2,-2)$ is the same as $(2,2)$; saddle.

b) if $x = y$, then $\frac{dy}{dx} = \frac{y^3 - y - 3x}{y^3 - 4x} = \frac{x^3 - 4x}{x^3 - 4x} = 1$

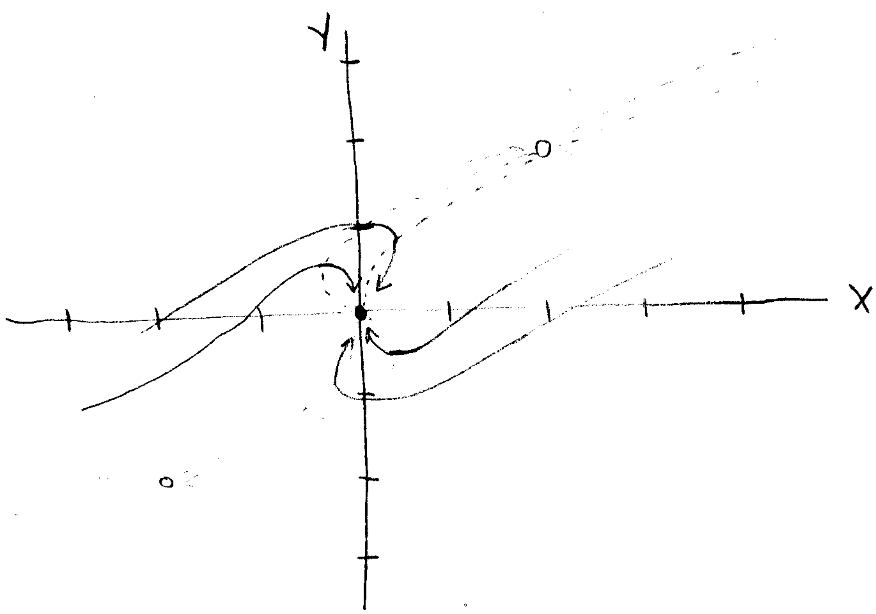
$\therefore$ trajectories that start on $y = x$ stay there

c) $\frac{d(x-y)}{dt} = \dot{x} - \dot{y} = \cancel{y^3} - 4x - \cancel{y^3} + y + 3x = y - x$

$\therefore \frac{d(x-y)}{dt} = -(x-y) \Rightarrow \int \frac{1}{(x-y)}\, d(x-y) = -\int dt$

$\Rightarrow \ln|x-y| = -t + c \Rightarrow |x-y| = Ce^{-t} \to 0$ as $t \to \infty$

d)



6.4.6 $\begin{cases} \dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2 \\ \dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2 \end{cases}$

a)

$$N_1 = x\, x_0$$
$$N_2 = y\, y_0$$
$$t = T\tau$$

$$\Rightarrow \begin{cases} \dfrac{x_0}{T}\dfrac{dx}{d\tau} = r_1 x_0 x \left(1 - x\dfrac{x_0}{K_1}\right) - b_1 x x_0 y y_0 \\[2ex] \dfrac{y_0}{T}\dfrac{dy}{d\tau} = r_2 y_0 y \left(1 - y\dfrac{y_0}{K_2}\right) - b_2 x x_0 y y_0 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{d\tau} = r_1 T x \left(1 - x\dfrac{x_0}{K_1}\right) - b_1 T x y y_0 \\[2ex] \dfrac{dy}{d\tau} = r_2 T y \left(1 - y\dfrac{y_0}{K_2}\right) - b_2 T x x_0 y \end{cases}$$

Let $T = \dfrac{1}{r_1}$, $x_0 = \dfrac{r_1}{b_2}$, $y_0 = \dfrac{r_1}{b_1}$

$$\Rightarrow \begin{cases} \dot{x} = x\left(1 - x\dfrac{r_1}{K_1 b_2}\right) - xy \\[2ex] \dot{y} = \dfrac{r_2}{r_1} y \left(1 - y\dfrac{r_1}{K_2 b_1}\right) - xy \end{cases}$$

Lastly,

Let $\rho = \frac{r_2}{r_1}$, $\alpha = \frac{r_1}{K_1 b_2}$, $\beta = \frac{r_1}{K_2 b_1}$

$$\Rightarrow \begin{cases} \dot{x} = x(1-\alpha x) - xy \\ \dot{y} = \rho y(1-\beta y) - xy \end{cases}$$

b) Nullclines:

$$\dot{x} = 0 \Rightarrow \begin{cases} x = 0 \\ y = 1 - \alpha x \end{cases}$$

$$\dot{y} = 0 \Rightarrow \begin{cases} y = 0 \\ y = 1/\beta \,(1 - x/\rho) \end{cases}$$

fixed points:

$(0, 0)$

$(1/\alpha, 0)$

$(0, 1/\beta)$

$$\left(\frac{1-\beta}{1/\rho - \alpha\beta},\ 1 - \alpha\left(\frac{1-\beta}{1/\rho - \alpha\beta}\right)\right)$$

Jacobian:

$$A = \begin{bmatrix} 1-2\alpha x - y & -x \\ -y & \rho - 2\rho\beta y - x \end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix} \qquad \lambda_1 = 1, \quad \lambda_2 = \rho$$

$\Rightarrow$ $(0,0)$ unstable $(\rho > 0)$

$$A_{(1/\alpha,0)} = \begin{bmatrix} -1 & -1/\alpha \\ 0 & \rho - 1/\alpha \end{bmatrix} \qquad \lambda_1 = -1/\alpha, \quad \lambda_2 = \rho - 1/\alpha$$

$\Rightarrow$ $(1/\alpha, 0)$ stable if $\rho < 1/\alpha$
saddle if $\rho > 1/\alpha$

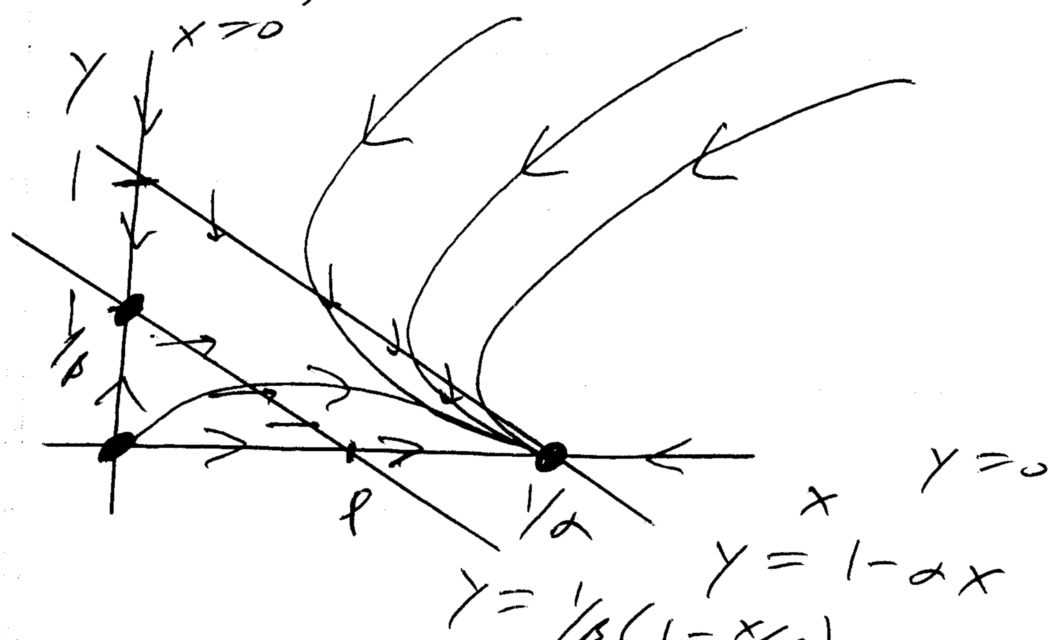
$$A_{(0,1/\beta)} = \begin{bmatrix} 1-1/\beta & 0 \\ -1/\beta & -\rho \end{bmatrix} \qquad \lambda_1 = 1 - 1/\beta, \quad \lambda_2 = -\rho$$

$\Rightarrow$ $(0, 1/\beta)$ stable if $1/\beta > 1$
saddle if $1/\beta < 1$

Depending on the relative values of ρ, α, and β we can see 4 different phase portraits:

1) suppose $1/\beta < 1$, $\rho < 1/\alpha$

$\Rightarrow$ $(1/\alpha, 0)$ is stable
$(0, 1/\beta)$ is a saddle

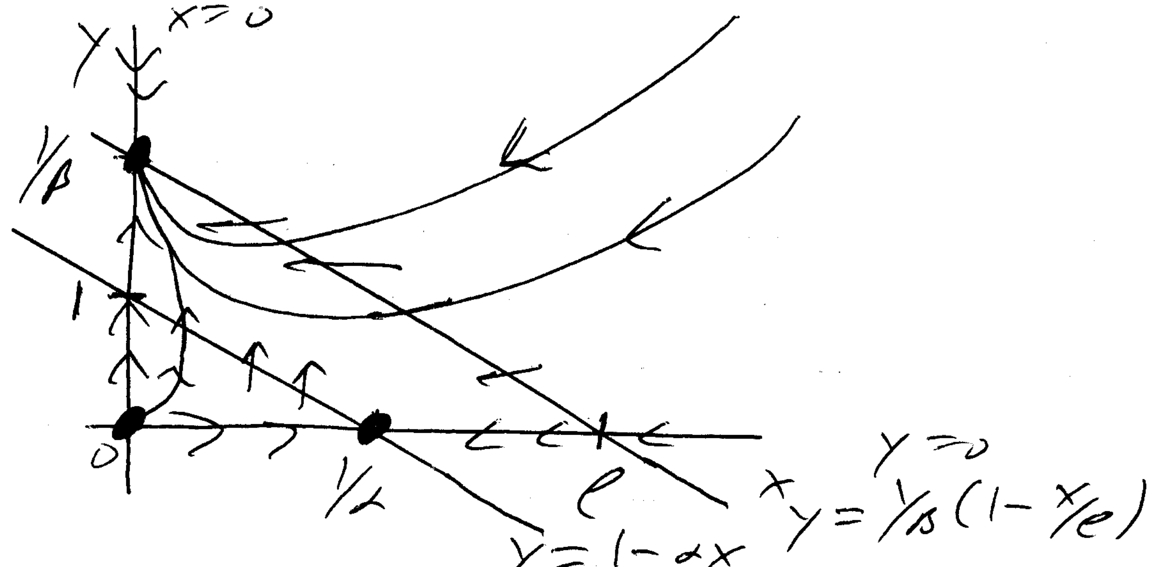


Only 3 fixed points.

Result is eventual elimination of species N_2.

2) Suppose $1/\beta > 1$, $e > 1/\alpha$

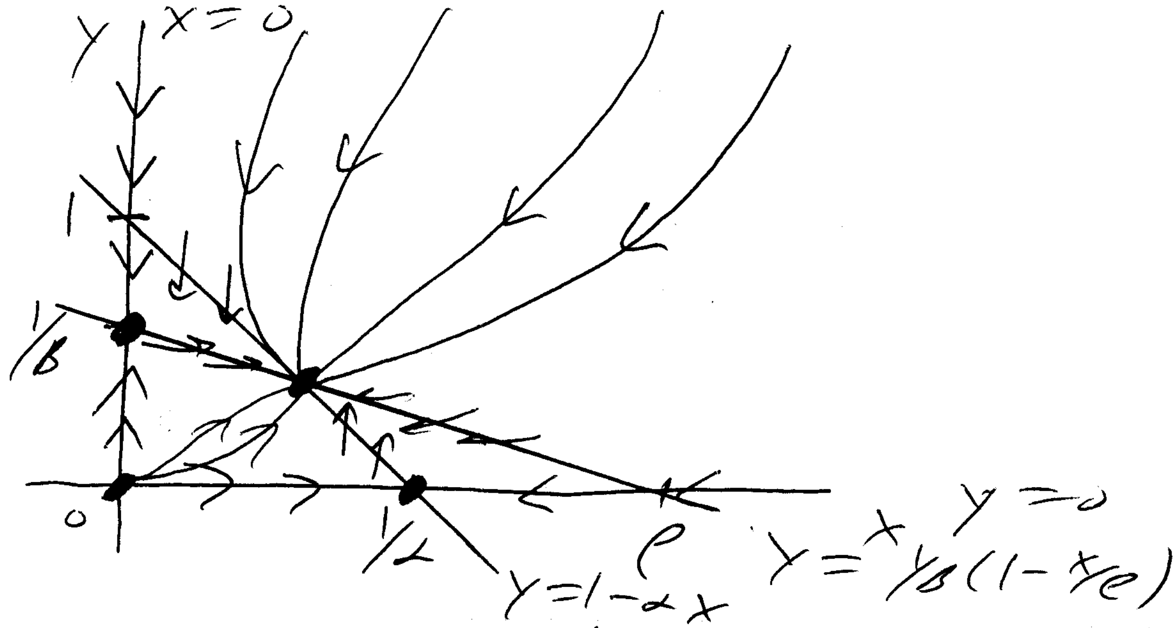
$\Rightarrow$ $(1/\alpha, 0)$ saddle
$(0, 1/\beta)$ stable



Result is ~~[illegible]~~ eventual ellimation of species N_1.

3) Suppose $1/\beta < 1$, $\rho > 1/\alpha$

$\Rightarrow$ $(1/\alpha, 0)$ saddle
$(0, 1/\beta)$ saddle



Using the nullclines, we can see that it looks like the fixed point at

$$\left(\frac{1-\beta}{1/\rho-\alpha\beta},\ 1-\alpha\left(\frac{1-\beta}{1/\rho-\alpha\beta}\right)\right) \text{ is stable.}$$

To confirm this, suppose

$\beta = 2$, $\rho = 3$, $\alpha = 1$

$\Rightarrow$ $1/\beta < 1$, $\rho > 1/\alpha$

The fixed point is then

$(.6, .4)$

$$A_{(.6,.4)} = \begin{bmatrix} 1-2(1)(.6)-.4 & -.6 \\ -.4 & 3-2(3)(2).4-.6 \end{bmatrix}$$

$$= \begin{bmatrix} -.6 & -.6 \\ -.4 & -2.4 \end{bmatrix}$$

$$(-.6-\lambda)(-2.4-\lambda) - .24 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 1.44 = 0$$

$$\Rightarrow \lambda = \frac{-3 \pm \sqrt{9-4(1.44)}}{2}$$

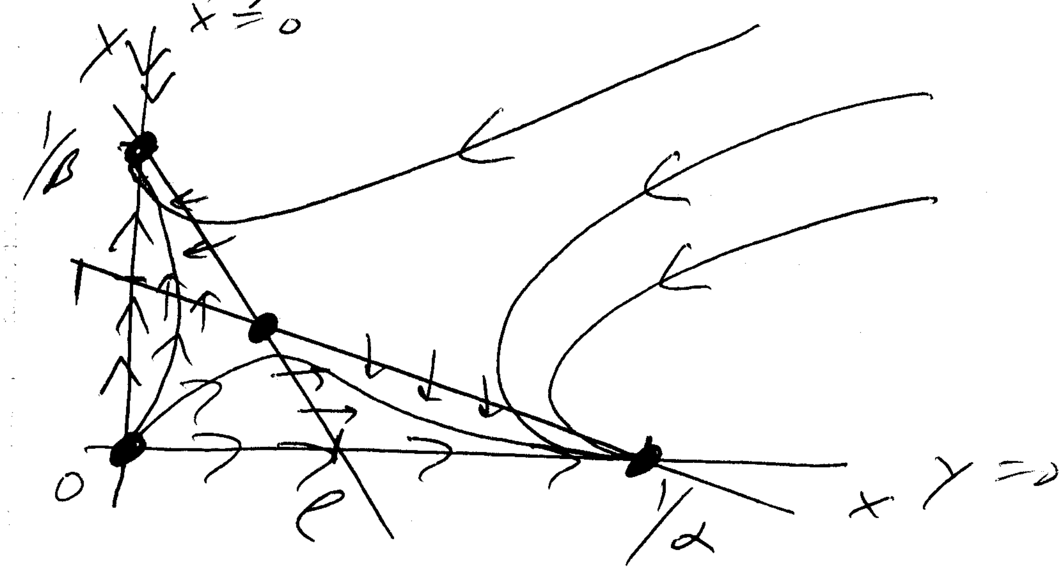
$$= \frac{-3 \pm 1.8}{2} < 0$$

So, $(.6, .4)$ is stable.

Thus, in this case, the two species can coexist.

4) Suppose $1/\beta > 1$, ~~[illegible]~~ $e < 1/\alpha$

$\Rightarrow$ $(1/\alpha, 0)$
$(0, 1/\beta)$ are both stable



Result is, depending upon your initial conditions, eventual elimination of either species N_1 or species N_2.

c) As seen from the four phase portraits, the two species can stably coexist if:

$$1/\beta < 1 \quad \text{and} \quad \rho > 1/\alpha$$

Recall that

$$\beta = \frac{r_1}{K_2 b_1} \qquad \alpha = \frac{r_1}{K_1 b_2} \qquad \rho = \frac{r_2}{r_1}$$

$$\Rightarrow \frac{r_1}{K_2 b_1} > 1 \qquad \frac{r_1}{r_2} < \frac{r_1}{K_1 b_2}$$

$$\Rightarrow \frac{1}{K_2 b_1} > \frac{1}{r_1} \qquad \frac{1}{r_2} < \frac{1}{K_1 b_2}$$

$$\Rightarrow K_2 b_1 < r_1 \quad + \quad r_2 > K_1 b_2$$

i.e.

$$r_1 > K_2 b_1$$
$$r_2 > K_1 b_2$$

Thus, the two species can stably coexist if their rates of growth are greater than the carrying capacity of the other species times the competition

6.4.7 $\dot{n}_1 = G_1 N n_1 - k_1 n_1$, $\dot{n}_2 = G_2 N n_2 - k_2 n_2$

$$N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2$$

a) $A_{00} = \begin{bmatrix} G_1 N_0 - k_1 & 0 \\ 0 & G_2 N_0 - k_2 \end{bmatrix} \Rightarrow (G_1 N_0 - k_1 - \lambda)(G_2 N_0 - k_2 - \lambda) = 0$

$\Rightarrow \lambda_1 = G_1 N_0 - k_1$, $\lambda_2 = G_1 N_0 - k_2$

So $(0,0)$ is stable if $k_i > G_i N_0$ for $i = 1, 2$,
$(0,0)$ is unstable if $k_i < G_i N_0$ for $i = 1, 2$, and
$(0,0)$ is a saddle otherwise.

b) $\dot{n}_1 = G_1 N_0 n_1 - G_1 \alpha_1 n_1^2 - G_1 \alpha_2 n_1 n_2 - k_1 n_1$

$= -G_1 \alpha_1 n_1^2 + (G_1 N_0 - G_1 \alpha_2 n_2 - k_1) n_1$

$= 0$ when $n_1 = 0$ or $n_1 = \dfrac{G_1 N_0 - G_1 \alpha_2 n_2 - k_1}{G_1 \alpha_1}$

$\dot{n}_2 = 0$ when $n_2 = 0$ or $n_2 = \dfrac{G_2 N_0 - G_2 \alpha_1 n_1 - k_2}{G_2 \alpha_2}$

$\therefore$ fixed points are $(0,0)$, $\left(0, \dfrac{G_2 N_0 - k_2}{G_2 \alpha_2}\right)$, $\left(\dfrac{G_1 N_0 - k_1}{G_1 \alpha_1}, 0\right)$

If $(0,0)$ is stable, the other 2 f.p.s are physically irrelevent.
If $(0,0)$ is unstable and $k_2 < k_1$, then $(0, n_2^*)$ is stable and $(n_1^*, 0)$ is a saddle -- we can see this by solving for the eigenvalues for $(0, n_2^*)$. They are $\sigma_1 = \lambda_1 - \frac{G_1}{G_2}\lambda_2$ and $\sigma_2 = -\lambda_2 - 4k_2$.
If $(0,0)$ is unstable and $k_2 < k_1$, then $(0, n_2^*)$ is a saddle and $(n_1^*, 0)$ is stable
If $(0,0)$ is a saddle, one other f.p. will be stable and the third will be physically irrelevent.

c) Looking at the stability discussion above we can see that there are 5 qualitatively different phase portraits.

(pics on next page)

6.5.3 $\ddot{x} = a - e^x = F(x;a)$

$$\begin{cases} \dot{x} = y \\ \dot{y} = a - e^x \end{cases}$$

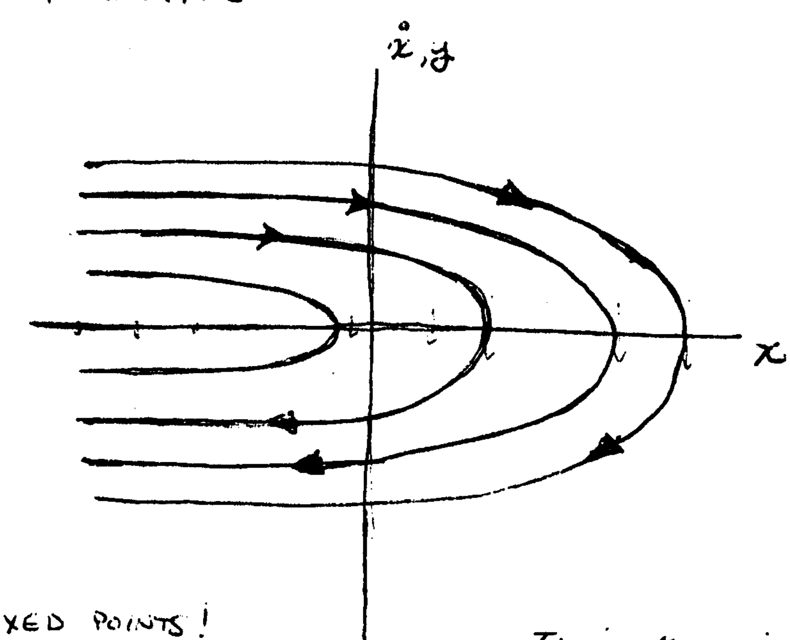
a) define $V(x) = -\int F(x;a)\,dx$
$= -ax + e^x$

(same argument as in 6.5.1 (b))

$E(x,y) = \frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}\dot{x}^2 - ax + e^x$
is a CONSERVED QUANTITY.

b) PHASE PORTRAITS

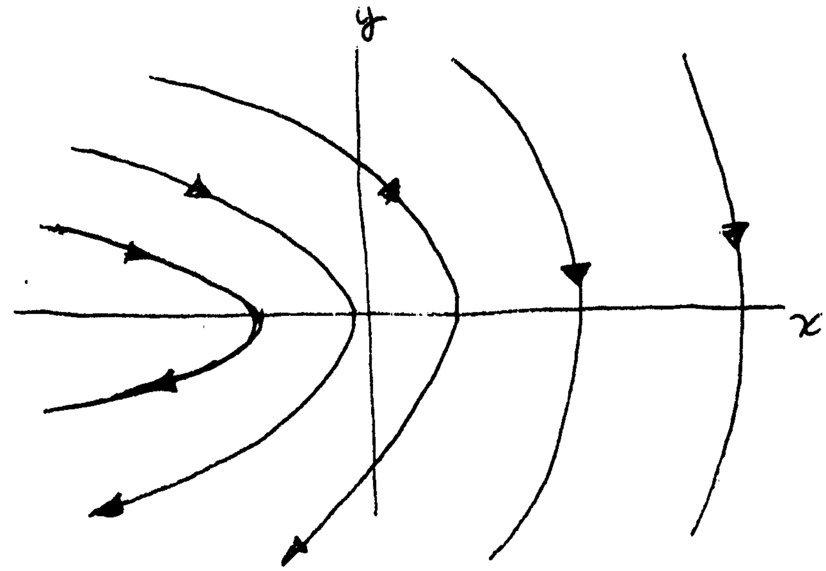
$a = 0$



NO FIXED POINTS!

trajectories are level sets of
$\frac{1}{2}y^2 + e^x = E(x,y)$

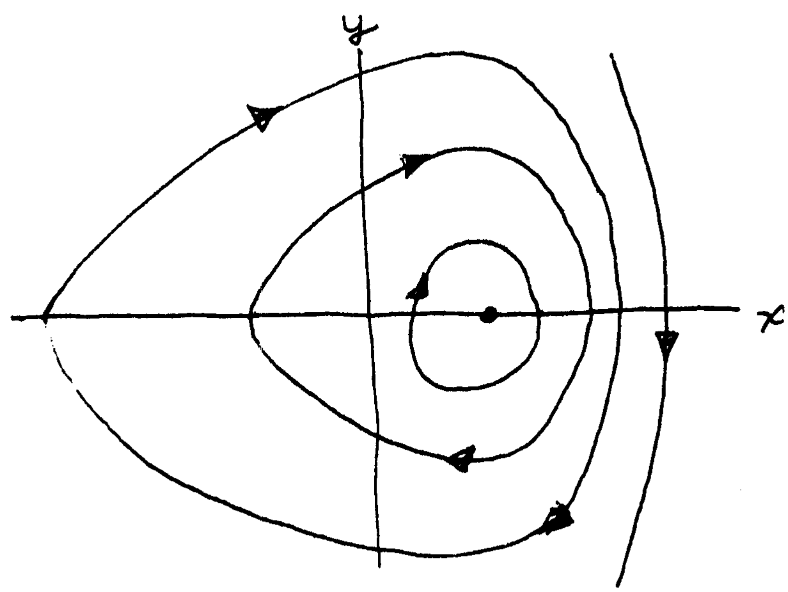
$a < 0$



NO FIXED PTS

traj. are level sets of $\frac{1}{2}y^2 - ax + e^x = E(x,y)$.

$a > 0$



FIXED PT at $y = 0$, $e^x = ax$

traj are level sets of $E(x,y) = \frac{1}{2}y^2 - ax + e^x$.

6.5.6 Kermack-Mckendrick Epidemic model

$$\dot{x} = -kxy \qquad x(t), y(t) \geq 0$$
$$\dot{y} = kxy - ly \qquad k, l > 0$$

a) FIXED PTS $\quad \dot{x} = 0 \Rightarrow x = 0, \; y = 0$

$\quad \dot{y} = 0 \Rightarrow x = \frac{l}{k}, \; y = 0$

$\therefore$ THERE IS A LINE OF FIXED PTS $y^* = 0$.

STABILITY

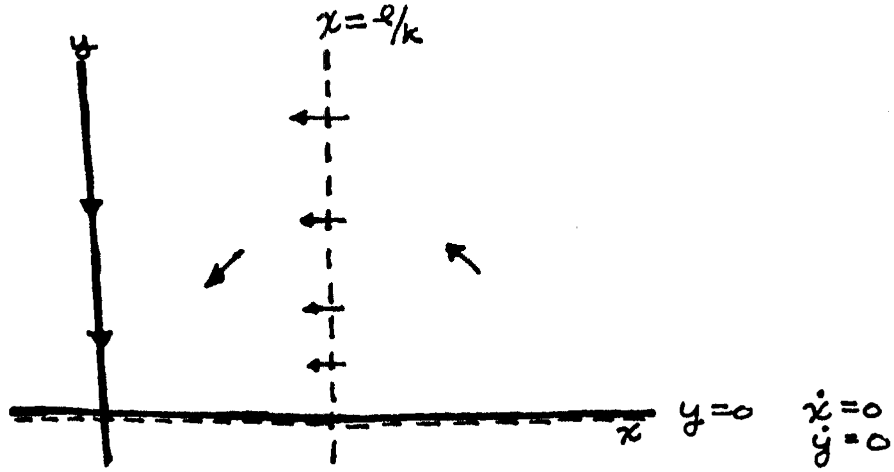
$$\det \begin{bmatrix} -\cancel{ky^*}^{\,0} - \lambda & -kx^* \\ \cancel{ky^*}^{\,0} & kx^* - l - \lambda \end{bmatrix} = 0$$

$$\lambda^2 - (k^* x^* - l)\lambda = 0$$

$\lambda_1 = 0 \quad \leftarrow$ NEUTRAL STABILITY ALONG $y = 0$.

$\lambda_2 = kx^* - l$

b)



c) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{kxy - \ell y}{-kxy} = -1 + \frac{\ell}{k}\frac{1}{x}$$

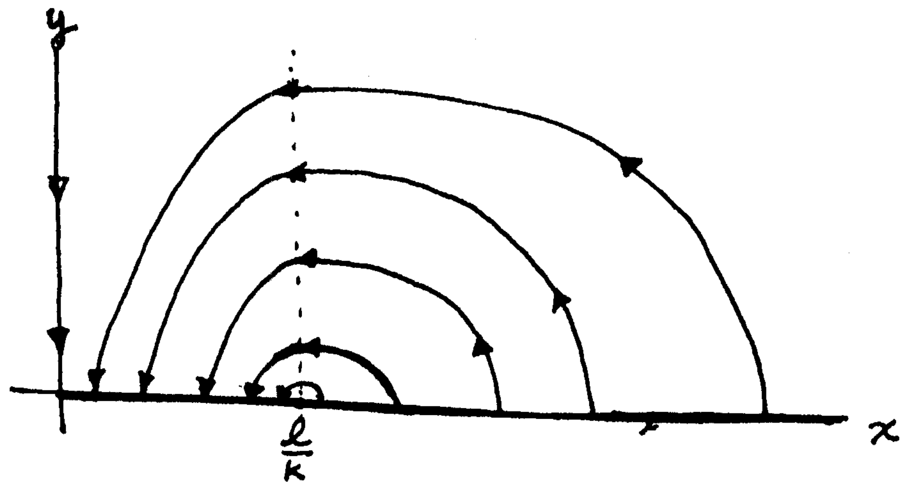
$$\int dy = \int \left(-1 + \frac{\ell}{k}\frac{1}{x}\right) dx + c$$

$$y = -x + \frac{\ell}{k} \ln x + c \quad , \quad x \neq 0$$

CONSERVED QUANTITY

$$y + x - \frac{\ell}{k} \ln x = E(x, y)$$

d)



- as $t \to \infty$, $x \to x^*$, $x^* \in [0, \frac{l}{k}]$ ...
  if $y > 0$

e) AN EPIDEMIC IS SAID TO OCCUR IF $y(t)$ INCREASES INITIALLY, THEREFORE THE MODEL PREDICTS THAT EPIDEMICS OCCUR WHEN $x(0) > \frac{l}{k}$, $y(0) > 0$.

i.e if $\dot{y} > 0$ initially

$\Rightarrow \quad kxy - ly > 0$

$\Rightarrow \quad xy - \frac{l}{k}y > 0$

$xy > \frac{l}{k}y$

$x > \frac{l}{k}$

6.5.7 $\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$, $\alpha > 0$ $0 < \varepsilon \ll 1$

a) $v = \frac{du}{d\theta}$

$\Rightarrow \frac{dv}{d\theta} = -u + \alpha + \varepsilon u^2$

b) $\dot{u} = 0 \Rightarrow v = 0$, $\dot{v} = 0 \Rightarrow u = \frac{1 \pm \sqrt{1 - 4\alpha\varepsilon}}{2\varepsilon}$

$\Rightarrow$ f.p.s: $\left(\frac{1 \pm \sqrt{1-4\alpha\varepsilon}}{2\varepsilon}, 0\right)$ assuming $1 - 4\alpha\varepsilon \geq 0$

c) $A = \begin{bmatrix} 0 & 1 \\ -1+2\varepsilon u & 0 \end{bmatrix}$ at $\left(\frac{1-\sqrt{1-4\alpha\varepsilon}}{2\varepsilon}, 0\right)$, we have $\begin{vmatrix} -\lambda & 1 \\ -\sqrt{1-4\alpha\varepsilon} & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 = -\sqrt{1-4\alpha\varepsilon} \Rightarrow \lambda$ is pure imaginary

$\therefore$ we have a center in the linearized system

Since $\frac{d^2u}{d\theta^2}$ is a function of u only, we have a conservative system. $\therefore$ we do have a nonlinear center.

d) Since $v = \frac{du}{d\theta} = 0$, $u = 1/r$ is a constant

$\therefore$ the corresponding orbit is circular b/c r is constant.

6.5.8 $$H = \frac{p^2}{2m} + \frac{kx^2}{2}$$

$$\Rightarrow \begin{cases} \dot{x} = \partial H/\partial p = \frac{2p}{2m} = p/m \\ \dot{p} = -\partial H/\partial x = -\frac{2xk}{2} = -kx \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = p/m \\ \dot{p} = -kx \end{cases}$$

$\dot{x} = p/m$

p = momentum = mass x velocity

$\dot{x}$ = velocity so,

$p = m\dot{x}$

$\Rightarrow \dot{x} = p/m$

Next,

$\dot{p} = -kx$ which is Hooke's Law

which is equivalent to

$F = ma$.

6.5.9 $H(x,p)$ is conserved.

Want to show $\dot{H} = 0$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt}$$

Recall,

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = \frac{-\partial H}{\partial x}$$

$$\Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial x}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial p}\left(\frac{-\partial H}{\partial x}\right) = 0$$

Thus, $\dot{H} = 0$

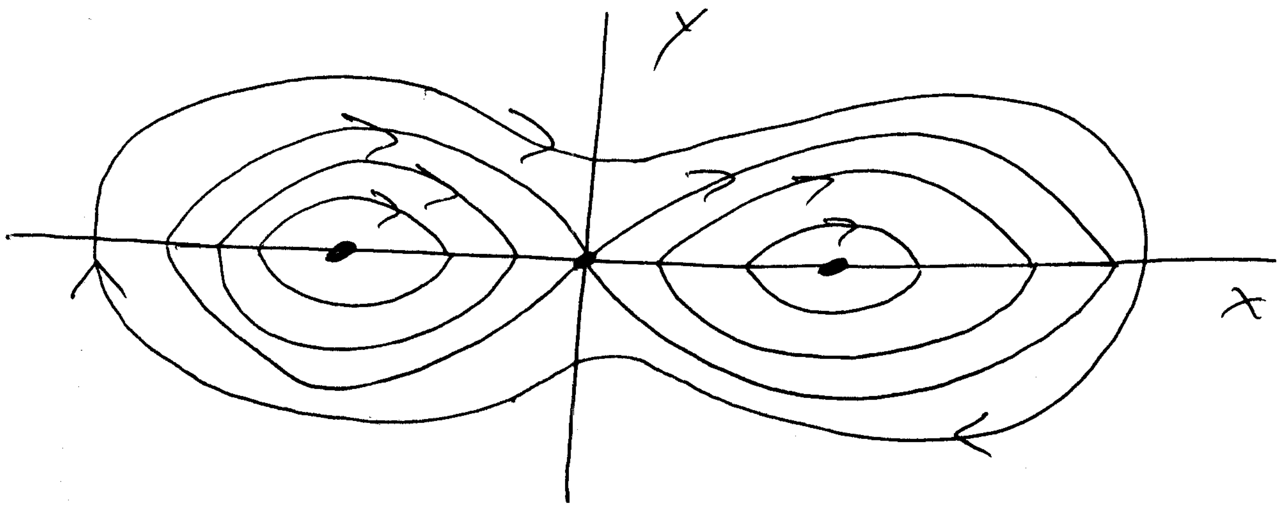
$\Rightarrow$ $H(x,p)$ is a conserved quantity.

6.5.11 $\begin{cases} \dot{x} = y \\ \dot{y} = -by + x - x^3 \end{cases}$ $0 < b \ll 1$

$$\Rightarrow \ddot{x} + \underline{\underline{b\dot{x}}} - x + x^3 = 0$$

(Damping term)

We know from example 6.5.2 that if $b = 0$ we have:



Now, to see how the b term affects this phase plane, we will look at the eigenvalues of the 3 fixed points.

Jacobian

$$A = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -b \end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -b \end{bmatrix}$$

$$(-\lambda)(-b-\lambda) - 1 = 0$$

$$\lambda^2 + \lambda b - 1 = 0$$

$$\Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2+4}}{2}$$

since $b \ll 1 \Rightarrow$ $(0,0)$ is a saddle

$$A_{(\pm 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -b \end{bmatrix}$$

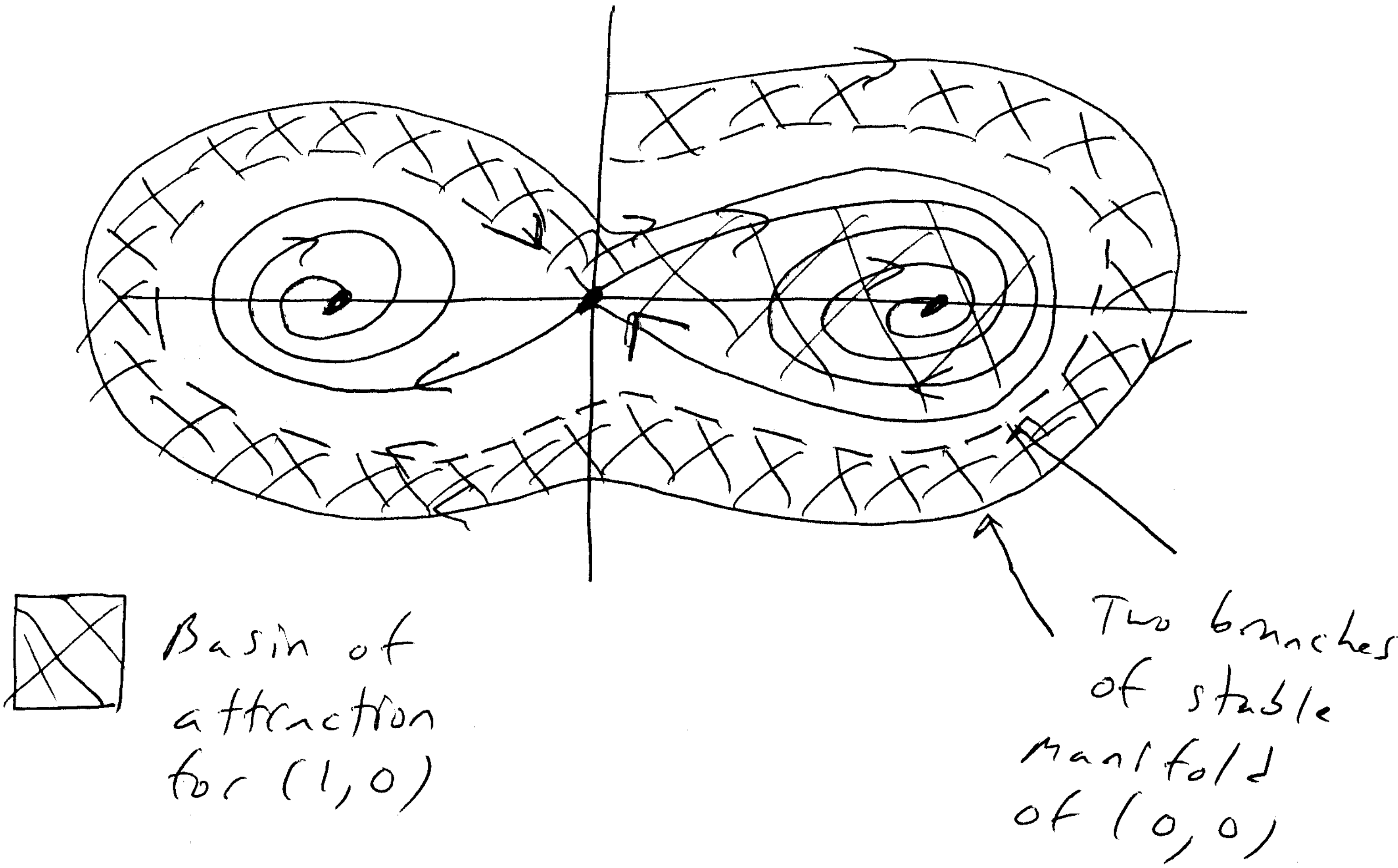
$$(-\lambda)(-b-\lambda) + 2 = 0$$

$$\lambda^2 + \lambda b + 2 = 0$$

$$\Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2-8}}{2}$$

$(\pm 1, 0)$ stable spirals.

So, our phase plane must look like:



6.5.19 $\quad \dot{R} = aR - bRF, \quad \dot{F} = -cF + dRF$

a) aR : without foxes, growth rate of rabbit population is proportional to that population.

$-cF$: without rabbits, death rate of foxes is proportional to fox population.

$-bRF$ and $+dRF$: foxes eat rabbits.

b) Let $\tau = at$ and $x = \frac{d}{c}R$ and $y = \frac{b}{a}F$

then $$\frac{dx}{d\tau} = \frac{d(\frac{d}{c}R)}{d(at)} = \frac{d}{ac}\dot{R} = \frac{d}{c}R - \frac{bd}{ac}RF = x - xy = x(1-y)$$

and $$\frac{dy}{d\tau} = \frac{d(\frac{b}{a}F)}{d(at)} = \frac{b}{a^2}\dot{F} = -\frac{bc}{a^2}F + \frac{c}{c}\frac{bd}{a^2}RF = -\frac{c}{a}y + \frac{c}{a}xy = \frac{c}{a}y(x-1)$$

so $\mu = \frac{c}{a}$

c) To find a conserved quantity, write $\frac{dy}{dx} = \frac{\mu y(x-1)}{x(1-y)}$ and separate variables:

$$\int \frac{1-y}{\mu y}\,dy = \int \frac{x-1}{x}\,dx$$

$$\Rightarrow \frac{1}{\mu}\ln y - \frac{y}{\mu} = x - \ln x + C$$

$\therefore$ the conserved quantity is $\frac{1}{\mu}(\ln y - y) + \ln x - x$

d) $A_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \mu & 0 \end{pmatrix} \Rightarrow \lambda = \pm i\sqrt{\mu}$ so linearization predicts a center at $(1,1)$, and since the system is conservative we actually have a nonlinear center.

$A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -\mu \end{pmatrix} \Rightarrow \lambda = 1, -\mu$ so $(0,0)$ is a saddle.

$\therefore$ the manifolds of $(0,0)$ are the only non-periodic [illegible]

6.7.4 $\ddot{\theta} + \sin\theta = 0$

a) conservation of energy:

$$\ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta = 0 \Rightarrow \frac{d}{dt}\left[\left(\frac{\dot{\theta}^2}{2}\right) - \cos\theta\right] = 0$$

$$\Rightarrow \frac{\dot{\theta}^2}{2} - \cos\theta + C = 0$$

$C = \cos\alpha$ just means we are setting 0 potential at $\theta = \alpha$.

$$\therefore \dot{\theta}^2 = (\cos\theta - \cos\alpha)2$$

If we separate variables and integrate over one period we get

$$T = 4\int_0^\alpha \frac{d\theta}{(2(\cos\theta - \cos\alpha))^{1/2}}$$

b) $$T = 4\int_0^\alpha \frac{d\theta}{\left[2(2\cos^2\frac{\theta}{2} - 1) - 2(2\cos^2\frac{\alpha}{2} - 1)\right]^{1/2}}$$

$$4\int_0^\alpha \frac{d\theta}{\left[4(1-\sin^2\frac{\theta}{2}) - 4(1-\sin^2\frac{\alpha}{2})\right]^{1/2}} = 4\int_0^\alpha \frac{d\theta}{\left[4(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2})\right]^{1/2}}$$

c) Let $(\sin\frac{\alpha}{2})\sin\varphi = \sin\frac{\theta}{2} \Rightarrow \sin\frac{\alpha}{2}\cos\varphi\, d\varphi = \frac{1}{2}\cos\frac{\theta}{2}\, d\theta$

then we have $$4\int_0^{\pi/2} \frac{d\theta}{2(\sin^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\sin^2\varphi)^{1/2}}$$

$$= 4\int_0^{\pi/2} \frac{d\theta}{2(\sin^2\frac{\alpha}{2}(1-\sin^2\varphi))^{1/2}} = 4\int_0^{\pi/2} \frac{d\theta}{2\sin\frac{\alpha}{2}\cos\varphi} = 4\int_0^{\pi/2} \frac{d\theta}{\cos\frac{\theta}{2}\frac{d\theta}{d\varphi}}$$

$$= 4\int_0^{\pi/2} \frac{d\varphi}{\cos\frac{\theta}{2}} = 4K(\sin^2\frac{\alpha}{2})$$

given

$$= 4\int_0^{\pi/2} \frac{d\varphi}{(1 - \sin^2\frac{\alpha}{2}\sin^2\varphi)^{1/2}}$$