$\boxed{6.1.2}$

$$\begin{cases} \dot{x} = x - x^3 \\ \dot{y} = -y \end{cases}$$

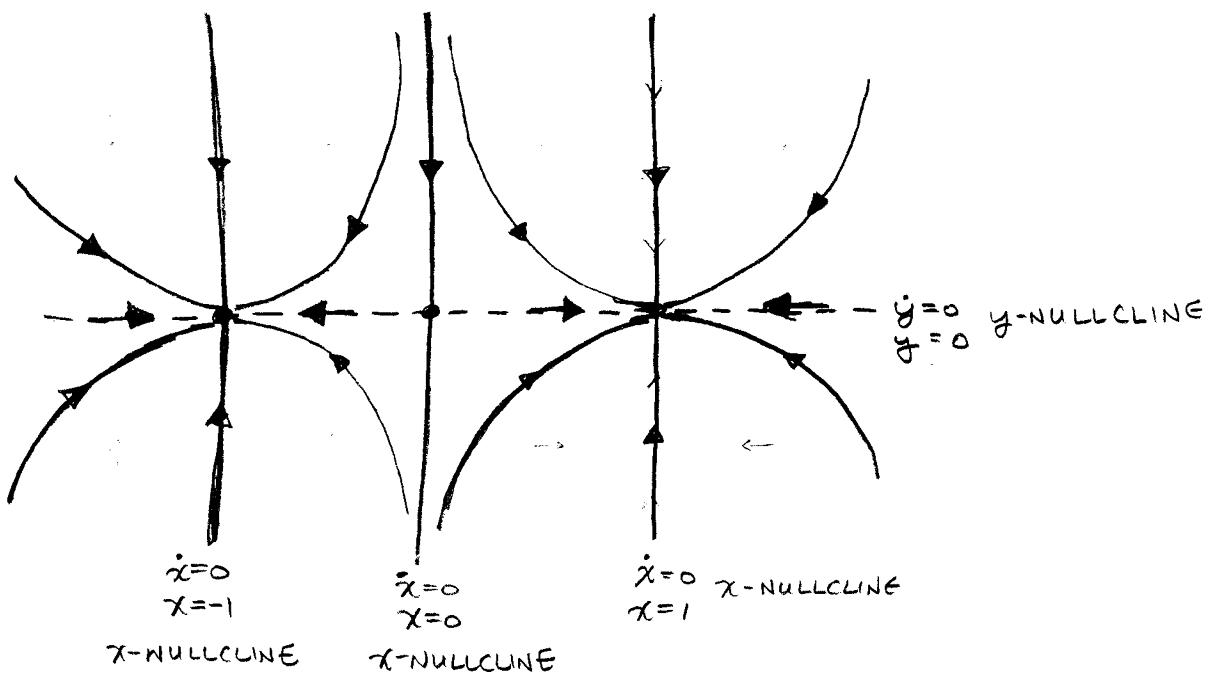
NULLCLINES $\quad \dot{x} = 0 \Rightarrow x - x^3 = 0 \quad x = 0,\ x = 1,\ x = -1$

$\dot{y} = 0 \Rightarrow y = 0$

FIXED POINTS $\quad \dot{x} = 0,\ \dot{y} = 0 \Rightarrow (x^*, y^*) = (0,0),\ (-1,0),\ (1,0)$

(0,0) SADDLE PT

(1, 0) STABLE
(-1, 0) NODES



* NOTE : system is UNCOUPLED. $\quad y(t) = y_0 e^{-t}$

$\dot{x} = x - x^3$

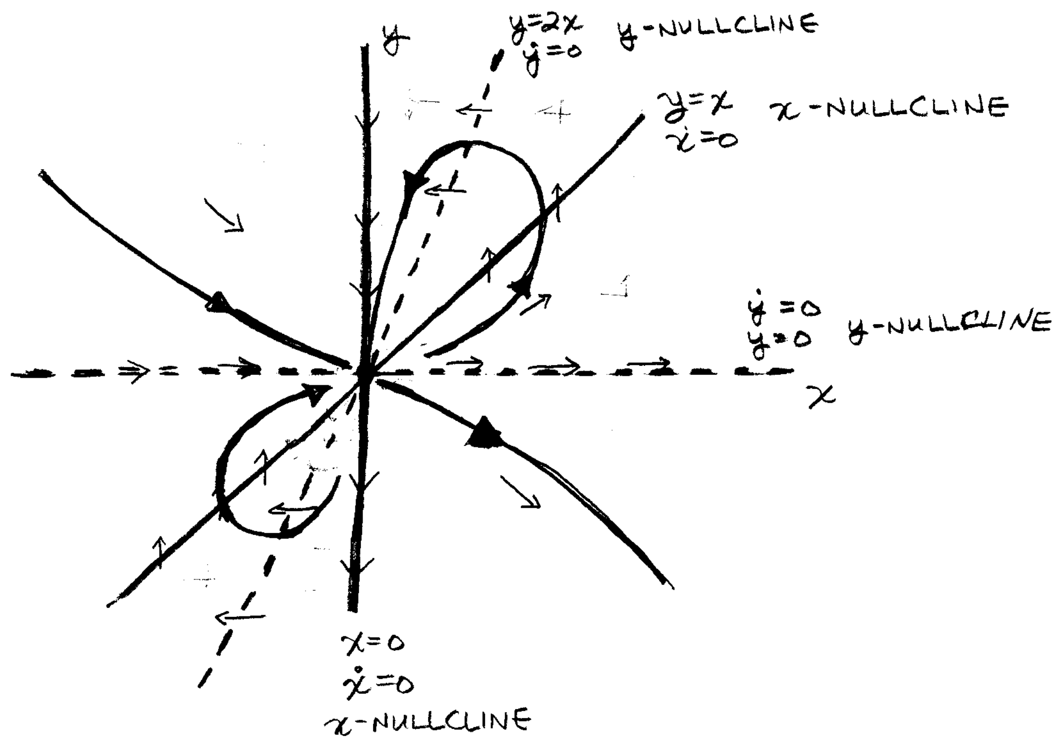
$\boxed{6.1.3}$

$$\begin{cases} \dot{x} = x(x-y) \\ \dot{y} = y(2x-y) \end{cases}$$

NULLCLINES $\quad \dot{x}=0 \Rightarrow x=0$ OR $y=x$

$\dot{y}=0 \Rightarrow y=0$ OR $y=2x$

FIXED POINTS $\quad \dot{x}=0,\ \dot{y}=0 \Rightarrow (x^*, y^*) = (0,0)$

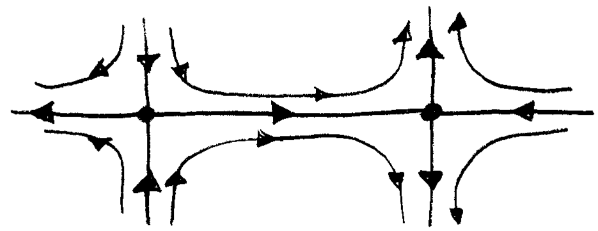


* NOTE: *linearize around* (0,0), $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

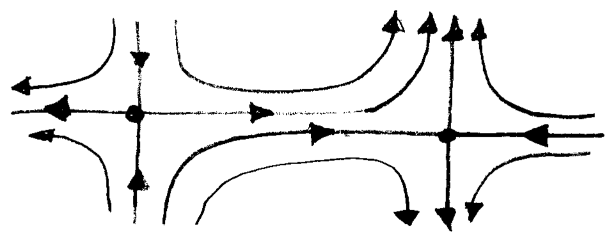
ie. LINEAR STABILITY ANALYSIS PREDICT ALL POINTS IN x-y PLANE ARE FIXED POINTS!

6.1.12

a)



b)



$\boxed{6.3.1}$

$$\begin{cases} \dot{x} = x - y = f(x, y) \\ \dot{y} = x^2 - 4 = g(x, y) \end{cases}$$

FIXED POINTS $\dot{x} = 0$, $\dot{y} = 0$

$$\begin{cases} x^* - y^* = 0 \\ x^{*2} - 4 = 0 \end{cases} \Rightarrow (x^*; y^*) = \begin{cases} (2, 2) \\ (-2, -2) \end{cases}$$

STABILITY

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_* = \begin{bmatrix} 1 & -1 \\ 2x^* & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2x^* & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 2x^* = 0$$

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 8x^*}$$

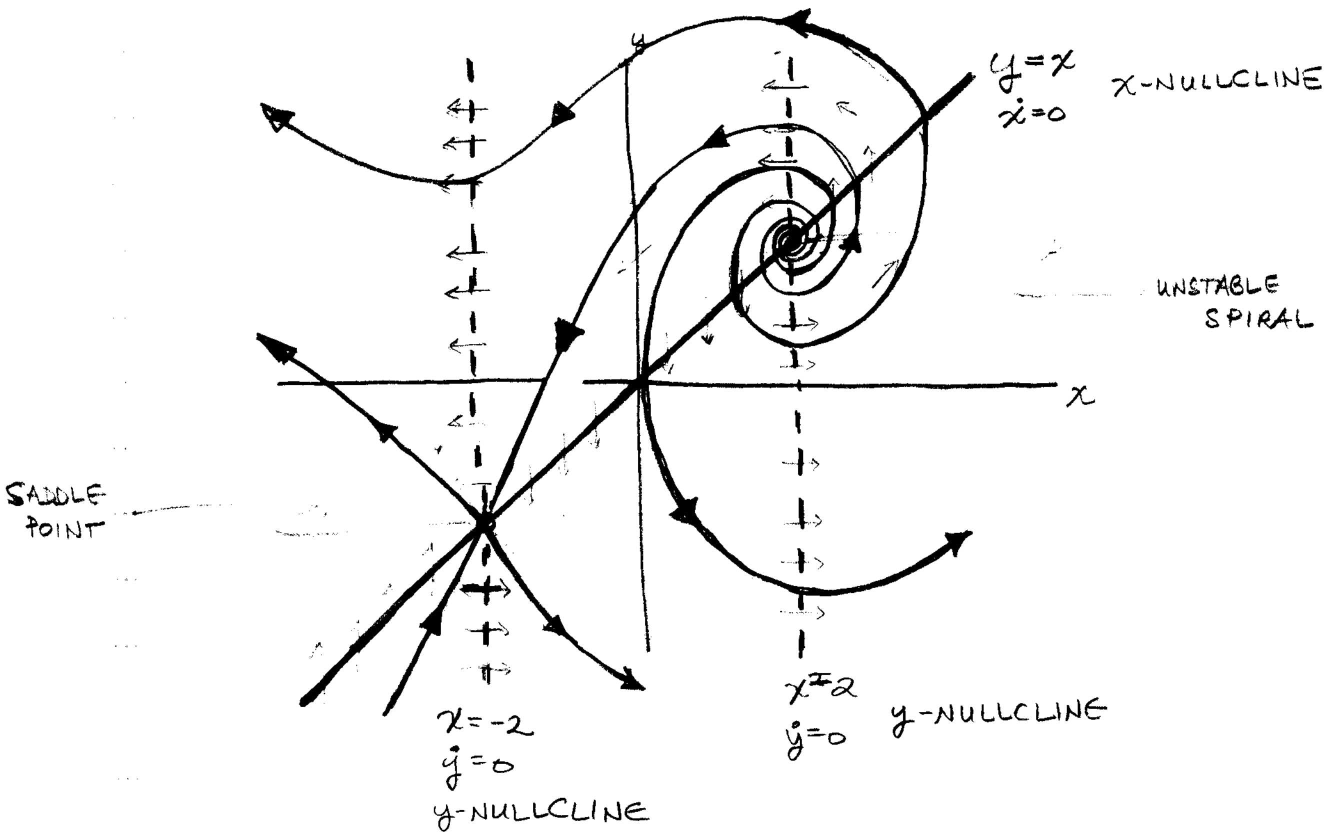
- for $(2,2)$, $\lambda = \frac{1}{2} \pm i\frac{\sqrt{15}}{2}$ $\Rightarrow$ $(2,2)$ IS AN UNSTABLE FOCUS

- for $(-2,-2)$, $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{17}$ $\Rightarrow$ $(-2,-2)$ IS A SADDLE POINT

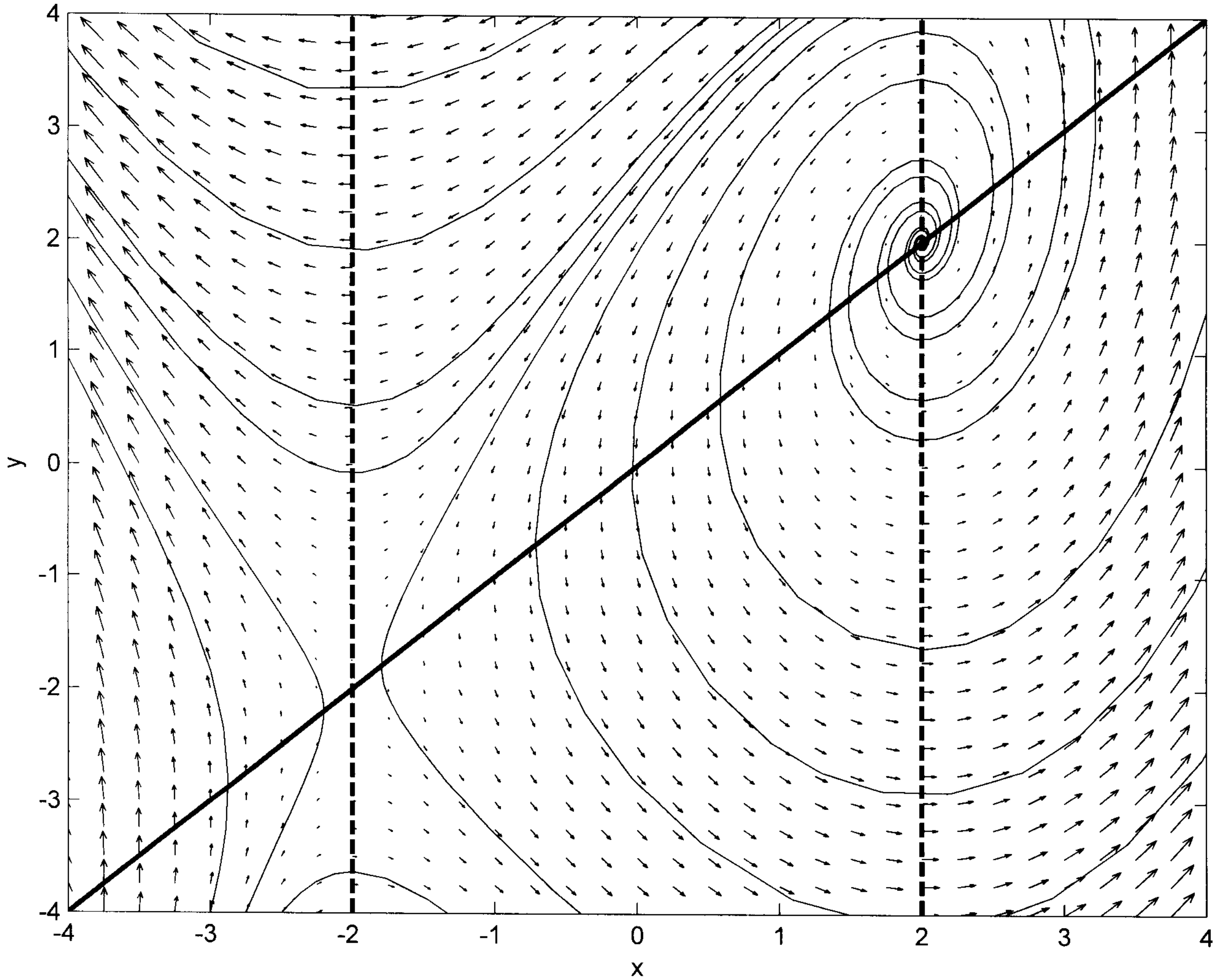
$$\left\{ \lambda_1 = \frac{1}{2}(1 + \sqrt{17}) , \bar{v}_1 = \begin{pmatrix} -\frac{1+\sqrt{17}}{8} \\ 1 \end{pmatrix} \right\}$$

$$\left\{ \lambda_2 = \frac{1}{2}(1 - \sqrt{17}) , \bar{v}_2 = \begin{pmatrix} -\frac{1-\sqrt{17}}{8} \\ 1 \end{pmatrix} \right\}$$

$-\frac{1+\sqrt{17}}{8} \sim -0.64$; $-\frac{1-\sqrt{17}}{8} = 0.39$


y
y=x
ẋ=0
x-NULLCLINE
UNSTABLE
SPIRAL
x
SADDLE
POINT
x=-2
ẏ=0
y-NULLCLINE
x=2
ẏ=0
y-NULLCLINE

(6)



$\boxed{6.3.3}$

$$\begin{cases} \dot{x} = 1 + y - e^{-x} = f(x,y) \\ \dot{y} = x^3 - y = g(x,y) \end{cases}$$

FIXED POINTS $\dot{x} = 0$, $\dot{y} = 0$

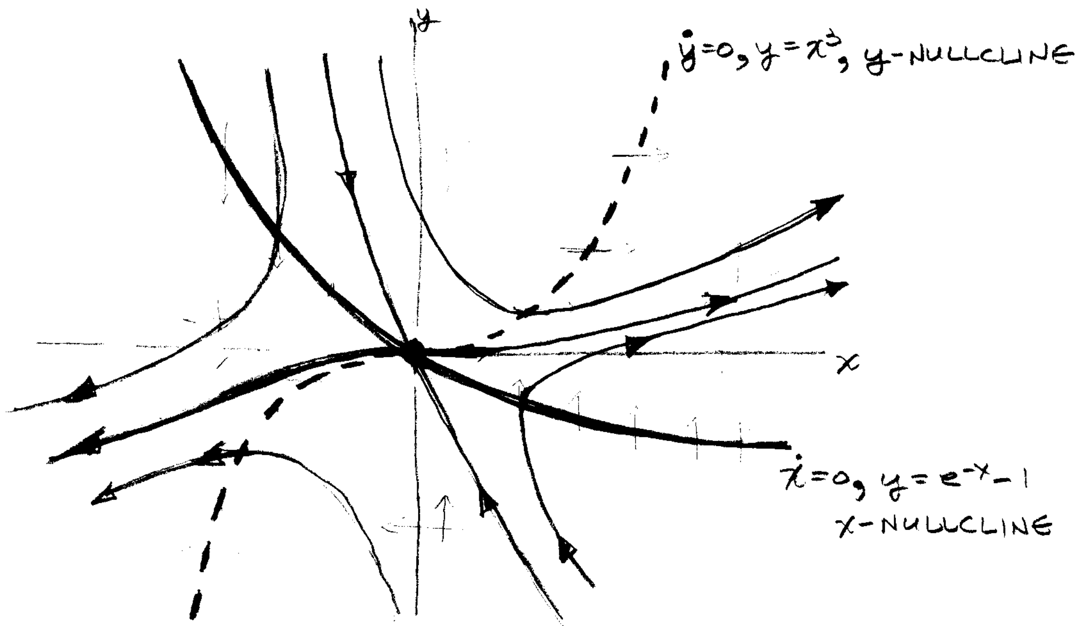
$$\begin{cases} y = e^{-x} - 1 \\ y = x^3 \end{cases} \Rightarrow (x^*, y^*) = (0,0)$$

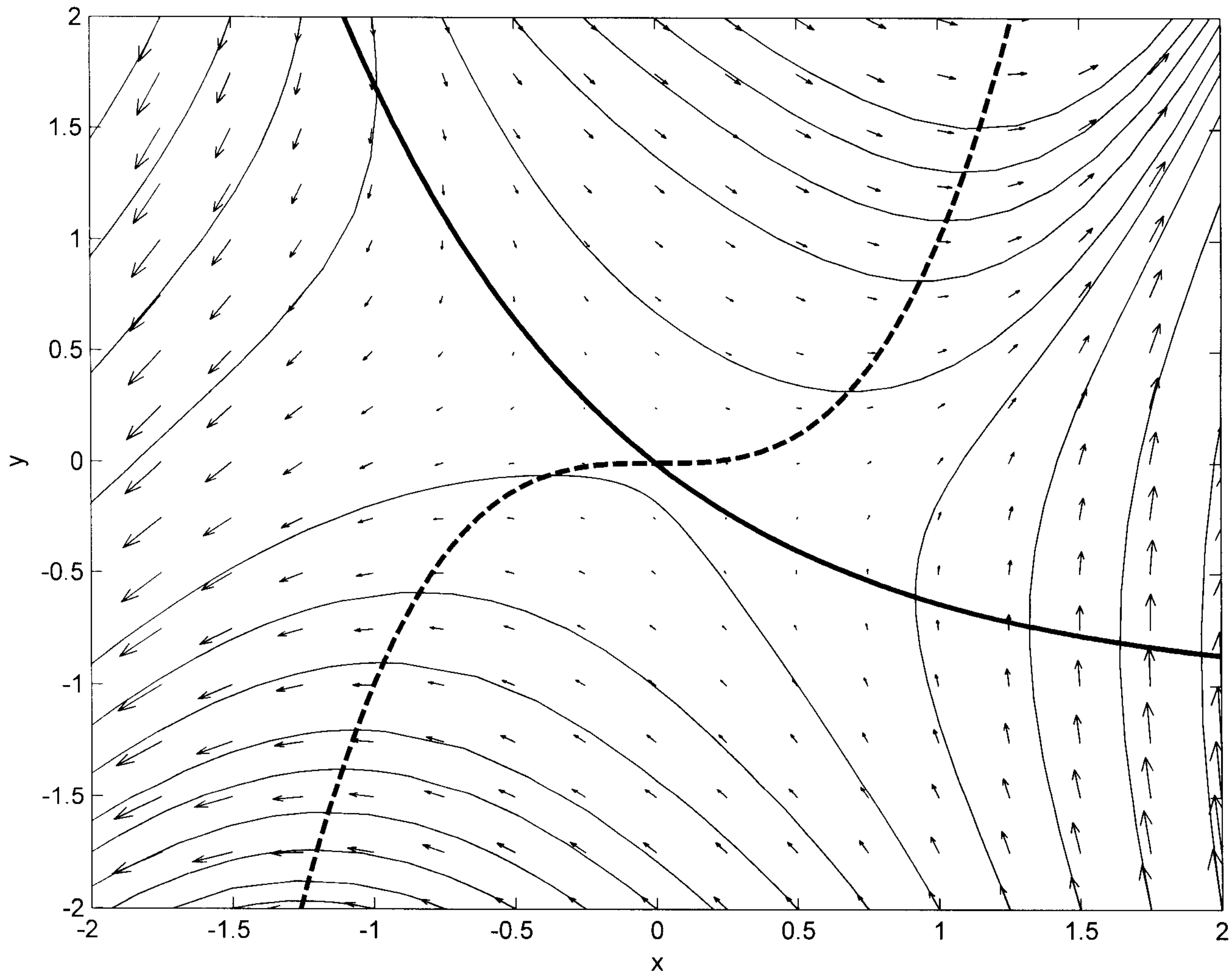
LINEAR STABILITY ANALYSIS

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(0,0)} = \begin{bmatrix} e^{-x^*} & 1 \\ 3x^{*2} & -1 \end{bmatrix}_{(0,0)}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda - 1)(\lambda + 1) = 0$$

$$\left\{ \lambda_1 = 1 , \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \left\{ \lambda_2 = -1 , \bar{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$





2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

$\boxed{6.3.6}$ $\begin{cases} \dot{x} = xy - 1 & = f(x,y) \\ \dot{y} = x - y^3 & = g(x,y) \end{cases}$

FIXED PTS $\dot{x} = 0$, $\dot{y} = 0$

$$\begin{matrix} x^* y^* = 1 \\ x^* = y^{*3} \end{matrix} \Rightarrow y^{*4} = 1 \qquad y^* = \pm 1$$

$$(x^*, y^*) = \begin{matrix} (1, 1) \\ (-1, -1) \end{matrix}$$

STABILITY

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_* = \begin{bmatrix} y^* & x^* \\ 1 & -3y^{*2} \end{bmatrix}$$

- $(x^*, y^*) = (1, 1)$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 4 = 0$$

$$\lambda = -1 \pm \sqrt{5}$$

$$\left\{ \lambda_1 = -1+\sqrt{5}, \; \bar{V}_1 = \begin{pmatrix} 1 \\ -2+\sqrt{5} \end{pmatrix} \right\},$$

$$\left\{ \lambda_2 = -1-\sqrt{5}, \; \bar{V}_2 = \begin{pmatrix} 1 \\ -2-\sqrt{5} \end{pmatrix} \right\}$$

$(1, 1)$ IS A SADDLE PT.

- $(x^*, y^*) = (-1, -1)$

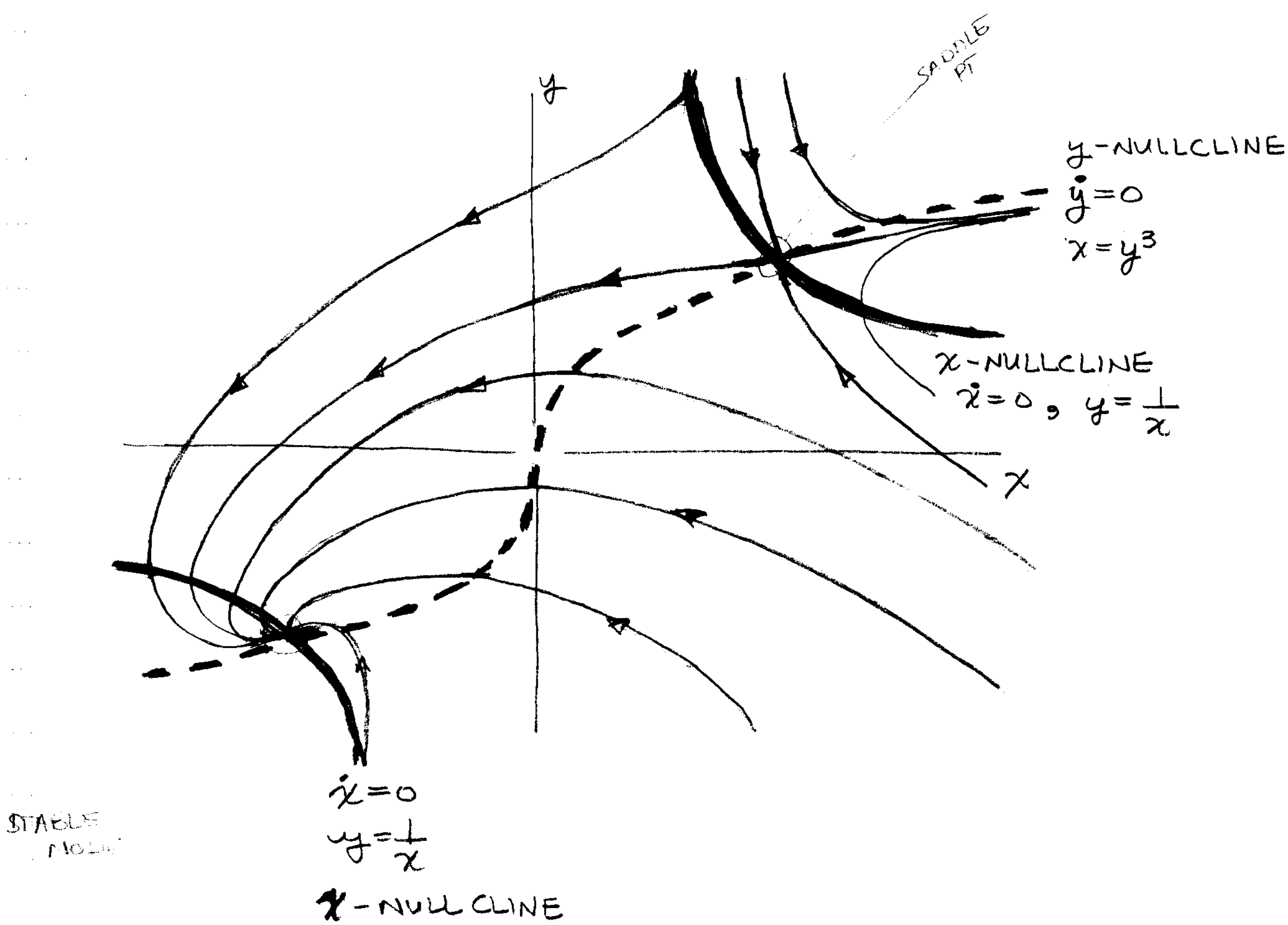
$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = 0$$

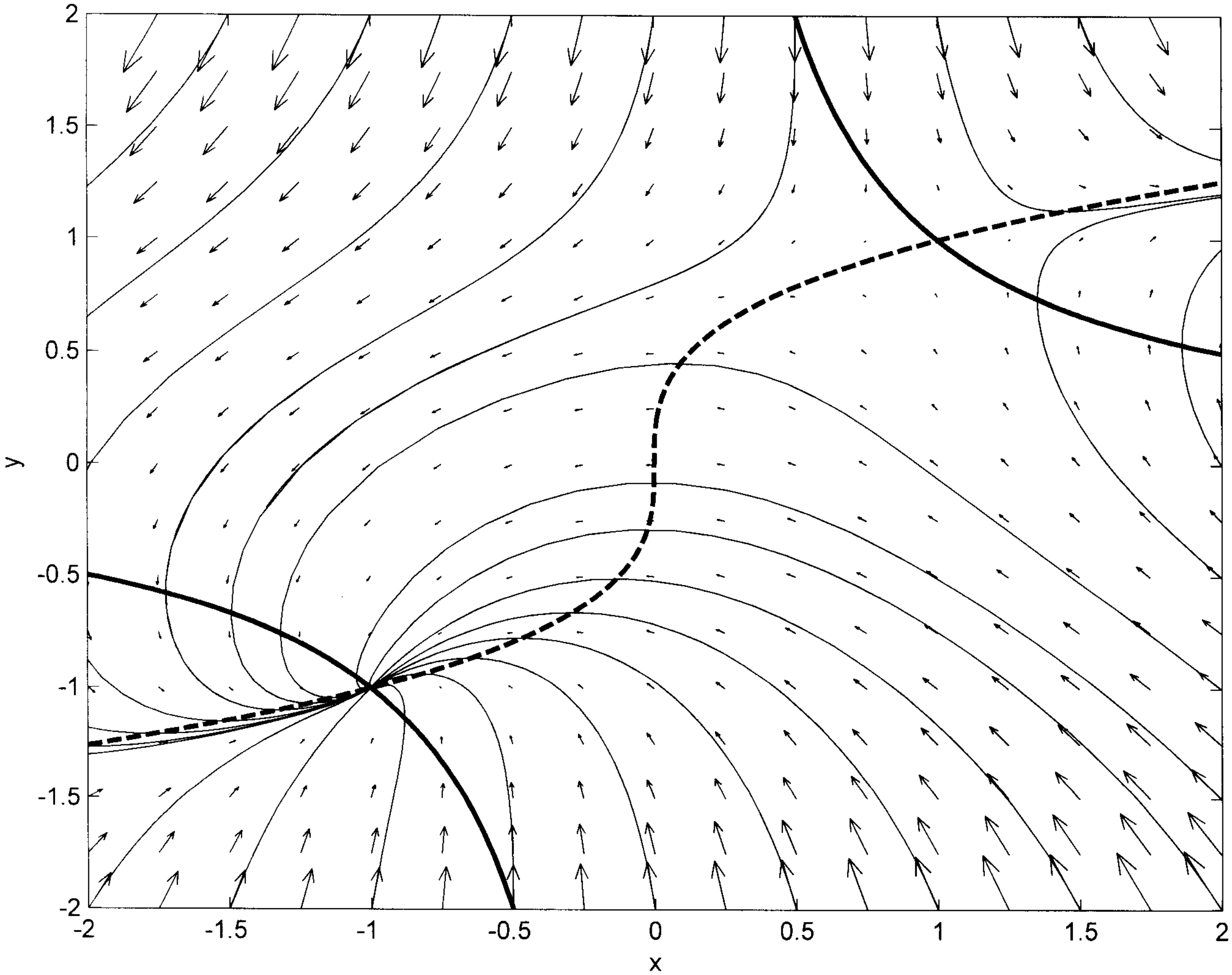
$$\left\{ \begin{array}{l} \lambda = -2 \\ \bar{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array} \right\}$$

(MULTIPLICITY = 2 !) $\lambda = -2$

ONLY ONE EIGENVECTOR

$(-1, -1)$ IS A DEGENERATE STABLE NODE.





y
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

$\boxed{6.3.10}$

$$\begin{cases} \dot{x} = xy \\ \dot{y} = x^2 - y \end{cases}$$

a) linearize system around the fixed point $(0,0)$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} y^* & x^* \\ 2x^* & -1 \end{bmatrix}_{\substack{(x^*,y^*) \\ =(0,0)}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

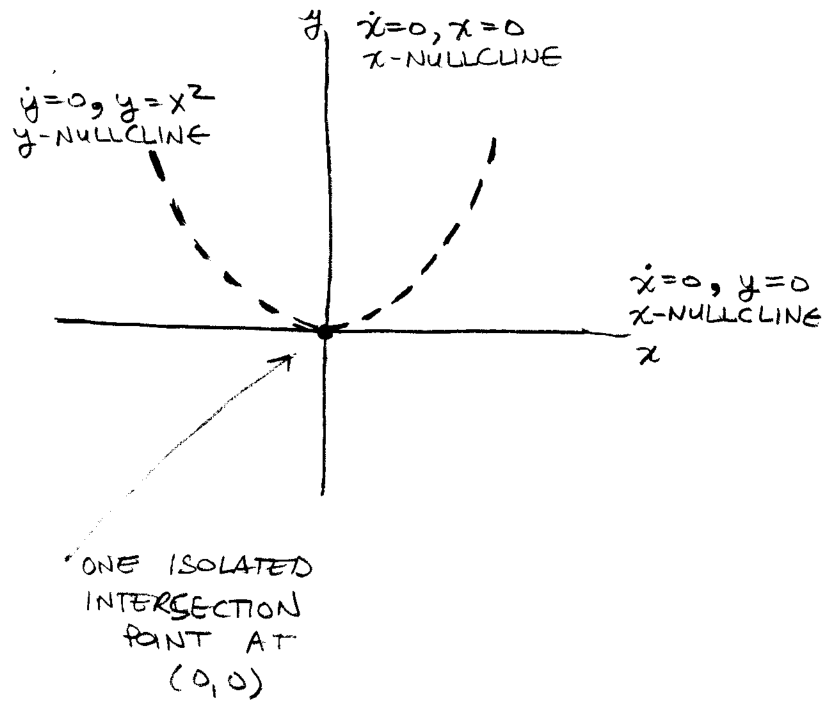
$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda - 1 \end{vmatrix} = \lambda(\lambda+1) = 0$$

$$\left\{ \lambda_1 = 0 \,, \; \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} , \left\{ \lambda_2 = -1 \,, \; \bar{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

therefore linear stability analysis suggest that there is a curve of ATTRACTING fixed points around $(0,0)$ [tangent to $\bar{v}_1 = (1,0)$ at $(0,0)$].

... BUT ...

b) plotting the nullclines ($\dot{x}=0$ and $\dot{y}=0$) shows that the fixed point at $(0,0)$ is an isolated fixed point.

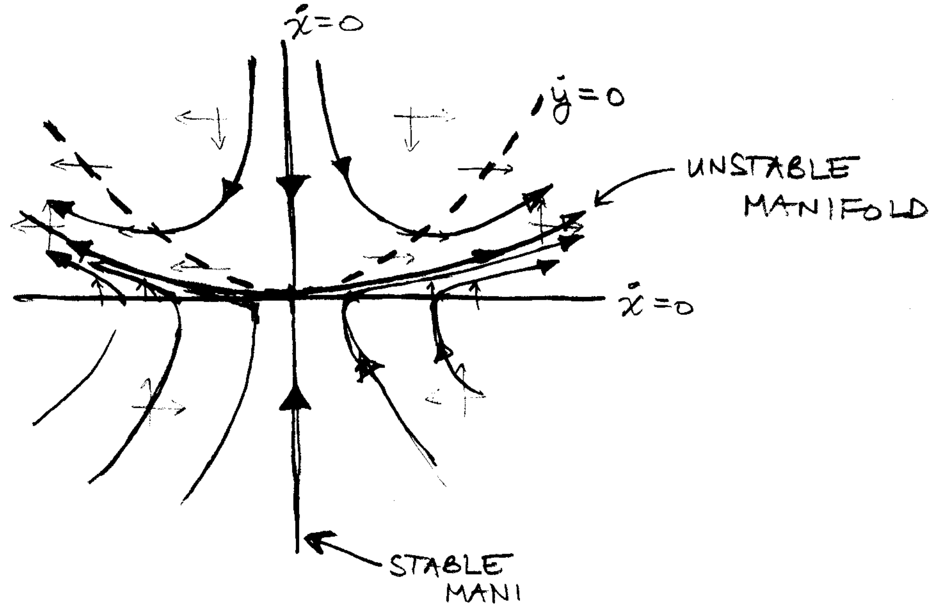


this can also be shown by solving $\{xy=0 \;,\; x^2-y=0\}$ i.e. $\dot{x}=0,\ \dot{y}=0$

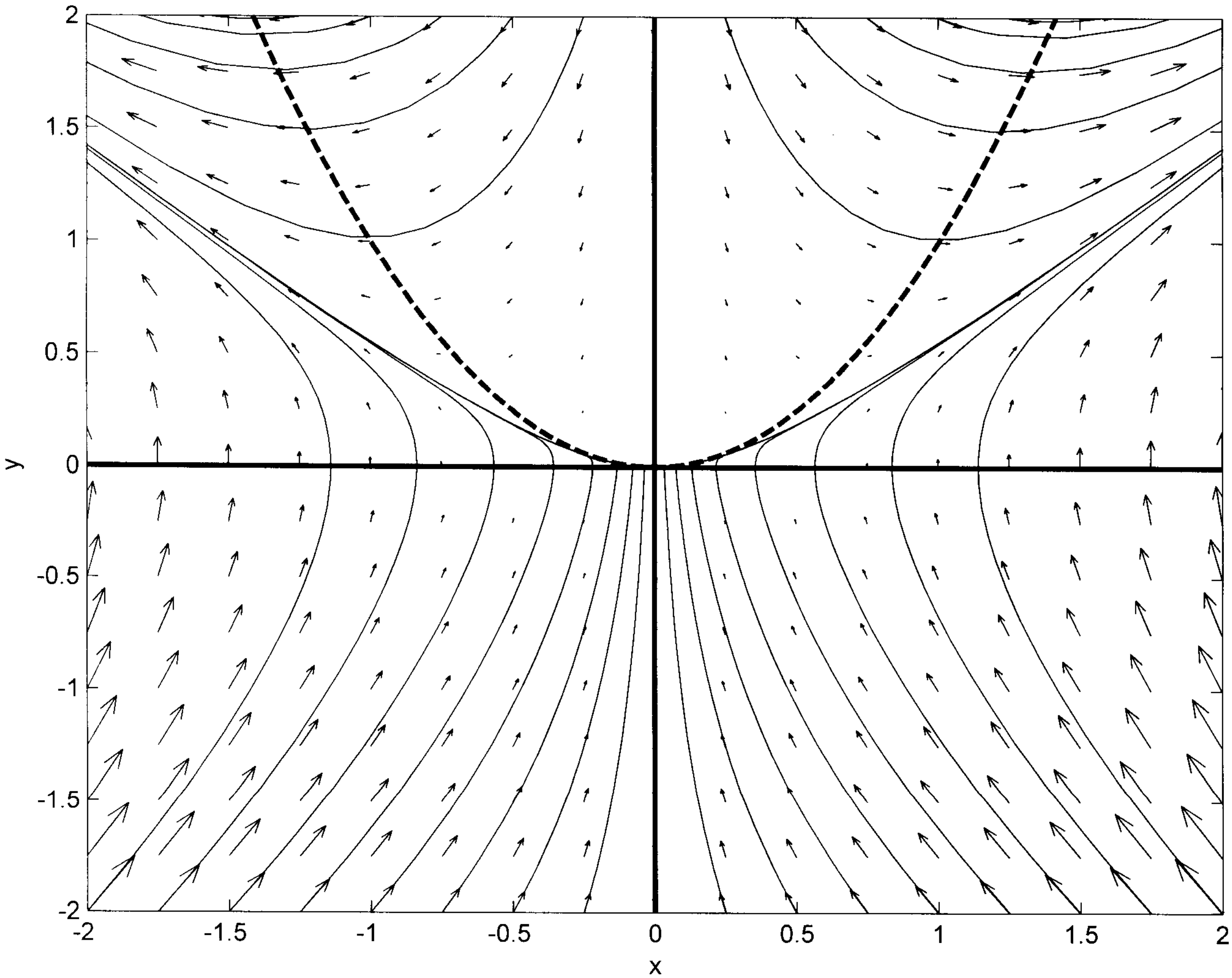
$$\Rightarrow \quad y=x^2 \;\rightarrow\; x\,x^2 = x^3 = 0 \quad \Rightarrow \quad (0,0)$$

IS ONLY SOLUTION.

c)



(0,0) IS A SADDLE POINT.


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

6.4.1
$$\begin{cases} \dot{x} = x(3-x-y) = f(x,y) \\ \dot{y} = y(2-x-y) = g(x,y) \end{cases}$$

FIXED POINTS $\quad \dot{x}=0, \ \dot{y}=0$

$$\begin{cases} x(3-x-y)=0 & x=0 \ \text{OR} \ y=-x+3 \\ y(2-x-y)=0 & y=0 \ \text{OR} \ y=-x+2 \end{cases}$$

$$\Rightarrow (x^*, y^*) = (0,0), \ (0,2), \ (3,0)$$

STABILITY

$$A = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}_{(x^*,y^*)} = \begin{bmatrix} 3-2x^*-y^* & -x^* \\ -y^* & 2-x-2y^* \end{bmatrix}$$

(i) $(0,0)$ $\quad A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$$\det(A-\lambda I) = (3-\lambda)(2-\lambda) = 0$$

$$\left\{ \lambda_1 = 3, \ \bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \ \left\{ \lambda_2 = 2, \ \bar{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$(0,0)$ IS AN UNSTABLE NODE

(ii) $(0, 2)$ $A = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)(-2-\lambda) = 0$$

$$\left\{ \lambda_1 = 1, \ \bar{V}_1 = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right\}, \ \left\{ \lambda_2 = -2, \ \bar{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

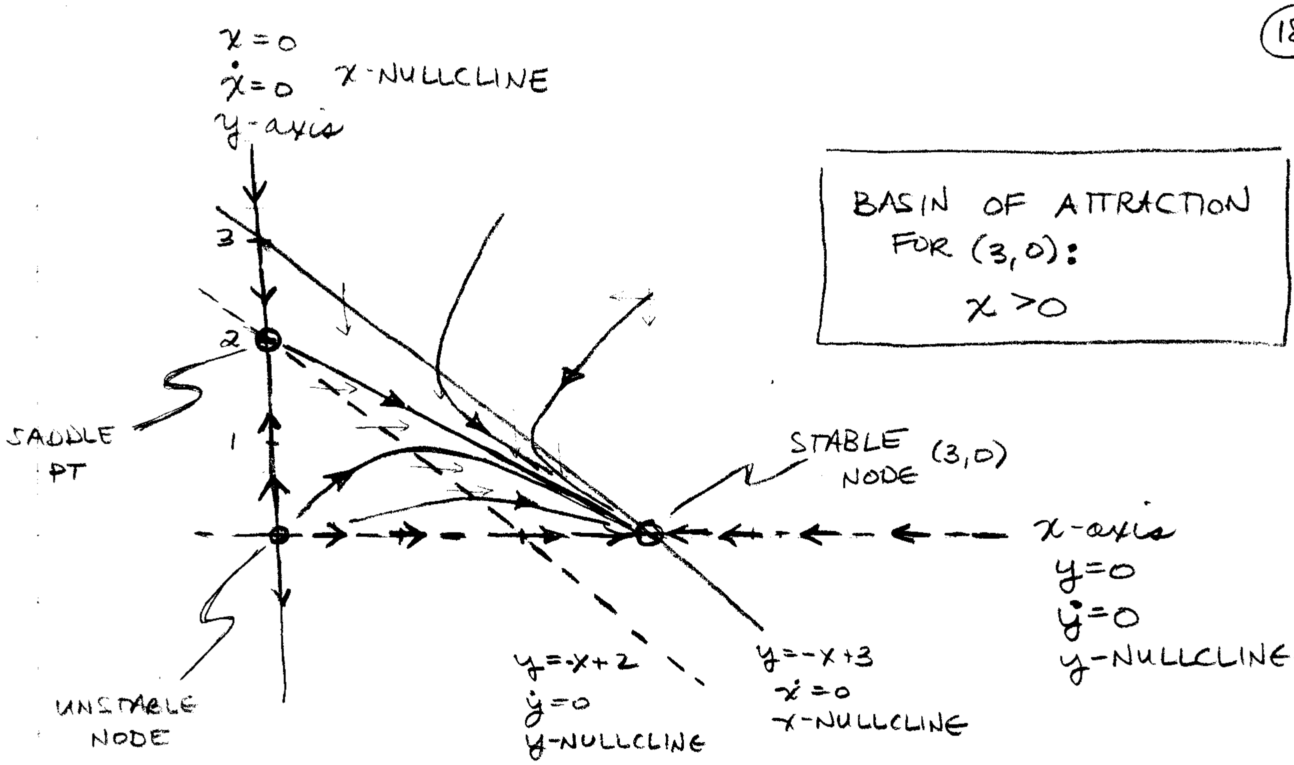
$(0,2)$ IS A SADDLE POINT

(iii) $(3, 0)$ $A = \begin{bmatrix} -3 & -3 \\ 0 & -1 \end{bmatrix}$

$$\det(A - \lambda I) = (-3-\lambda)(-1-\lambda) = 0$$

$$\left\{ \lambda_1 = -3, \ \bar{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \ \left\{ \lambda_2 = -1, \ \bar{V}_2 = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right\}$$

$(3,0)$ IS A STABLE NODE.



BASIN OF ATTRACTION FOR (3,0):

$x > 0$

* BASIN OF ATTRACTION FOR (0,2):

$x = 0, \; y > 0$