3.4.5

$$\boxed{\dot{x} = r - 3x^2} \longrightarrow$$

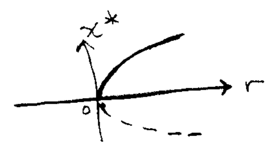
1. $\frac{1}{3}\frac{dx}{dt} = \frac{r}{3} - x^2$

let $R = \frac{r}{3}$, $\tau = 3t$

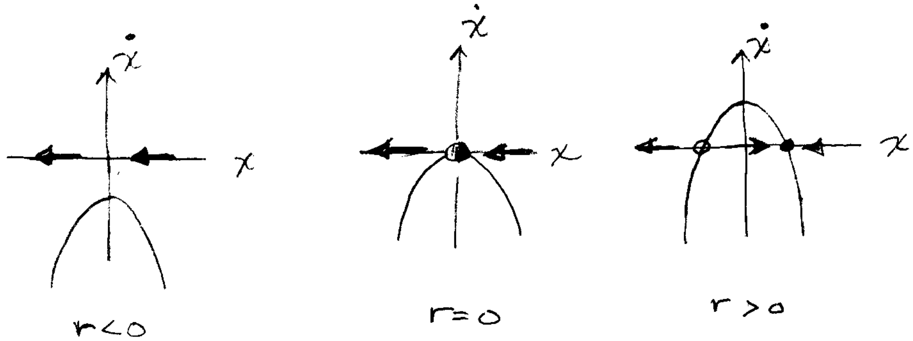
$$\frac{dx}{d\tau} = R - x^2$$

ie NORMAL FORM FOR SNB

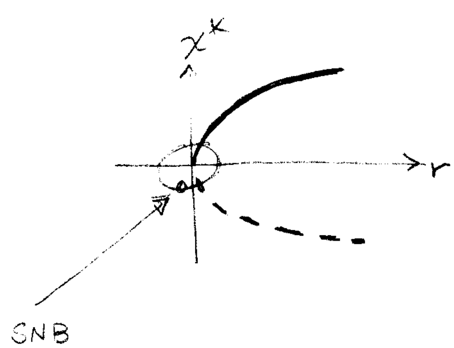
SNB at $x^* = 0$, $R = 0$
$r = 0$



2. graphically.



SNB OCCURS AT
$x^* = 0$, $r = 0$



ANALYTICALLY

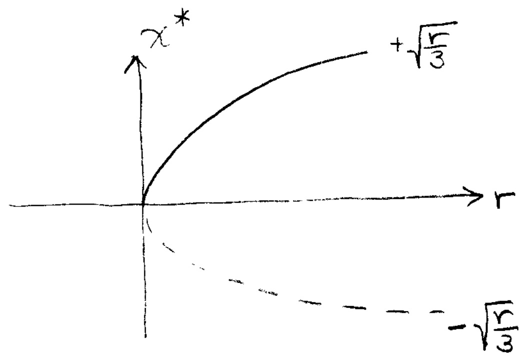
③ $\dot{x} = r - 3x^2 = f(x;r)$

F.P.s $\quad r - 3x^{*2} = 0 \Rightarrow x^* = \pm\sqrt{\frac{r}{3}}$ *exist only for* $r \geq 0$

Stability $\quad \frac{\partial f}{\partial x} = -6x \Rightarrow \frac{\partial f}{\partial x}\left(+\sqrt{\frac{r}{3}}\right) < 0, \quad x^* = \sqrt{\frac{r}{3}}$ STABLE

$\frac{\partial f}{\partial x}\left(-\sqrt{\frac{r}{3}}\right) > 0, \quad x = -\sqrt{\frac{r}{3}}$ UNSTABLE

$r > 0$



SOLUTION STRUCTURE SHOWS THAT THERE IS A SNB AT $x^* = 0, r = 0$.

NOTE ALSO

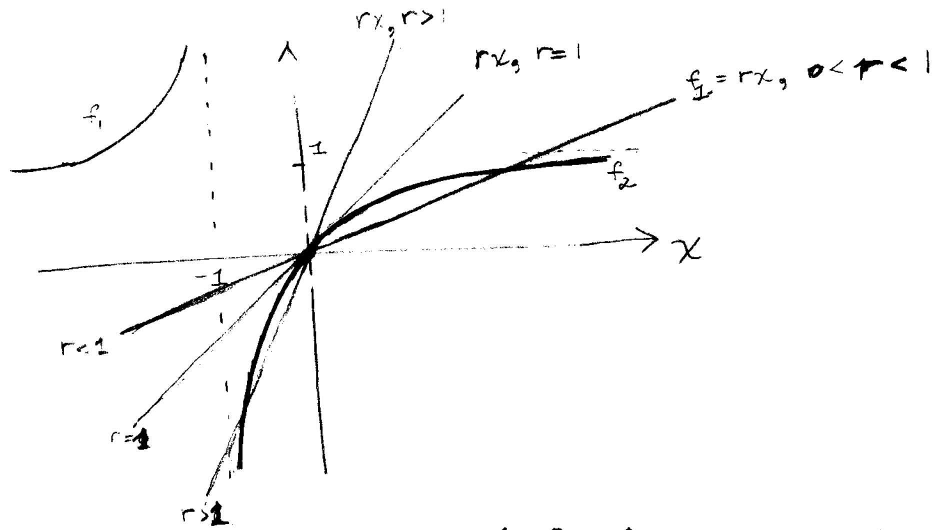
$$\left.\begin{aligned} &f(x^*=0, r=0) = 0 \\ &\frac{\partial f}{\partial x}(x^*=0, r=0) = 0 \\ &\frac{\partial^2 f}{\partial x^2}(x^*=0, r=0) = -3 \neq 0! \\ &\frac{\partial f}{\partial r}(x^*=0, r=0) = 1 \neq 0! \end{aligned}\right\} \Rightarrow \text{SNB at } \begin{matrix} x^*=0 \\ r=0. \end{matrix}$$

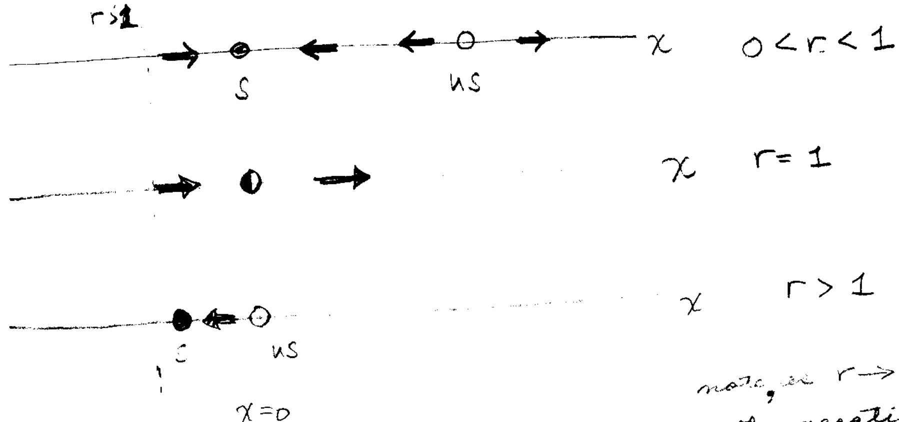
3.4.6

$$\boxed{\dot{x} = rx - \frac{x}{1+x}}$$

* we'll consider $r>0$ only. but see page 5

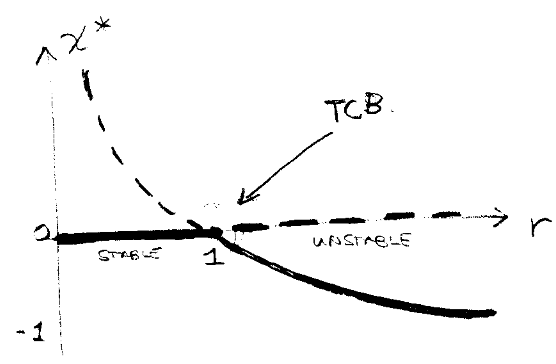
① graphically. $\dot{x} = \underbrace{rx}_{f_1(x)} - \underbrace{\frac{x}{1+x}}_{f_2(x)}$





note, as $r \to \infty$, the negative $x^* \to -1$.

TRANSCRITICAL BIFURCATION OCCURS AT $x^*=0$, $r=1$



(2) ANALYTICALLY

$$\dot{x} = rx - \frac{x}{1+x}$$

F.P.S $\quad \dot{x} = 0 \Rightarrow rx^* - \frac{x^*}{1+x^*} = 0$

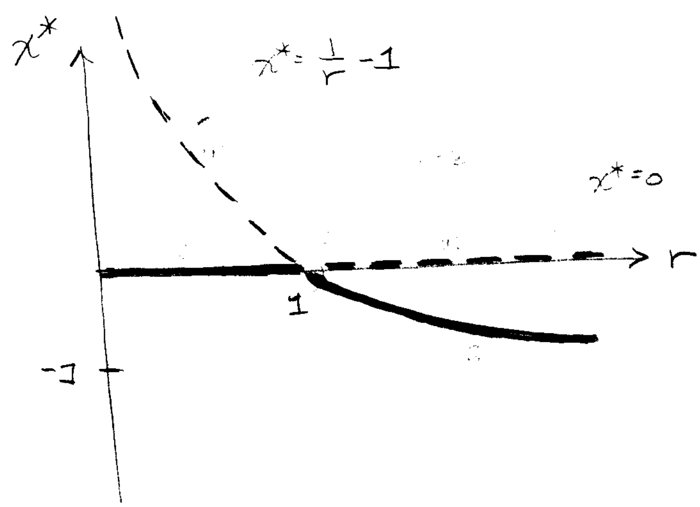
$$x^* = 0 \ , \quad r = \frac{1}{1+x^*} \quad \text{or} \quad x^* = \frac{1}{r} - 1 = \frac{1-r}{r}$$

STABILITY $\quad f'(x) = r - \frac{(1+x) - x}{(1+x)^2} = r - \frac{1}{(1+x)^2}$

$$f'(x^*=0) = r - 1 \begin{cases} >0, & r>1 \\ <0, & r<1 \end{cases} \Rightarrow \begin{array}{l} x^*=0 \text{ UNSTABLE FOR } r>1. \\ x^*=0 \text{ STABLE FOR } r<1. \end{array}$$

$$f'\left(x^* = \frac{1-r}{r}\right) = r - \frac{1}{\left(1 + \frac{1}{r} - 1\right)^2}$$

$$= r - r^2 = r(1-r) = \begin{cases} >0, & 0<r<1 \\ <0, & r>1 \end{cases}$$

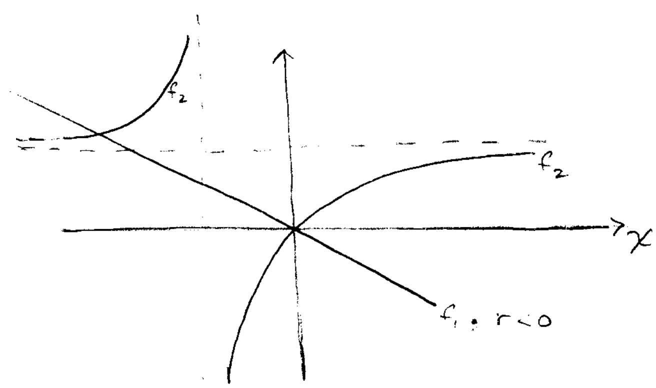


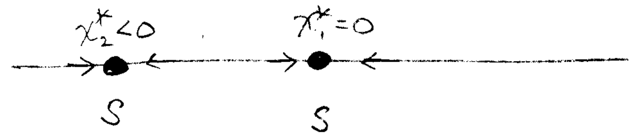
* NOTE $\quad f(x^*=0, r=1) = 0$, $\frac{\partial f}{\partial x}(x^*=0, r=1) = 0$, $\frac{\partial f}{\partial r}(x^*=0, r=1) = 0$

$\frac{\partial^2 f}{\partial x^2}(x^*=0, r=1) = \left.\frac{2}{(1+x^*)^3}\right|_{x^*=0} \neq 0$, $\frac{\partial f}{\partial r \partial x} = 1$ $\Bigg\} \Rightarrow$ TCB @ $x^*=0$, $r=1$

NOTE

$$\dot{x} = \underbrace{r x}_{f_1} - \underbrace{\frac{x}{1+x}}_{f_2} \qquad \text{for} \quad r < 0$$





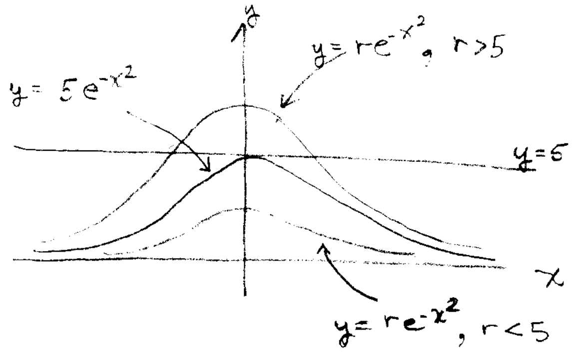
by inspection of f_1 and f_2, both $x_1^* = 0$ and $x_2^* < 0$ are BOTH stable. Can you figure out how this is possible?

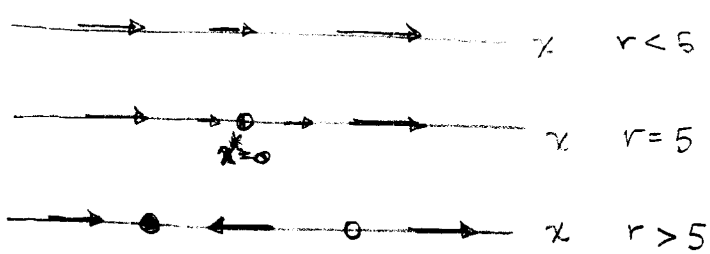
Also, what type of bifurcation is happening at $r = 0$? i.e. as r decreases to zero, the number of fixed pts the system has goes from 2 to 1.

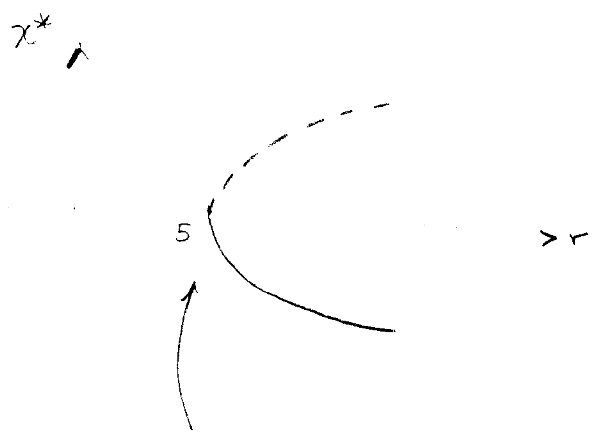
3.4.

$$\boxed{\dot{x} = 5 - re^{-x^2}}$$

Graphically







SNB at $x^* = 0$, $r_c = 5$

(c) analytically

F.P.s $\quad \dot{x} = 0 \qquad re^{-x^{*2}} = 5$

$$-x^2 = \ln \frac{5}{r} \Rightarrow x^* = \pm\sqrt{\ln \frac{r}{5}}$$

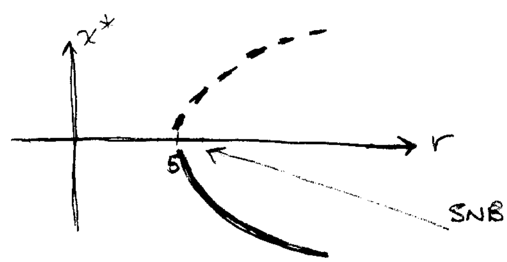
only exist for $r \geqslant 5$

STABILITY $\quad f'(x) = +2xre^{-x^2}$ $\qquad$ note $re^{-x^{*2}} = 5$

$$f'\left(\pm\sqrt{\ln \frac{r}{5}}\right) = \pm 10\sqrt{\ln \frac{r}{5}}$$

$\Rightarrow \quad x^* = +\sqrt{\ln \frac{r}{5}}$ is UNSTABLE $\qquad r > 5$

$\quad\;\; x^* = -\sqrt{\ln \frac{r}{5}}$ is STABLE



- Near at $x^* = 0$ ← F.P. at BIF. PT.

$$\frac{dx}{dt} = 5 - re^{-x^2}$$

$$\frac{dx}{dt} = \frac{5}{r} - e^{-x^2} = \frac{5}{r} - \left(1 - x^2 + O(x^4)\right)$$

$$= \left(\frac{5}{r} - 1\right) + x^2 + O(x^4)$$

let $\tau = 5t$, $R = \frac{5}{r} - 1$

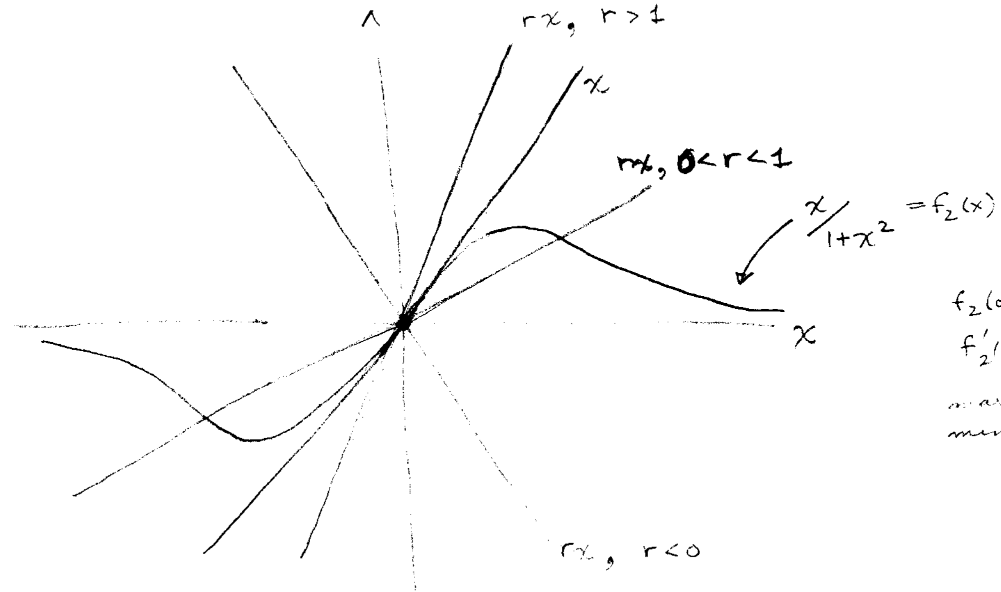
$$\frac{dx}{d\tau} = R + x^2$$ NORMAL FORM FOR SNB

BIF AT $R = 0 \Rightarrow \frac{5}{r} - 1 = 0$ or $r = 5$

AND $x^* = 0$.

$$\boxed{\dot{x} = rx - \frac{x}{1+x^2}}$$

① graphically

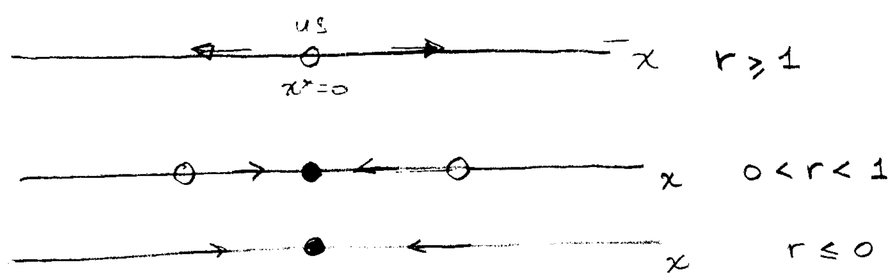


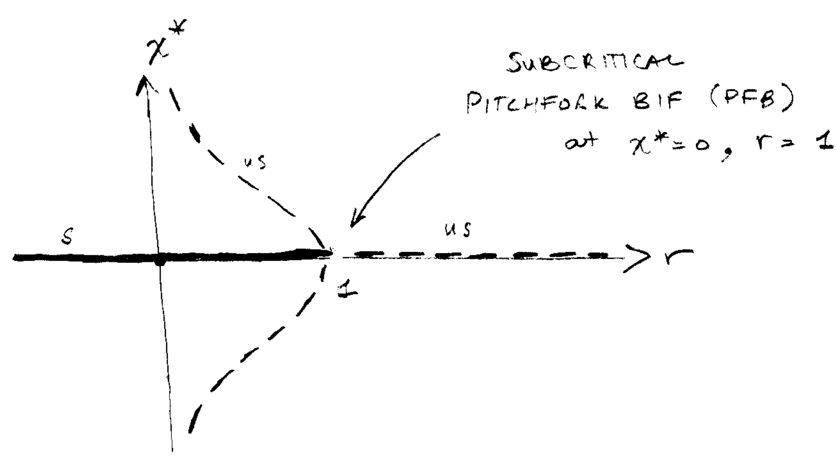
$f_2(0)=0$

$f_2'(0)=1$

max at $x=+1$

min at $x=-1$





* Q: what kind of bifurcation happens at $r=0$?!

2) analytically

F.P. $\dot{x} = 0$, $\quad rx - \frac{x}{1+x^2} = 0 \quad \Rightarrow \quad x^* = 0, \quad r = \frac{1}{1+x^{*2}}$

or $x^* = \pm\sqrt{\frac{1}{r} - 1}$

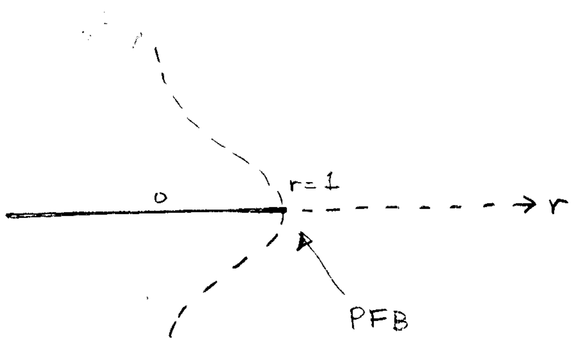
exists for $0 < r \leq 1$.

STABILITY $\quad f'(x) = r - \frac{1-x^2}{(1+x^2)^2}$

$$f'(0) = r - 1 \begin{cases} <0, & r<1 \\ >0, & r>1 \end{cases} \Rightarrow \begin{array}{l} x^*=0 \text{ STABLE FOR } r<1. \\ x^*=0 \text{ UNSTABLE FOR } r>0. \end{array}$$

$$f'\left(\pm\sqrt{\tfrac{1}{r}-1}\right) = r - \frac{1-\left(\frac{1}{r}-1\right)}{\left(1+\left(\frac{1}{r}-1\right)\right)^2} = 2r > 0 \text{ for } 0<r\leq 1$$

$\Rightarrow \quad x^* = \pm\sqrt{\frac{1}{r} - 1}$ are UNSTABLE when they exist.



NOTE: near $x=0$, $\dot{x} = rx - \left(x - x^3 + O(x^5)\right)$

$\approx (r-1)x + x^3$

ie, the NORMAL FORM of a SUBCRITICAL PITCHFORK BIF.

(BIF. PT at $r-1=0$, $x^*=0$)
$r=1$

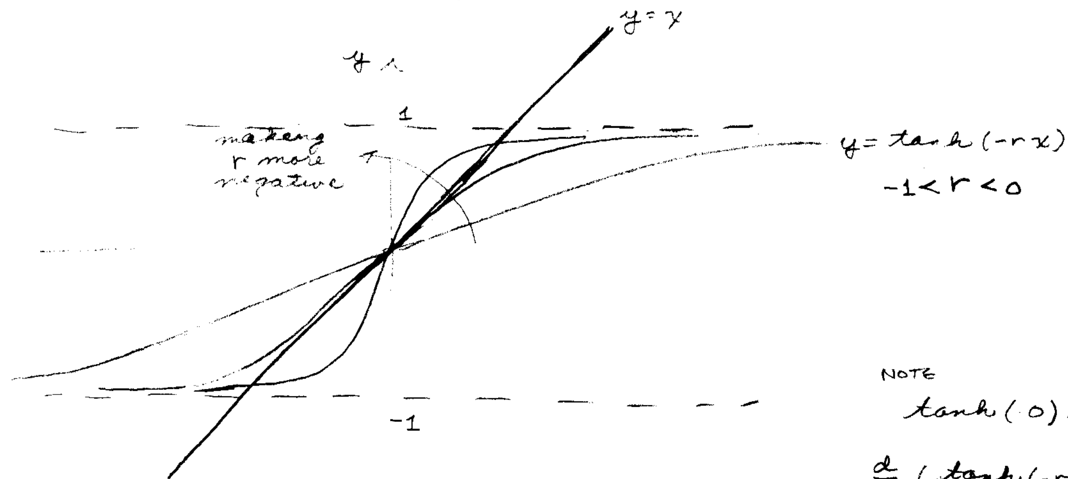
3.4.7

$$\boxed{\dot{x} = x + \tanh(rx) = f(x;r)}$$

⇒ graphically

$$\dot{x} = x - (-\tanh(rx))$$

$$= x - (\tanh(-rx))$$

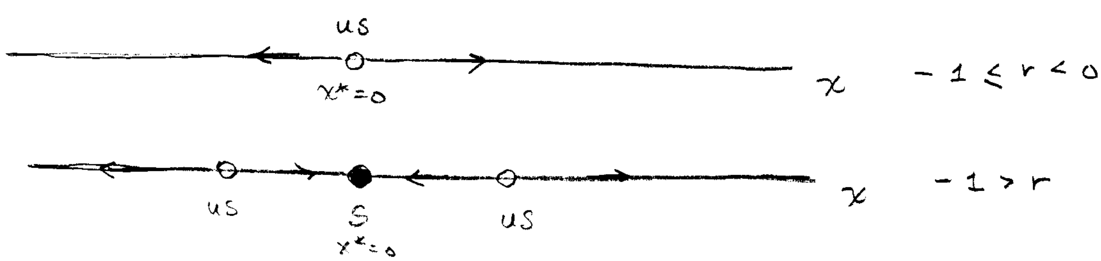


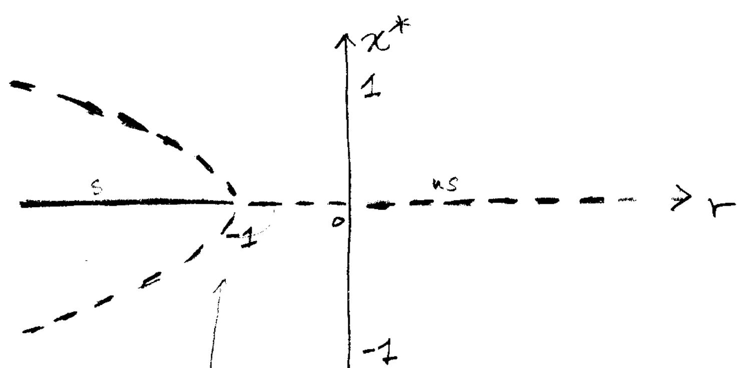
NOTE

$\tanh(0) = 0$

$\frac{d}{dx}(\tanh(-rx))\Big|_{x=0} = -r$

∘ note that when $r \geq 0$, only $x^* = 0$ is a fixed pt and it is UNSTABLE





SUBCRITICAL PITCHFORK BIF.

(c) analytically

F.P. $\dot{x}=0 \Rightarrow x = -\tanh(rx)$

$x^*=0$ always a F.P., but we can't solve (analytically) for any other F.P.s.

HOWEVER, we can try to indentify BIF. PTS. along the $x^*=0$ solution branch.

<u>STABILITY</u>

$$\frac{\partial f}{\partial x} = 1 + r\,\mathrm{sech}^2(rx)$$

$$\frac{\partial f}{\partial x}(x^*=0) = 1 + r \quad \Rightarrow \quad x^*=0 \text{ is STABLE for } r<-1 \text{ and UNSTABLE for } r>-1$$

also, there is a BIFURCATION PT at $r=r_c=-1$, $x^*=0$

<u>WHAT KIND OF BIF OCCURS</u>?

expand $f(x,r)$ about $x^*=0$, $r=-1=r_c$

$$f(x^*,r_c)=0 \qquad \frac{\partial f}{\partial x}(x^*,r_c)=0$$

$$\frac{\partial f}{\partial r}(x^*,r_c) = x^*\,\mathrm{sech}^2(r_c x^*) = 0$$

$$\frac{\partial^2 f}{\partial x^2}(x^*,r_c) = -2r_c^2\tanh(r_c x^*)\,\mathrm{sech}^2(r_c x^*) = 0$$

$$\frac{\partial^2 f}{\partial x\partial r}(x^*,r_c) = \mathrm{sech}^2(r_c x^*) - 2r_c x^*\tanh(r_c x^*)\,\mathrm{sech}^2(r_c x^*) = 1 \neq 0$$

$$\frac{\partial^3 f}{\partial x^3}(x^*,r_c) = r_c^3\, 2\,\mathrm{sech}^2(r_c x^*)\left(-1+3\tanh^2(r_c x^*)\right) = +2 \neq 0$$

$$\Rightarrow \quad \dot{x} \simeq (r+1)x + \frac{2}{3}x^3$$

comparing to normal forms $\rightarrow$ SUBCRITICAL PFB at $x^*=0$, $r+1=0$

3.4.10

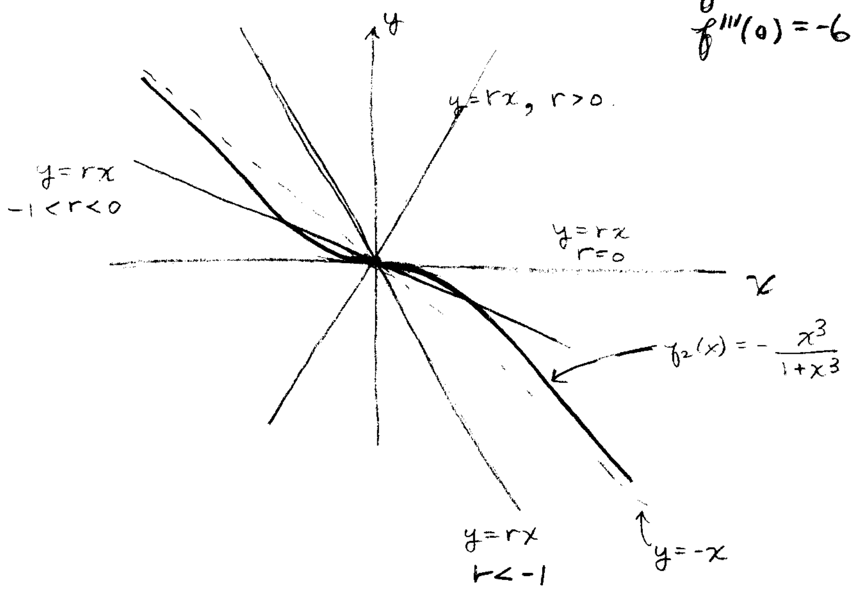
$$\boxed{\dot{x} = rx + \frac{x^3}{1+x^2} = f(x,r)}$$

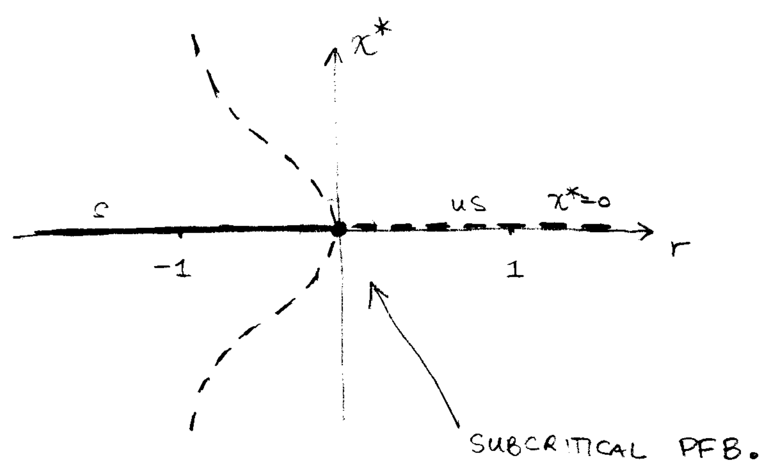
① graphically $\quad \underbrace{\dot{x} = rx}_{f_1(x)} - \underbrace{\left(-\frac{x^3}{1+x^2}\right)}_{f_2(x)}$

$f(0)=0$
$f'(0)=0$
$f''(0)=0$
$f'''(0)=-6$

$\frac{-x^3}{1+x^2} = -x + \frac{x}{1+x^2}$

i.e. asymptotes to $\underline{y=-x}$.





(1) analytically

F.P. $\quad rx + \dfrac{x^3}{1+x^2} = 0 \quad \Rightarrow \quad x^* = 0$ always exists. $\quad , \quad r = -\dfrac{x^{*2}}{1+x^{*2}}$

or

$$r(1+x^{*2}) = -x^{*2}$$

$$r = (r-1)x^{*2}$$

$$x^* = \pm\sqrt{\frac{r}{r-1}}$$

exist if $-1 < r \leq 0$

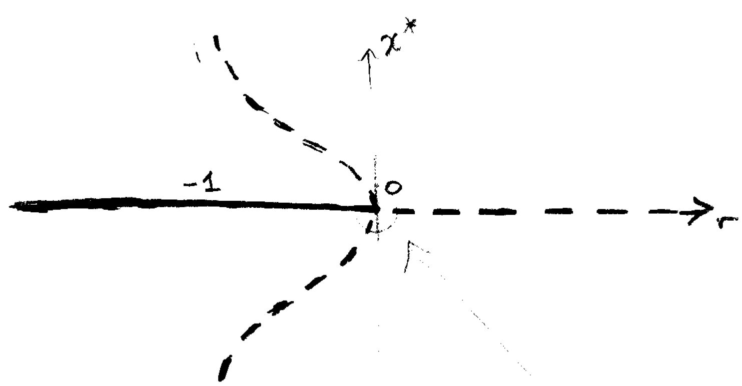
STABILITY

$$f'(x) = r + \frac{x^2(3+x^2)}{(1+x^2)^2}$$

$f'(0) = r \quad \Rightarrow \quad x^* = 0$ is STABLE for $r < 0$

and UNSTABLE for $r > 0$.

$f'\left(\pm\sqrt{\frac{r}{r-1}}\right) > 0$ for $-1 < r \leq 0$



SUBCRITICAL
PFB.
@ $r = 0,\ x^* = 0$

NOTE at $x^* = 0$, $\quad \dot{x} \simeq rx + x^3$

ie NORMAL FORM FOR A SUBCRITICAL PFB AT $x^* = 0,\ r = 0$.

IMPERFECT TRANSCRITICAL BIFURCATION (TCB)

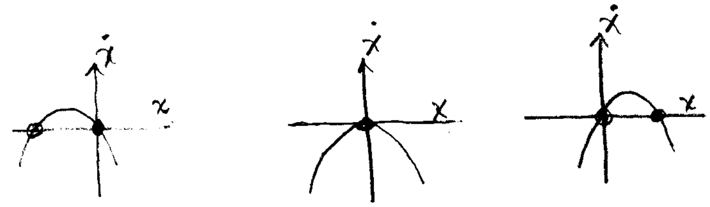
$$\dot{x} = h + rx - x^2$$

a)

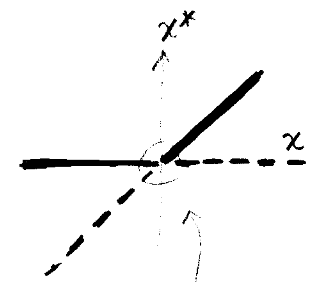
EQU'N | PHASE PORTRAITS | BIF. DIAGRAMS.

$h = 0$

$\dot{x} = rx - x^2$



$r<0$ $r=0$ $r>0$

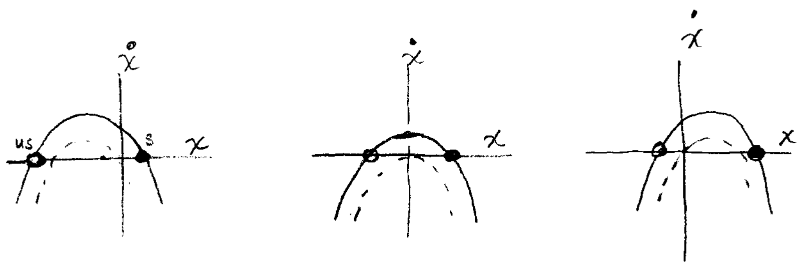


TCB.
at $x^*=0, r=0$

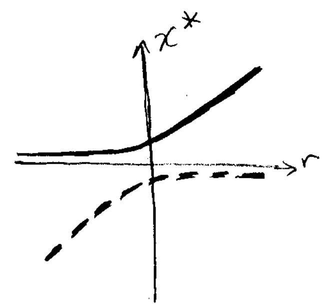
$h > 0$

$\dot{x} = h + rx - x^2$

(shifts parabola up)



$r<0$ $r=0$

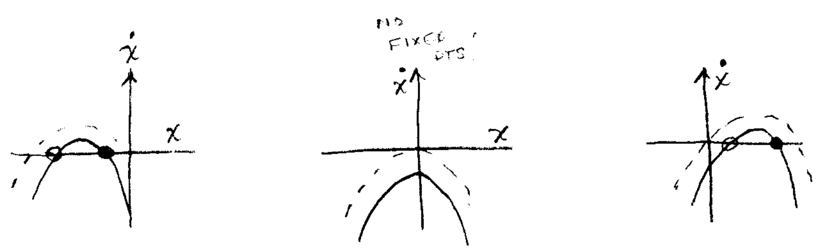


NO BIFS!

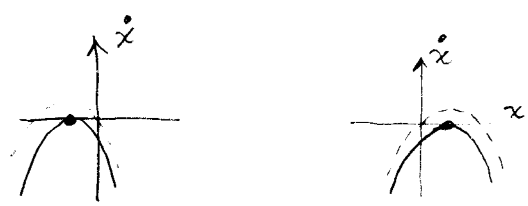
$h < 0$

$\dot{x} = h + rx - x^2$

(shifts parabola down)

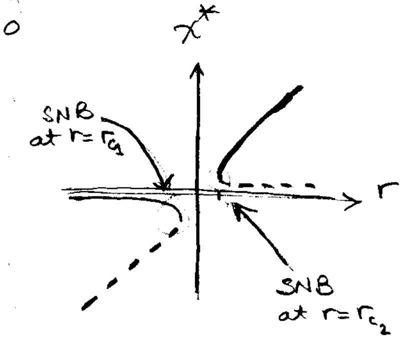


$r < r_{c_1} < 0$ $r = 0$ $r > r_{c_2} > 0$



SNB! at $r = r_{c_1} < 0$

SNB at $r = r_{c_2}$



region of no f.ps.

b) 2-PARAMETER BIF DIAGRAM for $\dot{x} = h + rx - x^2 = f(x; r, h)$

fixed pts when $h + rx^* - x^{*2} = 0$ $\qquad f(x^*; r, h) = 0$

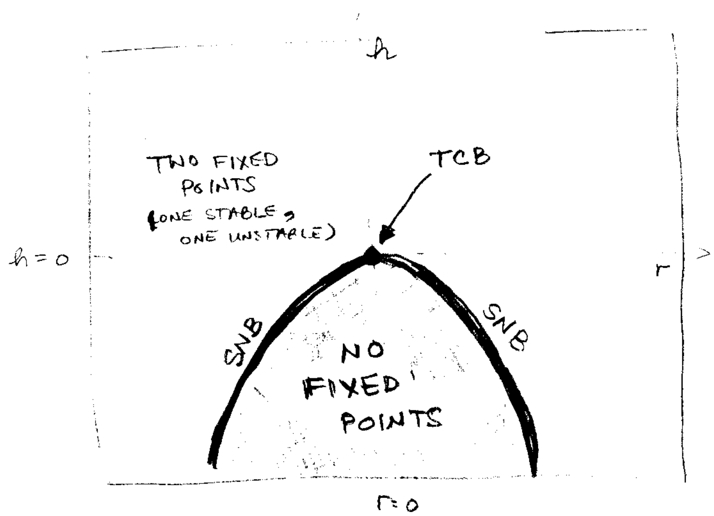
bifurcation when $r - 2x^* = 0$ $\qquad \frac{\partial f}{\partial x}(x^*; r, h) = 0$

$$\Rightarrow \quad x^* = \frac{r}{2}$$

$$h + r\left(\frac{r}{2}\right) - \left(\frac{r}{2}\right)^2 = 0$$

$$h = -\left(\frac{r}{2}\right)^2$$

curve where bifurcations occur.



3.6.4 what happens if you add small imperfections to a system that has a saddle-node bif.?

NORMAL FORM FOR SNB.

$$\dot{x} = r - x^2 + O(x^3)$$

ie. SNB occurs at $x^* = 0$, $r = 0$

add imperfections:

(i) $$\dot{x} = \varepsilon + r - x^2 = R - x^2$$

merely shift bifurcation pt.

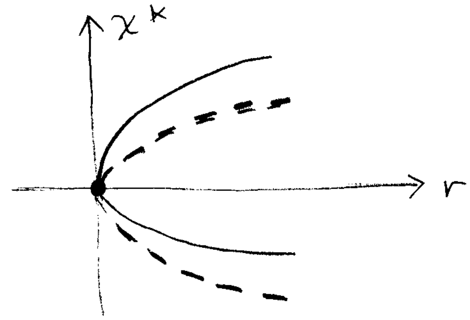
(ii) same for $\frac{dx}{dt} = r - (1+\varepsilon)x^2$

ie. $\frac{1}{1+\varepsilon}\frac{dx}{dt} = \left(\frac{r}{1+\varepsilon}\right) - x^2$

$\tau = (1+\varepsilon)$
$R = \frac{r}{1+\varepsilon}$ $\Rightarrow$ $\frac{dx}{d\tau} = R - x^2$

$\Rightarrow$ SADDLE-NODE BIFURCATIONS ARE ROBUST TO IMPERFECTIONS, I.E. THEY ARE STRUCTURALLY STABLE.

BONUS



3.4.12

"QUADFURCATION"

$$\dot{x} = (x^2 - \alpha_1 r)(x^2 - \alpha_2 r), \quad \alpha_2 > \alpha_1 > 0$$

fixed points

$$x^* = \pm\sqrt{\alpha_1 r}, \ \pm\sqrt{\alpha_2 r}$$

$\Rightarrow$ $r < 0$, no fixed pts exist.
$r > 0$, 4 fixed pts exist

GENERALIZATION

(i) $\dot{x} = (x^2 - \alpha_1 r)(x^2 - \alpha_2 r) \cdots (x^2 - \alpha_N r), \quad \alpha_N > \cdots > \alpha_2 > \alpha_1 > 0$

$r < 0$, no fixed points exist
$r > 0$, $2N$ fixed points exist.

ie. Saddle-node bifurcation-like

(ii) $\dot{x} = x(x^2 - \alpha_1 r)(x^2 - \alpha_2 r) \cdots (x^2 - \alpha_N r), \quad \alpha_N > \cdots > \alpha_2 > \alpha_1 > 0$

$r < 0$, one fixed point exists $x^* = 0$.
$r > 0$, $2N+1$ fixed point exist.

ie. pitchfork-like bifurcation