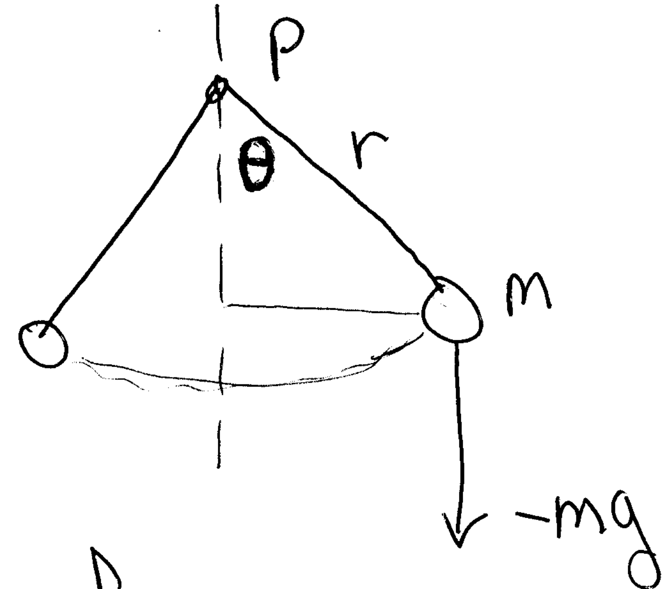
Solution (§5) | 119A W02 Temple 

Consider the nonlinear pendulum:

"mass $m$ at end of rigid massless rod swinging in plane at frictionless pivot $p$"




(i) Derive the 2nd order equations for the nonlinear pendulum using Lagranges Principles

(ii) Non-dimensionalize & show the nonlinear eqn's are equivalent to $\ddot{\theta} + \sin\theta = 0$. Linearize @ $\theta = 0$ to obtain Harmonic osc.

(iii) Write Hamilton's 1st order eqn's

Find the Hamiltonian

$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L$ &

(2)

Soln (i): $KE = \frac{1}{2}(\dot{x}^2+\dot{y}^2)$ (in plane)

$$\vec{F} = (0, -mg) = -\nabla(mgy)$$

$$L = KE - PE = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + mgy$$

$$x = r\sin\theta \qquad \dot{x} = -r\cos\theta\,\dot{\theta}$$

$$y = -r\cos\theta \qquad \dot{y} = r\sin\theta\,\dot{\theta}$$

$$L = \frac{1}{2}\left\{(r\cos\theta)^2\dot{\theta}^2 + (r\sin\theta)^2\dot{\theta}^2\right\} - mgr\cos\theta$$

$$= \frac{1}{2}r^2\dot{\theta}^2 - mgr\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = \frac{d}{dt}r^2\dot{\theta} - mgr\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{mg}{r}\sin\theta = 0.}$$

Soln (ii): write $\tau = \frac{t}{T}$ so

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\theta'$$

$$\ddot{\theta} = \frac{d}{dt}\frac{1}{T}\frac{d\theta}{d\tau} = \frac{1}{T}\frac{d\tau}{dt}\frac{d^2}{d\tau^2}\theta = \frac{1}{T^2}\theta''$$

$$\therefore \quad \frac{1}{T^2}\theta'' + \frac{mg}{r}\sin\theta = 0$$

$$\theta'' + \frac{mgT^2}{r}\sin\theta = 0$$

choose $\frac{mgT^2}{r} = 1 \iff T = \sqrt{\frac{r}{mg}}$

$$\boxed{\theta'' + \sin\theta = 0}$$

Soln (iii):

$$L = \frac{1}{2} r^2 \dot{\theta}^2 - mgr\cos\theta$$

$$E = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \dot{\theta}\, r^2 \dot{\theta} - \frac{1}{2} r^2 \dot{\theta}^2 + mgr\cos\theta$$

$$= \frac{1}{2} r^2 \dot{\theta}^2 + mgr\cos\theta$$

$$p = \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta} \qquad \dot{\theta} = \frac{p}{r^2} = f(x, p)$$

$$x = \theta, \ \dot{x} = \dot{\theta}$$

$$E(x, f(x,p)) = \frac{1}{2}\frac{p^2}{r^2} + mgr\cos x = H(x, p)$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{r^2}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = + mgr\sin x$$