Homework Solution: $L(x,\dot{x}) = x^2\dot{x}^4 - \sin x$

Find: (i) Lagranges 2nd order equation

(ii) generalized momentup p

(iii) the energy (and show its const along sol'ns)

(iv) Hamilton's First order eqn.

Soln (i): $\frac{d}{dt}\frac{\partial}{\partial \dot{x}} L - \frac{\partial L}{\partial x} = 0$

$$\frac{d}{dt} 4x^2\dot{x}^3 - 2x\dot{x}^4 + \cos x = 0$$

$$\cancel{8}\,6\dot{x}^4 + 12x^2\dot{x}^2\ddot{x} - \cancel{2x\dot{x}^4} + \cos x = 0$$

$$\boxed{12x^2\dot{x}^2\ddot{x} + 6\dot{x}^4 + \cos x = 0}$$

Soln (ii) $p = \frac{\partial L}{\partial \dot{x}} = 4x^2\dot{x}^3$

Soln (iii):

$$E = \dot{x}\, p - L$$

$$= 4x^2\dot{x}^4 - x^2\dot{x}^4 + \sin x$$

$$= 3x^2\dot{x}^4 + \sin x$$

$$\frac{dE}{dt} = \frac{d}{dt}\left\{(3x^2\dot{x}^4) - \sin x\right\}$$

$$= 6x\dot{x}^5 + 12x^2\dot{x}^3\ddot{x} + \cos x\, \dot{x}$$

$$= \dot{x}\,\underbrace{\left\{12x^2\dot{x}^2\ddot{x} + 6x\dot{x}^4 + \cos x\right\}}_{\text{Equation} = 0}$$

$$= 0 \checkmark$$

Soln (iv): $H(x,p) = E(x, \underbrace{f(x,p)}_{\dot{x} = f(x,p)})$

$$p = 4x^2 \dot{x}^3$$

$$\dot{x} = \sqrt[3]{\frac{p}{4x^2}} = f(x,p)$$

$$H(x,p) = 3x^2 \dot{x}^4 + \sin x$$

$$= 3x^2 \left(\frac{p}{4x^2}\right)^{4/3} + \sin x$$

$$\dot{x} = H_p = \cancel{3} \cdot \frac{4}{\cancel{3}} \cancel{x^2} \left(\frac{p}{4x^2}\right)^{1/3} \frac{1}{\cancel{4x^2}}$$

$$\dot{p} = -H_x = 6x \left(\frac{p}{4x^2}\right)^{4/3} + \cancel{3} \cdot \frac{\overset{2}{\cancel{4}}}{\cancel{3}} \cancel{x^2} \left(\frac{p}{4x^2}\right)^{1/3} \cdot \left(-\frac{1}{\cancel{2}x^{\cancel{3}}}\right) + \cos x$$

OR

$$\dot{x} = \left(\frac{p}{4x^2}\right)^{1/3}$$

$$\dot{p} = 6x\left(\frac{p}{4x^2}\right)^{4/3} - \frac{2}{x}\left(\frac{p}{4x^2}\right)^{1/3} + \omega x.$$