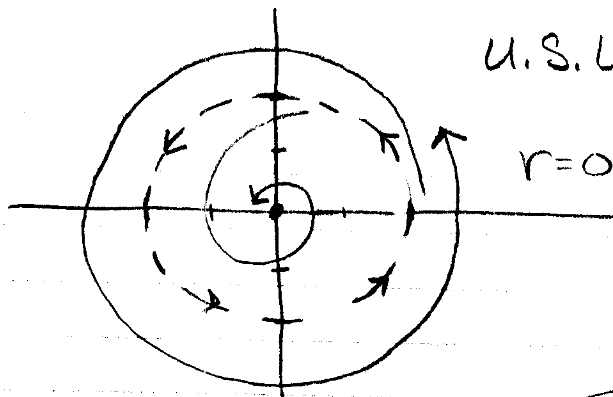
7.1.1 $\dot{r} = r^3 - 4r,\ \dot{\theta} = 1$

r-null: $\dot{r} = r(r^2-4) = 0 \Rightarrow r = 0, 2 \Rightarrow$ f.p. at $r=0$

θ-null: DNE



U.S. Limit cycle at $r=2$.

$r=0$ is stable

7.1.2 $\dot{r} = r(1-r^2)(9-r^2),\ \dot{\theta} = 1$

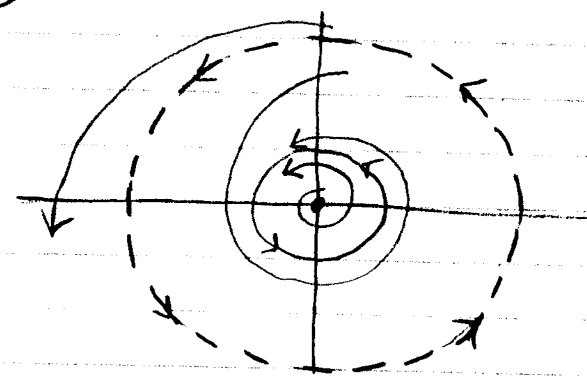
Limit cycles at $r = 1$ and 3

fixed point at $r=0$

$r=1$ is a stable L.C.

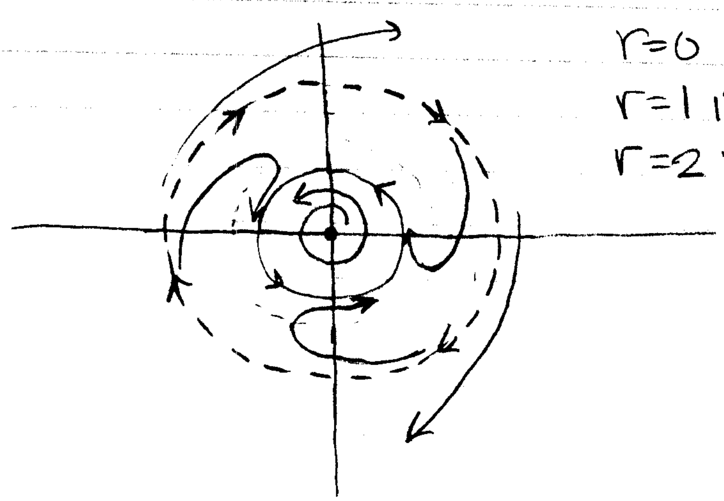
$r=3$ is an unstable L.C.

$r=0$ is U.S.



7.1.3 $\dot{r} = r(1-r^2)(4-r^2),\ \dot{\theta} = 2-r^2$

$\dot{r} = 0$ at $r = 0, 1, 2$
$\dot{\theta} = 0$ at $r = \sqrt{2}$ } fixed pt. at $r=0$



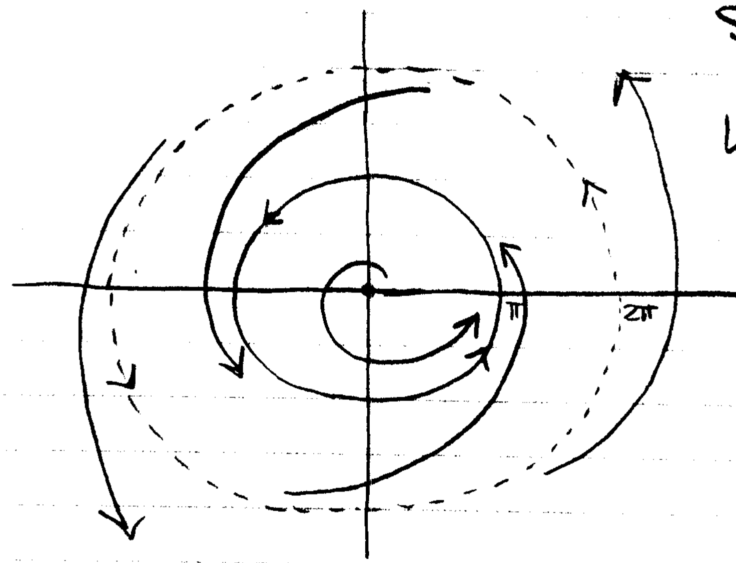
$r=0$ is unstable

$r=1$ is a stable L.C.

$r=2$ is an unstable L.C.

7.1.4 $\dot{r} = r\sin r,\ \dot{\theta} = 1$

$\left.\begin{array}{l}\dot{r}=0 \Rightarrow r=0 \text{ or } n\pi,\ n\in\mathbb{N} \\ \dot{\theta}\neq 0\end{array}\right\} \Rightarrow r=0$ is the only f.p.



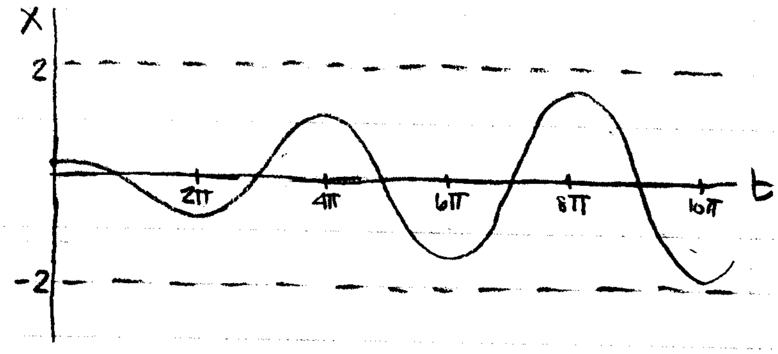
Stable limit cycles when $r=(2n+1)\pi$;
U.S. limit cycles when $r=2n\pi$
$(n\in\mathbb{N})$.

7.1.7 $\dot{r} = r(4-r^2)$, $\dot{\theta} = 1$

$r=0$ is an unstable f.p. and $r=2$ is a stable L.c.

Period of cycles is 2π

$\therefore$ If $x(0) = .1$ and $y(0) = 0$, then $x(t)$ will look something like



(the amplitude of oscillations will approach 2)

7.1.8 $\ddot{x} + a\dot{x}(x^2+\dot{x}^2-1)+x=0$

a) $\begin{cases} \dot{x} = y \\ \dot{y} = -ay(x^2+y^2-1)-x \end{cases}$

fixed point. $(0,0)$

Jacobian

$$A = \begin{bmatrix} 0 & 1 \\ -a2xy-1 & -ax^2-3ay^2+a \end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}$$

$\Rightarrow$ $-\lambda(a-\lambda)+1=0$

$\lambda^2 - a\lambda + 1 = 0$

$\Rightarrow \lambda = \dfrac{a \pm \sqrt{a^2-4}}{2}$

$\Rightarrow$ if $0<a<2$ $\Rightarrow$ $(0,0)$ is unstable spiral

if $a>2$ $\Rightarrow$ $(0,0)$ unstable node

b) Convert to polar coordinates

$$r\dot{r} = x\dot{x} + y\dot{y}$$
$$= xy + y(-ay(x^2+y^2-1)-x)$$
$$= \cancel{xy} - ay^2(x^2+y^2-1) - \cancel{xy}$$
$$= -ay^2(x^2+y^2-1)$$
$$= -a(r\sin\theta)^2(r^2-1)$$

$$\Rightarrow \quad \dot{r} = -ar(r^2-1)\sin^2\theta$$

$$r^2\dot{\theta} = \underline{x\dot{y} - y\dot{x}}$$
$$= x(-ay(x^2+y^2-1)-x) - y(y)$$
$$= -axy(x^2+y^2-1) - x^2 - y^2$$
$$= -axy(x^2+y^2-1) - (x^2+y^2)$$
$$= -ar^2\cos\theta\sin\theta(r^2-1) - r^2$$

$$\Rightarrow \quad \dot{\theta} = -a\cos\theta\sin\theta(r^2-1) - 1$$

$$\Rightarrow \quad \begin{cases} \dot{r} = -ar(r^2-1)\sin^2\theta \\ \dot{\theta} = -a\cos\theta\sin\theta(r^2-1) - 1 \end{cases}$$

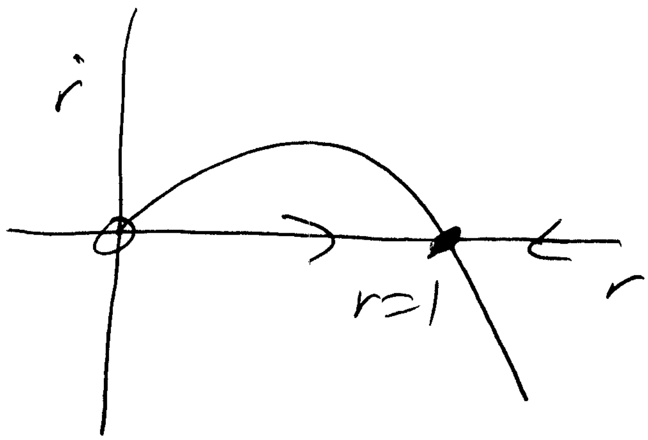
Note that $\sin^2\theta \geq 0$

So, if $\sin^2\theta = 0$

$\Rightarrow \dot{r} = 0$, so at 0 or 2π, trajectories are strictly vertical

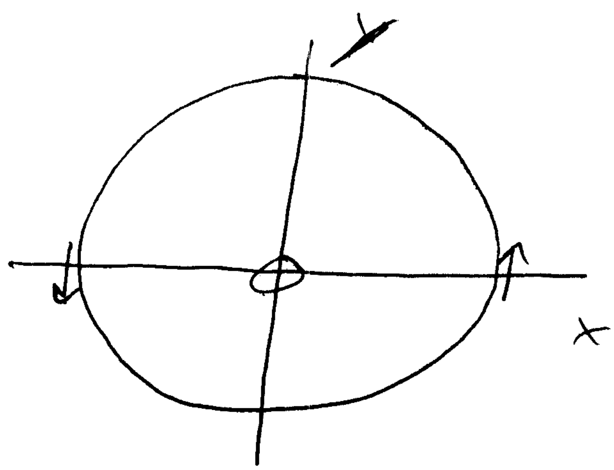
So, we can plot
(absorb $a\sin^2\theta$ into $\tilde{a}$)

$-r(r^2-1)\tilde{a}$



c) $\Rightarrow$ existence of a stable limit cycle at $r=1$ and at $r=1$
$\dot{\phi} = -1$

$\Rightarrow$

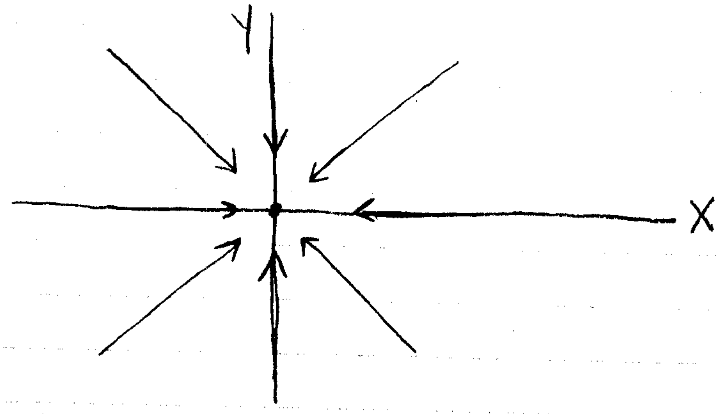


$T = 2\pi$

amplitude $= 1$

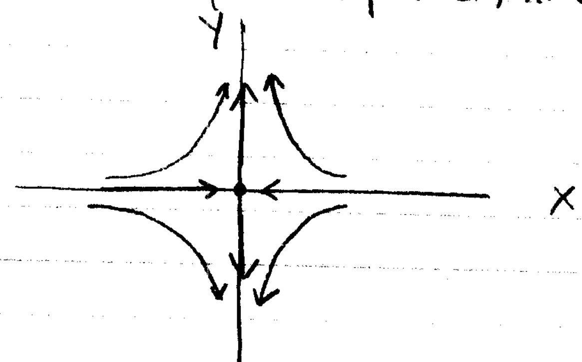
7.2.1 $V = x^2 + y^2 \Rightarrow \dot{x} = -2x,\ \dot{y} = -2y$

$\Rightarrow (0,0)$ is the only fixed point, and it's stable



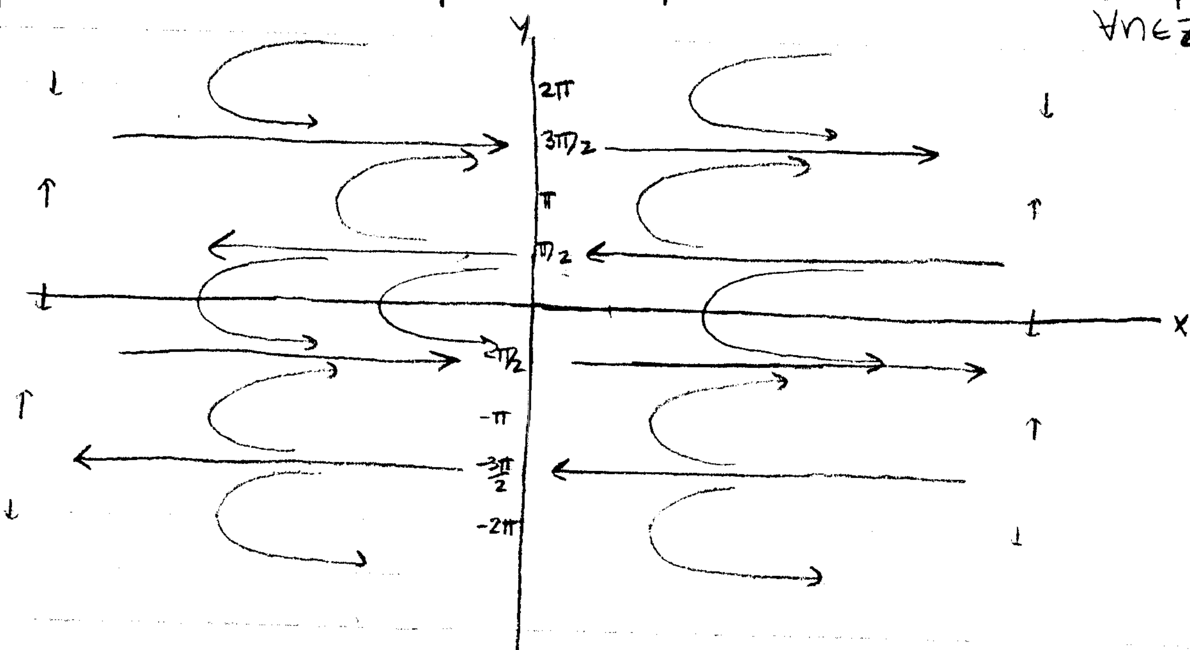
7.2.2 $V = x^2 - y^2 \Rightarrow \dot{x} = -2x,\ \dot{y} = 2y$

$\Rightarrow (0,0)$ is the only fixed point, and it's a saddle.



7.2.3 $V = e^x \sin y \Rightarrow \dot{x} = -e^x \sin y,\ \dot{y} = -e^x \cos y$

$\Rightarrow$ no fixed points $y = 3\pi/2 + 2n\pi$ is a stable asymptote $\forall n \in \mathbb{Z}$



7.2.6 a) $\dot{x} = y^2 + y\cos x$

$\dot{y} = 2xy + \sin x$

$\dot{x} = -dV/dx$

$\dot{y} = -dV/dy$

$\int -\dot{x}\, dx = -xy^2 - y\sin x$

$\int -\dot{y}\, dy = -xy^2 - y\sin x$

Thus,

$V(x,y) = -xy^2 - y\sin x$

b) $\dot{x} = 3x^2 - 1 - e^{2y}$

$\dot{y} = -2xe^{2y}$

$\int -\dot{x}\, dx = -x^3 + x + xe^{2y}$

$\int -\dot{y}\, dy = xe^{2y}$

Thus,

$V(x,y) = -x^3 + x + xe^{2y}$

7.2.10 $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$, $V = ax^2 + by^2$

Show no closed orbits.

We need $V > 0$, $V(x^*, y^*) = 0$, $\dot{V} < 0$ along trajectories:

- $a, b > 0 \Rightarrow V > 0$
- $V(0,0) = 0$
- $\dot{V} = 2ax\dot{x} + 2by\dot{y} = 2ax(y - x^3) + 2by(-x - y^3)$

$= 2axy - 2ax^4 - 2bxy - 2by^4 < 0$ if $a = b$

- $\therefore$ $V(x,y) = a(x^2 + y^2)$ is a Liapunov fcn $\forall a > 0$.

$\Rightarrow$ the system has no closed orbits.

?.2.11 $V = ax^2 + 2bxy + cy^2$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$

Assume $a > 0$ & $ac - b^2 > 0$

$$V = ax^2 + 2bxy + cy^2$$

$$= a\left(x^2 + 2\frac{b}{a}xy\right) + cy^2$$

$$= a\left(x^2 + 2\frac{b}{a}xy + \left(\frac{b}{a}y\right)^2 - \left(\frac{b}{a}y\right)^2\right) + cy^2$$

$$= a\left(x + \frac{b}{a}y\right)^2 - a\left(\frac{b}{a}y\right)^2 + cy^2$$

$$= a\left(x + \frac{b}{a}y\right)^2 + y^2\left(c - \frac{b^2}{a}\right)$$

but $\left(x + \frac{b}{a}y\right)^2 > 0$

$ac - b^2 > 0 \Rightarrow \left(c - b^2/a\right) > 0$

$a > 0$

$y^2 > 0$

Thus,

$V > 0$. (positive definite)

Next, suppose V is positive definite

then

$$V = a\left(x + \frac{b}{a}y\right)^2 + y^2\left(c - \frac{b^2}{a}\right) > 0$$

so, we need

$a > 0$

& $c - b^2/a > 0$

since $\left(x + \frac{b}{a}y\right)^2 > 0$ & $y^2 > 0$

$\Rightarrow$ $a > 0$

$ac - b^2 > 0$.

7.3.1 $\dot{x} = x - y - x(x^2+5y^2)$

$\dot{y} = x + y - y(x^2+y^2)$

a)
$$A = \begin{bmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-x^2-3y^2 \end{bmatrix}$$

$$|A|_{0,0} - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 + 1 = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = 1 \pm i$$

$\therefore$ (0, 0) is an unstable spiral.

b) $r\dot{r} = x\dot{x} + y\dot{y}$

$$= x^2 - \cancel{xy} - x^4 - 5x^2y^2 + \cancel{xy} + y^2 - x^2y^2 - y^4$$

$$= x^2 + y^2 - (x^4 + y^4 + 2x^2y^2) - 4x^2y^2$$

$$= r^2 - r^4 - 4r^2\cos^2\theta \cdot r^2\sin^2\theta$$

$$\Rightarrow \dot{r} = r - r^3 - r^3\sin^2 2\theta$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

$$= x^2 \cancel{+xy} - \cancel{x^3y} - xy^3 - \cancel{xy} + y^2 + \cancel{x^3y} + 5xy^3$$

$$= x^2 + y^2 + 4xy^3$$

$$= r^2 + 4r\cos\theta\, r^3\sin^3\theta$$

$$\Rightarrow \dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta$$

7.3.1 c) $\dot{r} > 0 \Rightarrow r - r^3(1+\sin^2 2\theta) > 0$

$$\Rightarrow 1 > r^2(1+\sin^2 2\theta)$$

$$\Rightarrow \frac{1}{\sqrt{1+\sin^2 2\theta}} > r > 0$$

$\therefore$ all trajectories have a radially outward component on the circle $r_1 = 1/\sqrt{2} - \varepsilon$

d) $\dot{r} < 0 \Rightarrow r - r^3(1+\sin^2 2\theta) < 0$

$$\Rightarrow 1 < r^2(1+\sin^2 2\theta)$$

$$\Rightarrow \frac{1}{\sqrt{1+\sin^2 2\theta}} < r$$

$\therefore$ all trajectories have a radially inward component on the circle $r_2 = 1+\varepsilon$

e) Since the region $1/\sqrt{2} - \varepsilon \le r \le 1+\varepsilon$ is a trapping region that contains no fixed points, the Poincare-Bendixon thm implies there is a periodic orbit inside the region.

$*$ $\dot{r} = 0 \Rightarrow r^2 = \frac{1}{1+\sin^2 2\theta} \Rightarrow \dot{\theta} = 1 + 4\,\frac{\cos\theta \sin^3\theta}{1+\sin^2 2\theta} = 0$

$\Rightarrow 1+\sin^2 2\theta + 2\sin 2\theta \sin^2\theta = 0$

$\Rightarrow 1 + (\sin 2\theta + \sin^2\theta)^2 - \sin^4\theta = 0$

$\Rightarrow \sin^2\theta\,(2\cos\theta + \sin\theta)^2 + \cos^2\theta(1-\sin^2\theta) = 0$

$\Rightarrow \sin\theta = \cos\theta = 0$ $\divideontimes$ $\therefore$ $\dot{\theta} \neq 0$

So no fixed points in the trapping region

7.3.4 $\dot{x} = x(1-4x^2-y^2) - \frac{1}{2}y(1+x)$

$\dot{y} = y(1-4x^2-y^2) + 2x(1+x)$

a) $A = \begin{bmatrix} 1-12x^2-y^2-\frac{1}{2}y & -2xy-\frac{1}{2} \\ -8xy+2+4x & 1-4x^2-3y^2 \end{bmatrix}$

at $(0,0)$, $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1/2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2+1=0 \Rightarrow \lambda^2-2\lambda+2=0$

$\Rightarrow \lambda = \dfrac{2\pm\sqrt{4-4(2)}}{2} = 1\pm i$

$\Rightarrow (0,0)$ is an u.s. spiral

b) $V=(1-4x^2-y^2)^2$ use $\dot{V}$ to show trajectories approach $4x^2+y^2=1$.

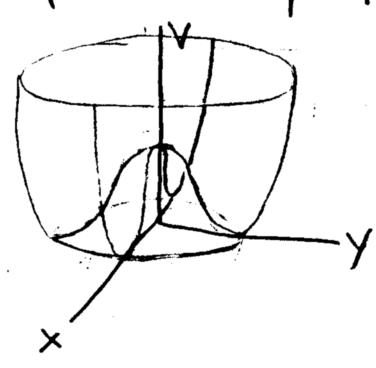
$\dot{V} = 2(1-4x^2-y^2)(-8x\dot{x}-2y\dot{y})$

$= 2(1-4x^2-y^2)(-8x^2+32x^4+8x^2y^2+\cancel{4xy}+\cancel{4x^2y}$
$\quad -2y^2+8x^2y^2+2y^4-\cancel{4xy}-\cancel{4x^2y})$

$= -4(1-4x^2-y^2)(4x^2+y^2-8x^2y^2-16x^4-y^4)$

$= -4(1-4x^2-y^2)^2(4x^2+y^2) < 0$ unless $x=y=0$ or $4x^2+y^2=1$

$\therefore$ V decreases along trajectories until it reaches the ellipse $4x^2+y^2=1$ (b/c $(0,0)$ is a max. of V).



So all trajectories approach the ellipse -- except $(0,0)$.

7.5.4 $\ddot{x} + \mu f(x)\dot{x} + x = 0$

$$f(x) = \begin{cases} -1 & |x| < 1 \\ 1 & |x| \geq 1 \end{cases}$$

a)

$$\int f(x)\,dx = \begin{cases} \int -dx & |x| \leq 1 \\ \int dx & x \geq 1 \\ \int dx & x \leq -1 \end{cases}$$

$$F(x) = \int f(x)\,dx$$

$$F(x) = \begin{cases} -x & |x| \leq 1 \\ x + c & x \geq 1 \\ x + c & x \leq -1 \end{cases}$$

to get c want

$x + c = -x$ at $x = 1$

$\Rightarrow c = -2$

$x + c = -x$ at $x = -1$

$\Rightarrow c = 2$

So,

$$F(x) = \begin{cases} x+2, & x \le -1 \\ -x, & |x| \le 1 \\ x-2, & x \ge 1 \end{cases}$$

$$\ddot{x} + \mu \dot{x} f(x) = \frac{d}{dt}\left(\dot{x} + \mu F(x)\right)$$

$$w = \dot{x} + \mu F(x)$$

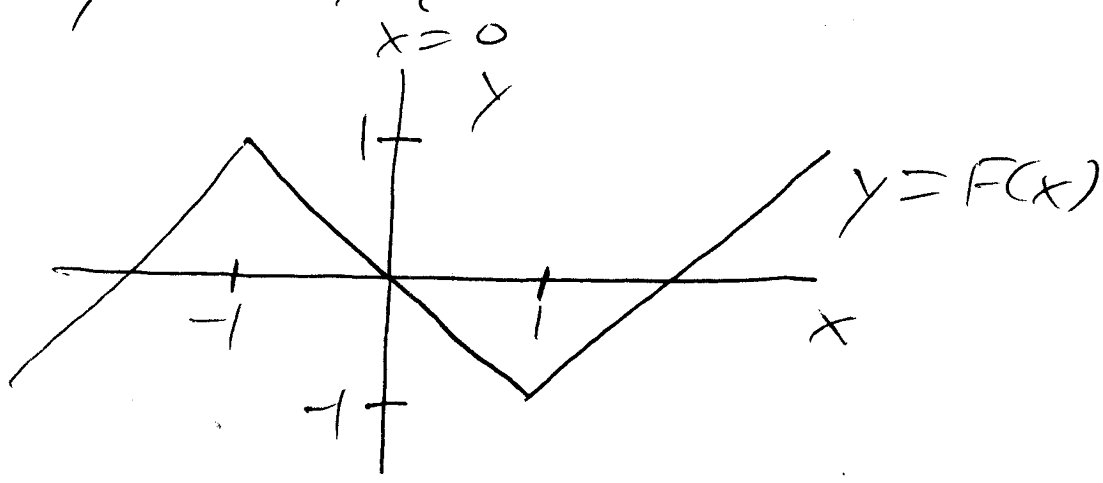
$$\Rightarrow \quad \dot{w} = -x$$

$$\begin{cases} \dot{x} = w - \mu F(x) \\ \dot{w} = -x \end{cases}$$

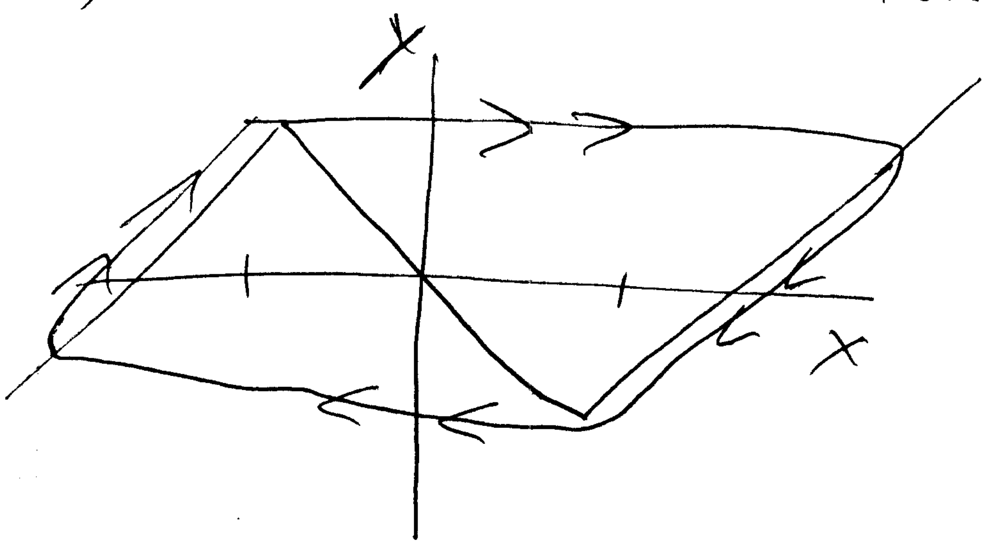
let $y = w/\mu$

$$\Rightarrow \begin{cases} \dot{x} = \mu\left(y - F(x)\right) \\ \dot{y} = -x/\mu \end{cases}$$

b)



c) The oscillation would look like



We can justify this picture by saying:

suppose that the initial condition is not too close to the piece-wise linear nullcline, i.e. suppose $y - F(x) \sim O(1)$. Then

$$\begin{cases} \dot{x} = \mu(y - F(x)) \\ \dot{y} = -x/\mu \end{cases}$$

implies that $|\dot{x}| \sim O(\mu) >> 1$

and $|\dot{y}| \sim O(\mu^{-1}) << 1$

hence, the velocity is enormous in the horizontal direction and tiny in the vertical direction, so trajectories move practically horizontally.

If the initial condition is above the nullcline, then $y - F(x) > 0$ and $\dot{x} > 0$; thus the trajectories moves sideways toward the nullcline. If the initial condition is below the nullcline, then $y - F(x) < 0$ and $\dot{x} < 0$. However, once the trajectory gets so close that $y - F(x) \sim O(\mu^{-2})$, then $\dot{x}$ and $\dot{y}$ become comparable and the trajectory crosses the nullcline vertically.

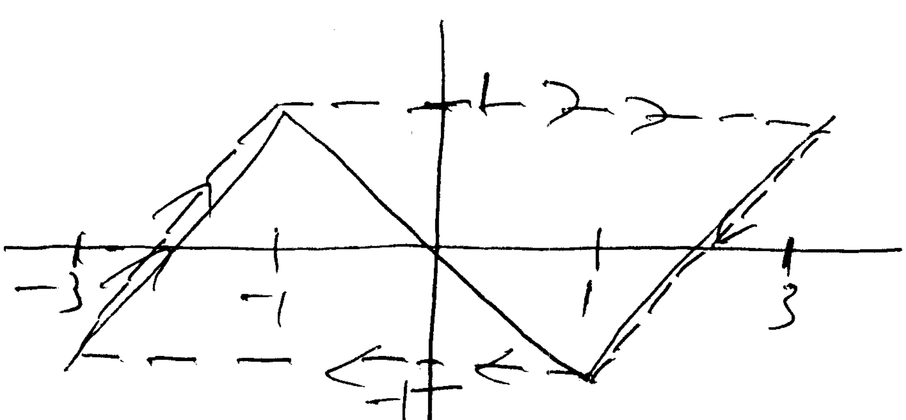
d) The period of the oscillations is the integral of the two linear function $x - 2$ and $x + 2$.

[redacted]

This is true since

$y \approx F(x)$ on the slow branches of the nullcline

To find the limits of integration



Find x when

$x - 2 = 1$

$\Rightarrow x = 3$

$x + 2 = -1$

$\Rightarrow x = -3$

Next,

$$y \approx F(x)$$

$$\dot{y} \approx F'(x)\,\frac{dx}{dt} = (1)\,\frac{dx}{dt}$$

$$\dot{y} = -x/u$$

$$\Rightarrow \quad -x/u = \frac{dx}{dt}$$

$$\Rightarrow \quad dt \approx -\frac{u}{x}\,dx$$

$$T \approx 2\int_3^1 -\frac{u}{x}\,dx = 2\int_1^3 \frac{u}{x}\,dx$$

$$= 2u\,[\ln(x)]_1^3$$

$$= 2u\ln(3)$$