<u>6.5.1</u>

$$\ddot{x} = x^3 - x = F(x)$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = x^3 - x \end{cases}$$

a) FIXED POINTS

$$\dot{x} = 0 \,,\ \dot{y} = 0 \quad \Rightarrow \quad y^* = 0 \,, \quad x^{*3} - x^* = 0$$

$$x^* = 0, \pm 1$$

i.e. $(x^*, y^*) = (0,0)\,,\ (1,0)\,,\ (-1,0)$

STABILITY

$$\det \begin{bmatrix} -\lambda & 1 \\ 3x^{*2} - 1 & -\lambda \end{bmatrix} = \lambda^2 - (3x^{*2} - 1) = 0$$

$$\lambda_{1,2} = \pm \sqrt{3x^{*2} - 1}$$

- $(x^*, y^*) = (0,0)$

  $\lambda_{1,2} = \pm i \quad \Rightarrow$ CENTER

  (* see below)

- $(x^*, y^*) = (\pm 1, 0)$

  $\lambda_{1,2} = \pm \sqrt{2} \quad \Rightarrow$ SADDLE PT.

b) CONSERVED QUANTITY

define $$-\frac{dV}{dx}(x) = F(x) = x^3 - x$$

i.e. $$V(x) = -\int F(x)\,dx = \underline{\underline{-\frac{x^4}{4} + \frac{x^2}{2}}}$$

$$\ddot{x} = F(x) = x^3 - x$$

$$\int (\ddot{x})\,\dot{x}\,dt = \int F(x)\,\dot{x}\,dt + C$$

$$\int \frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) dt = -\int \frac{dV}{dx}(x)\,\frac{dx}{dt}\,dt + C$$

$$= -\int \frac{dV}{dx}\,dx + C$$

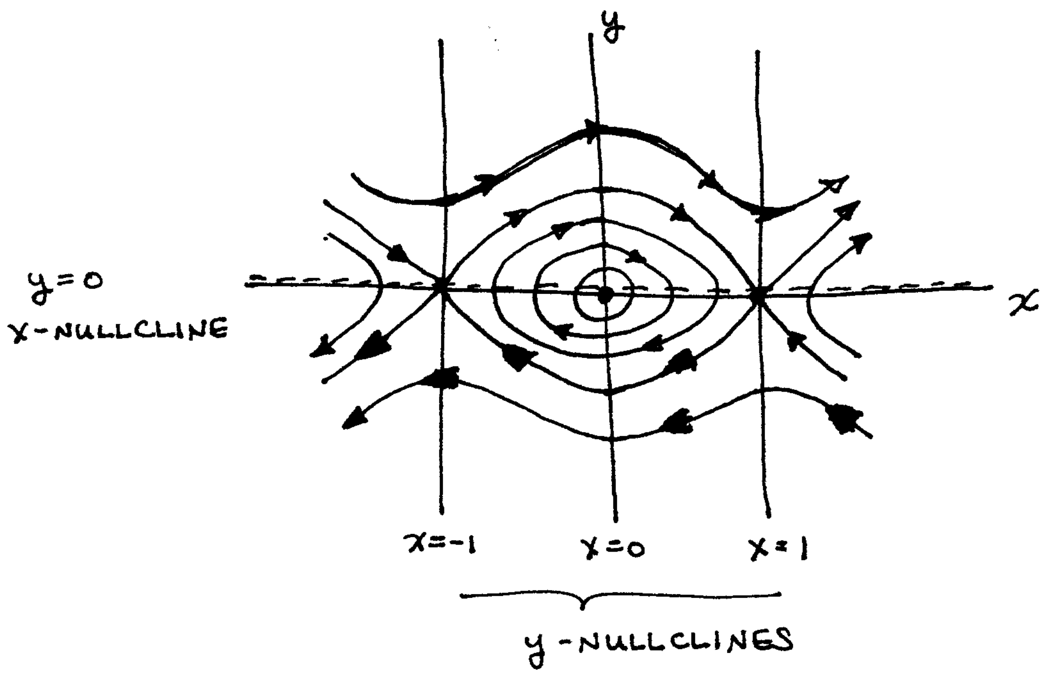
$$\frac{\dot{x}^2}{2} = -V(x) + C$$

$E(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$ IS A CONSERVED QUANTITY.

i.e. THE SYSTEM IS CONSERVATIVE.

* NOTE THAT THIS IMPLIES THAT (0,0) IS INDEED A NON-LINEAR CENTER.

c) PHASE· PORTRAIT.



6.5.3

$$\ddot{x} = a - e^x = F(x;a)$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = a - e^x \end{cases}$$

a) define $V(x) = -\int F(x;a)\,dx$

$$= -ax + e^x$$

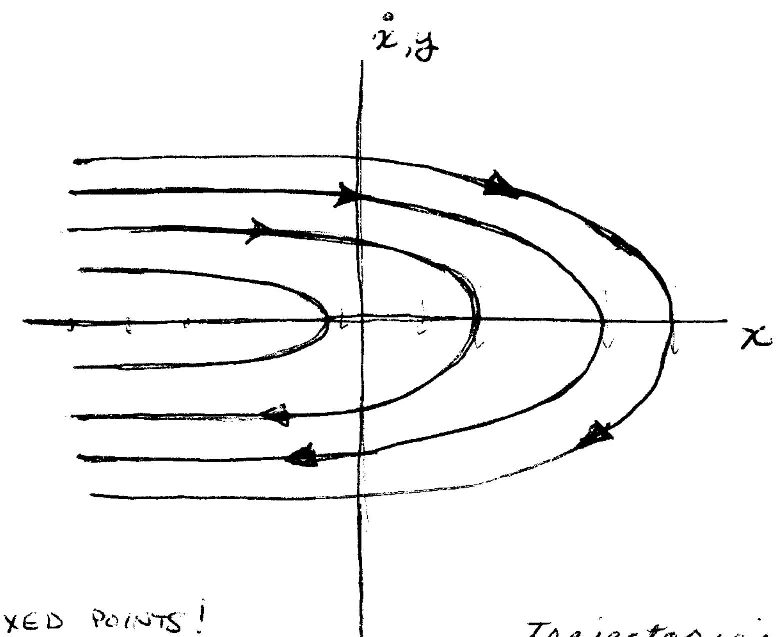
(same argument as in 6.5.1 (b))

$$E(x,y) = \frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}\dot{x}^2 - ax + e^x$$

is a CONSERVED QUANTITY.

b) PHASE PORTRAITS

$a = 0$

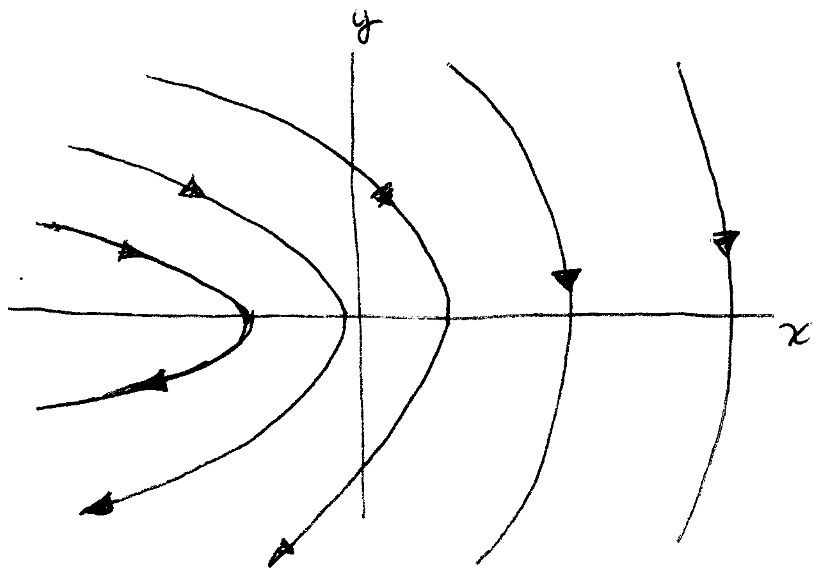


NO FIXED POINTS!

trajectories are level sets of

$$\frac{1}{2}y^2 + e^x = E(x,y)$$

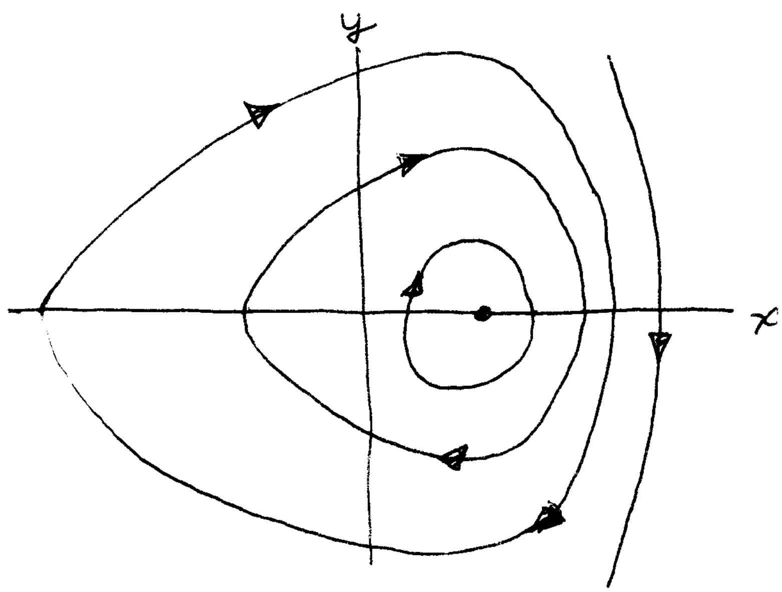
$a < 0$



NO FIXED PTS

traj. are level sets of $\frac{1}{2}y^2 - ax + e^x = E(x,y)$.

$a > 0$



FIXED PT
at $y = 0$
$e^x = ax$

traj are level sets of $E(x,y) = \frac{1}{2}y^2 - ax + e^x$.

6.5.6 Kermack-Mckendrick Epidemic model

$$\dot{x} = -kxy \qquad\qquad x(t), y(t) \geq 0$$
$$\dot{y} = kxy - ly \qquad\qquad k, l > 0$$

a) FIXED PTS $\quad \dot{x} = 0 \Rightarrow x = 0 \, , \, y = 0$

$\quad \dot{y} = 0 \Rightarrow x = \frac{l}{k} \, , \, y = 0$

$\therefore$ THERE IS A LINE OF FIXED PTS $y^* = 0$.

STABILITY

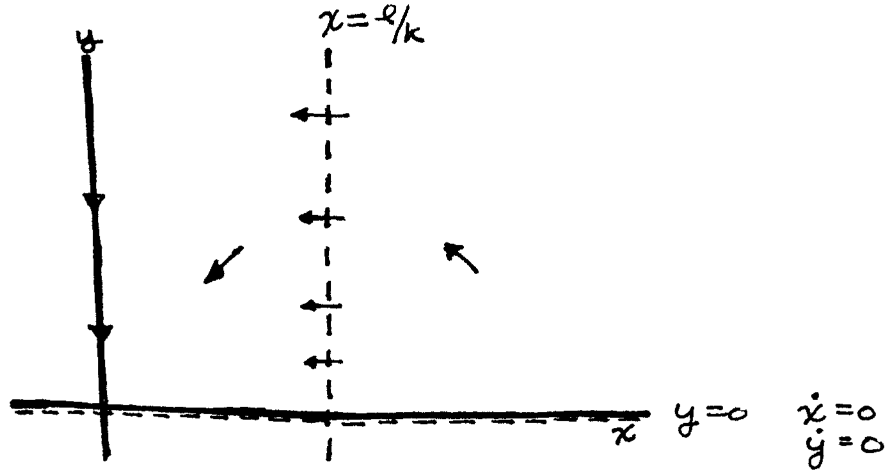
$$\det \begin{bmatrix} -k\cancel{y^*}^{\,0} - \lambda & -kx^* \\ k\cancel{y^*}^{\,0} & kx^* - l - \lambda \end{bmatrix} = 0$$

$$\lambda^2 - (k^*x^* - l)\lambda = 0$$

$\lambda_1 = 0 \quad \leftarrow$ NEUTRAL STABILITY ALONG $y = 0$.

$\lambda_2 = kx^* - l$

b)



c) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{kxy - \ell y}{-kxy} = -1 + \frac{\ell}{k}\frac{1}{x}$$

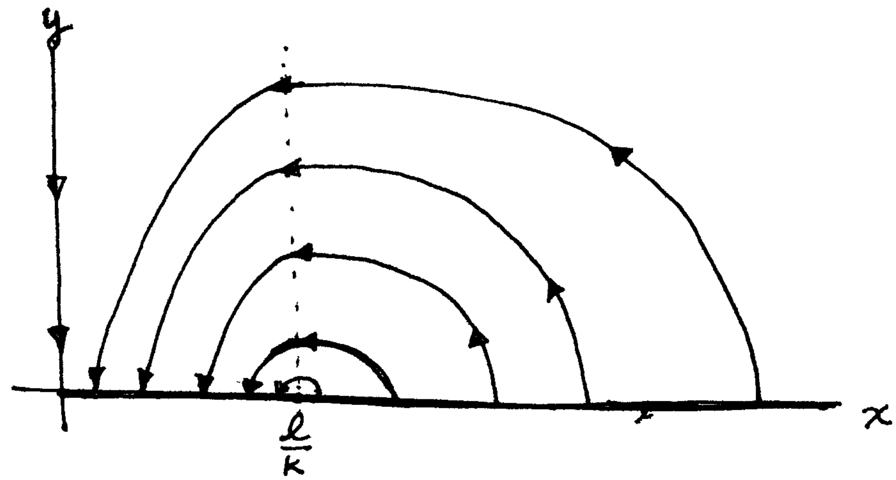
$$\int dy = \int \left(-1 + \frac{\ell}{k}\frac{1}{x}\right) dx + c$$

$$y = -x + \frac{\ell}{k}\ln x + c \quad , \quad x \neq 0$$

CONSERVED QUANTITY

$$y + x - \frac{\ell}{k}\ln x = E(x,y)$$

d)



- as $t \to \infty$, $x \to x^*$, $x^* \in [0, \frac{\ell}{k}]$ ...
  if $y > 0$

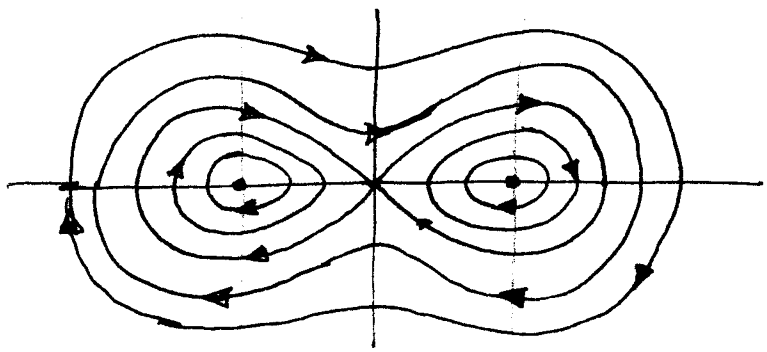
e) AN EPIDEMIC IS SAID TO OCCUR IF $y(t)$ INCREASES INITIALLY, THEREFORE THE MODEL PREDICTS THAT EPIDEMICS OCCUR WHEN
$x(0) > \frac{\ell}{k}$, $y(0) > 0$.

6.5.11

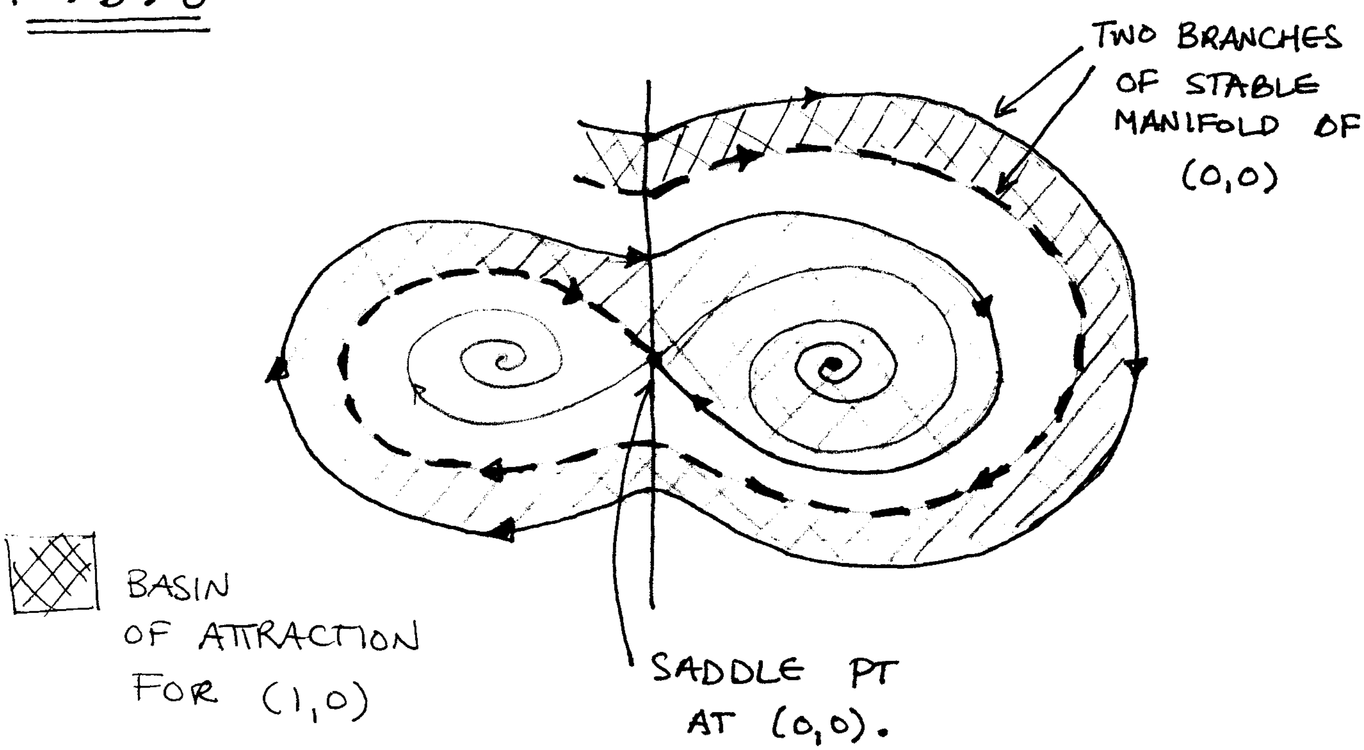
$$\begin{cases} \dot{x} = y \\ \dot{y} = -by + x - x^3 \end{cases} \qquad 0 < b \ll 1$$

i.e. $\ddot{x} + b\dot{x} - x + x^3 = 0$

$b = 0$ (EXAMPLE 6.5.2)



$1 \gg b > 0$



<u>6.5.14</u>

$$\dot{v} = -\sin\theta - Dv^2$$
$$v\dot{\theta} = -\cos\theta + v^2$$

a) $D=0$.

let $E(\theta, v) = v^3 - 3v\cos\theta$

$$\frac{dE}{dt} = \frac{dE}{dv}\frac{dv}{dt} + \frac{dE}{d\theta}\frac{d\theta}{dt}$$

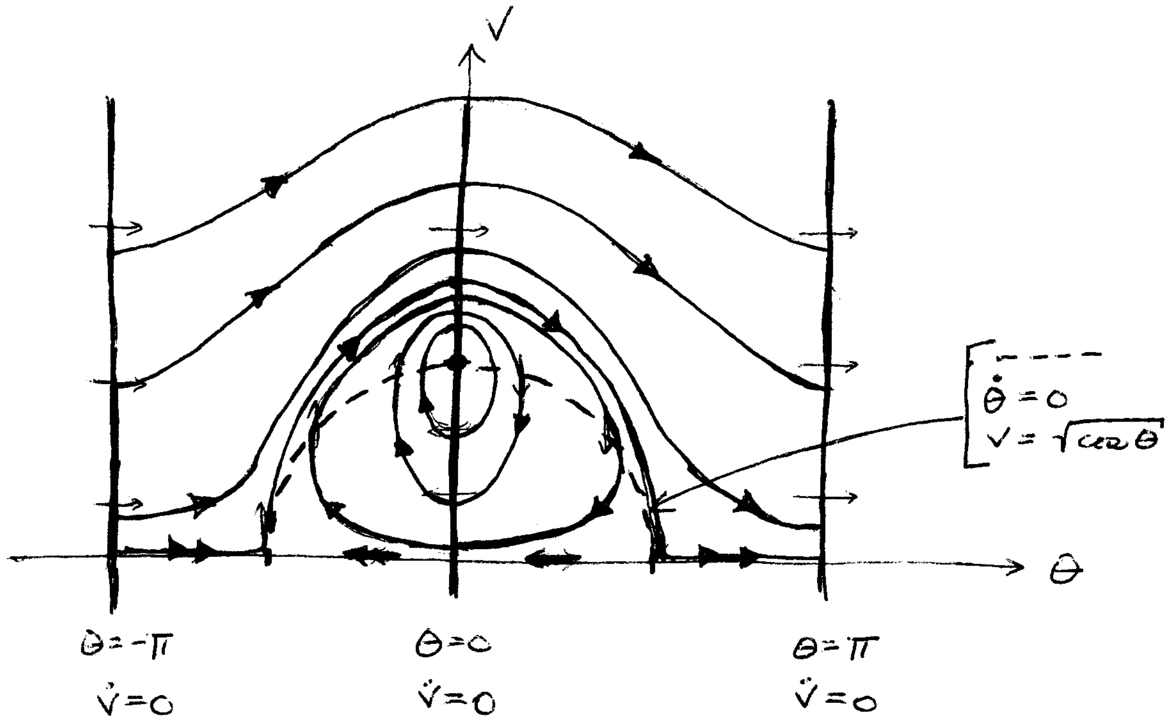
$$= (3v^2 - 3\cos\theta)(-\sin\theta) + (\sin\theta\ 3v)\left(\frac{-\cos\theta + v^2}{v}\right)$$

$$= -3v^2\sin\theta + 3\cos\theta\sin\theta - 3\sin\theta\cos\theta + 3v^2\sin\theta$$

$$= 0$$

$\Rightarrow$ $E(\theta, v)$ is a CONSERVED QUANTITY.

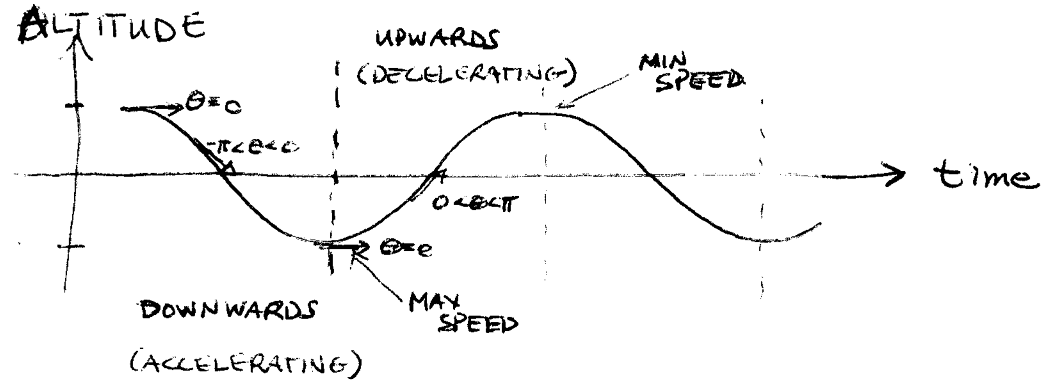
PHASE PORTRAIT



* note that as $V \to 0$, $\dot{\theta} \to \infty$!

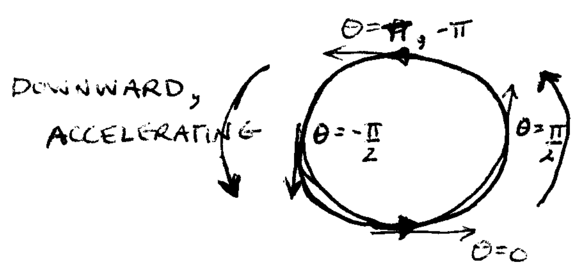
* system is 2π - periodic in θ.
* singularity in flow at $V = 0$, consider $V > 0$.
* FIXED PT at $(\theta^*, V^*) = (0, 1)$ is A NON-LINEAR CENTER.

(i) CLOSED (ORBIT) LOOP TRAJECTORIES correspond to flight paths that are oscillatory - accelerating as glider heads downward $(-\pi < \theta < 0)$ ~~and~~, hitting a minimum ~~accelerating~~ altitude and then heading upward $(0 < \theta < \pi)$ while decelerating. The glider then reaches a max. altitude and starts to head downwards again.

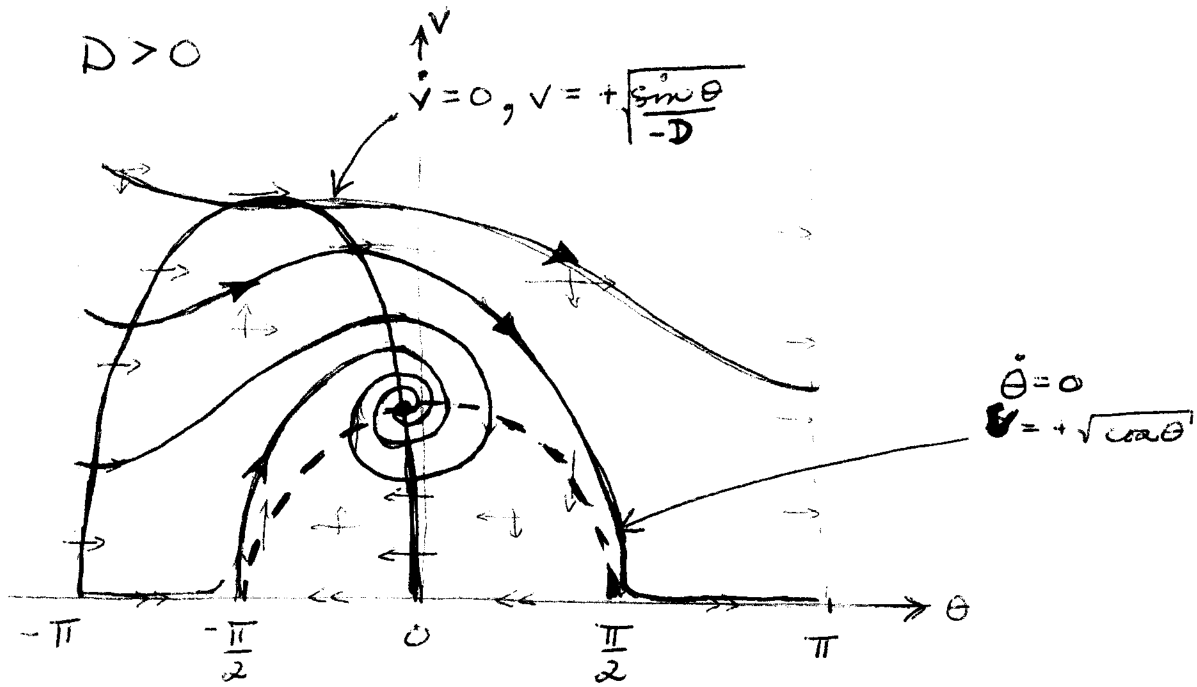


(ii) FIXED POINT corresponds to horizontal flight at a constant altitude and a constant velocity.

(iii) TRAJECTORIES ABOVE THE REGION WITH THE CLOSED ~~ORBITS~~ ORBITS THAT RUN FROM $\theta = -\pi$ TO $\theta = \pi$ (AND AROUND AGAIN ie. 2π-PERIODIC SYSTEM) correspond to the glider doing loop-the-loops (continually).



b) $D > 0$



* 2π-period FLOW.

THE NON-LINER CENTER BECOMES A STABLE SPIRAL THAT IS A GLOBAL ATTRACTOR FOR $V(0) > 0$.

THE GLIDER CAN TRANSIENTLY DO LOOP-THE-LOOP AND OSCILLATORY FLIGHT BUT EVENTUALLY APPROACHES A STEADY DECLINE AT A FIXED SPEED (i.e. $-\pi < \theta^* < 0$ AT FIXED POINT).

6.6.1 $\begin{cases} \dot{x} = y(1-x^2) = f(x,y) \\ \dot{y} = 1 - y^2 = g(x,y) \end{cases}$

(1) $f(x,-y) = -y(1-x^2) = -f(x,y).$

(2) $g(x,-y) = 1-(-y)^2 = (1-y^2) = g(x,y)$

$\Rightarrow$ f is ODD in y and g is even in y, therefore system is TIME REVERSIBLE. ie. invariant under $t \to -t$, $y \to -y$.

more generally:

$$\xrightarrow[\substack{t \to -t \\ y \to -y}]{} \left[\begin{array}{l} -x' = (-y)(1-x^2) \\ -(-y)' = 1-(-y)^2 \end{array}\right. \longrightarrow \begin{cases} x' = y(1-x^2) \\ y' = 1-y^2 \end{cases}$$

FIXED PTS $\quad \dot{x} = 0 \quad \dot{y} = 0 \qquad \left.\begin{array}{l} y(1-x^2) = 0 \\ 1-y^2 = 0 \end{array}\right\} \begin{array}{l} y^* = \pm 1 \\ x^* = \pm 1 \end{array}$

STABILITY

$$\det \begin{bmatrix} -2x^*y^* - \lambda & 1-x^{*2} \\ 0 & -2y^* - \lambda \end{bmatrix} = 0$$

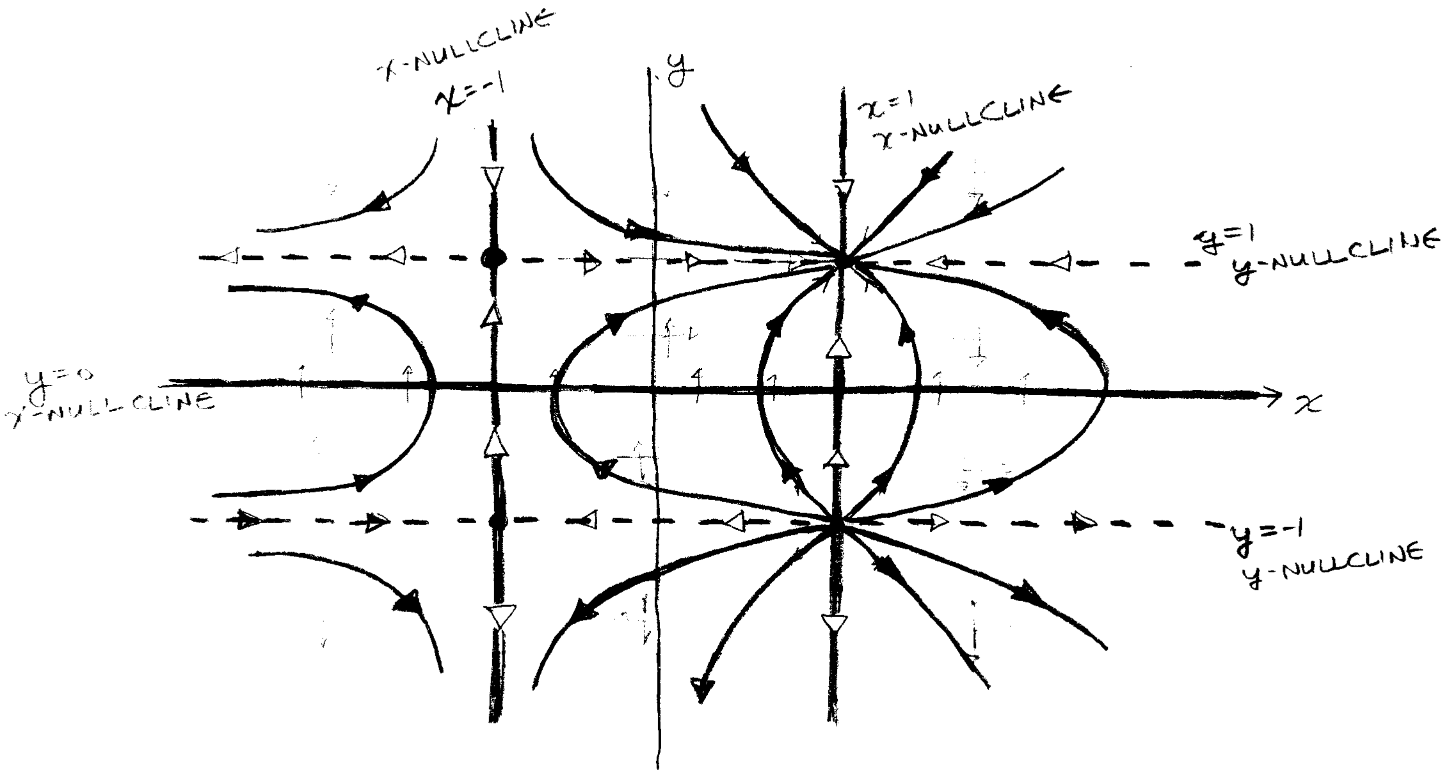
$$\lambda^2 + (x^*+1)\, 2y^* \lambda + 4x^*y^{*2} = 0$$

$$\lambda = -y^*(x^*+1) \pm \sqrt{y^{*2}(x^*+1)^2 - 4x^*y^{*2}}$$

$$= -y^*\left(x^*+1 \mp (x^*-1)\right)$$

$$\lambda_1 = -2y^* \quad , \quad \lambda_2 = -2y^*x^*$$

$(x^*, y^*) =$	$(1, 1)$	STABLE NODE
	$(1, -1)$	UNSTABLE NODE
	$(-1, 1)$	SADDLE POINT
	$(-1, -1)$	SADDLE POINT



* NOTE: ALL TRAJECTORIES $(x(t), y(t))$ in $y > 0$ HAVE "TWIN" TRAJECTORIES $(x(-t), -y(-t))$ (that live in the bottom half on phase portrait).

6.6.5 $\ddot{x} + a(\dot{x}) + b(x) = 0$

a is an even function.

a, b are smooth functions.

$$\begin{cases} \dot{x} = y = f(x,y) \\ \dot{y} = -a(y) - b(x) = g(x,y) \end{cases}$$

a) PURE TIME-REVERSAL SYMMETRY

$\tau = -t$

$$\frac{d^2x}{d\tau^2} + a\left(-\frac{dx}{d\tau}\right) + b(x) = 0$$

a is even $\Rightarrow$ $$\frac{d^2x}{d\tau^2} + a\left(\frac{dx}{d\tau}\right) + b(x) = 0$$ ✓

OR

$f(x,-y) = -y = -f(x,y)$

a is even

$g(x,-y) = -a(-y) - b(x) = -a(y) - b(x)$
$= g(x,y)$

* f is odd in y and g is even in y

$\Rightarrow$ TIME-REVERSIBLE

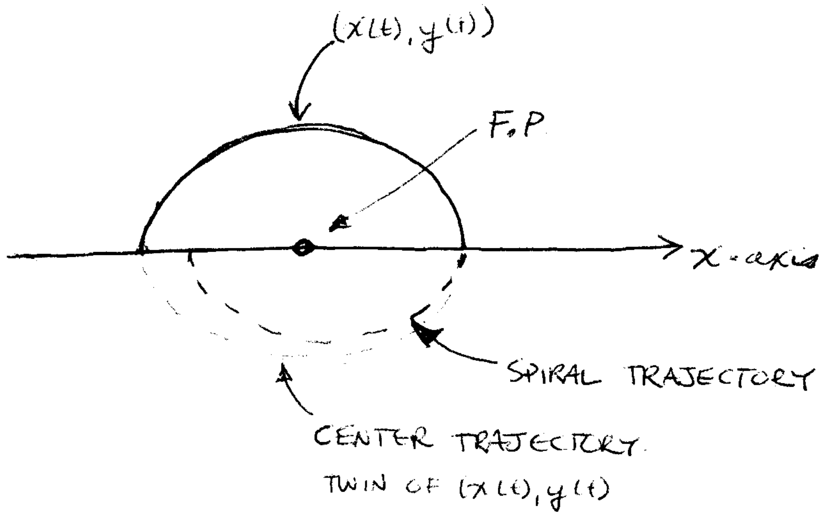
b) STEADY STATES CANNOT BE NODES OR SPIRALS.

note that all fixed points are on the x-axis ($y=0$).

ie. $\dot{x}=0$, $\dot{y}=0 \Rightarrow \begin{cases} y^* = 0 \\ b(x^*) = -a(0) \end{cases}$

because the system is time reversible, every trajectory $(x(t), y(t))$ has a twin trajectory $(x(-t), -y(-t))$.
THIS PROPERTY IS IMPOSSIBLE FOR NODES AND SPIRALS.

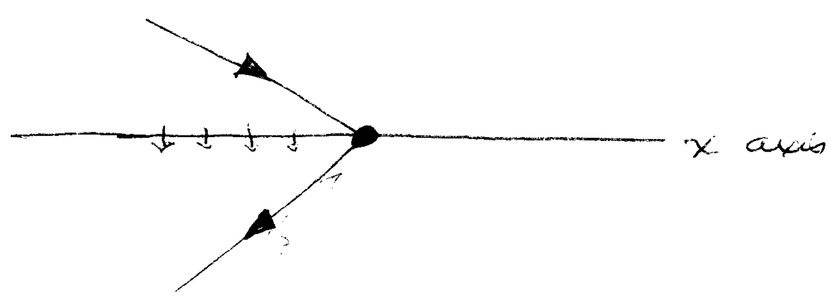
(i) trajectories near a spiral wind around the fixed point (into or out of the fixed point depending on the stability).
a trajectory in $y \geq 0$ would be an arc from $(x(t_1) = x_1, 0)$ to $(x(t_2) = x_2, 0)$.
The fact that it has a twin trajectory in $y \leq 0$ implies that a CLOSED ORBIT is formed, not a spiral.
Trajectories spiralling into or out of a fixed point would violate the symmetry of the system around the x-axis.



(ii) trajectories cannot approach (or come out of) fixed points of the system along the x-axis (where $\dot{x}=0$).

any trajectory going into ([redacted] coming out of) a fixed in $y>0$ must have a twin trajectory coming out of (going into) the fixed point in $y<0$.

Therefore fixed points cannot be nodes



$\boxed{6.6.7}$ $\ddot{x} + x\dot{x} + x = 0$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -xy - x \end{cases} \qquad \ldots (1)$$

- $t \to -t$ and $y \to -y$

$$\begin{cases} +\dot{x} = +y \\ +\dot{y} = +xy - x \end{cases}$$

NOT EQUIVALENT TO ONE

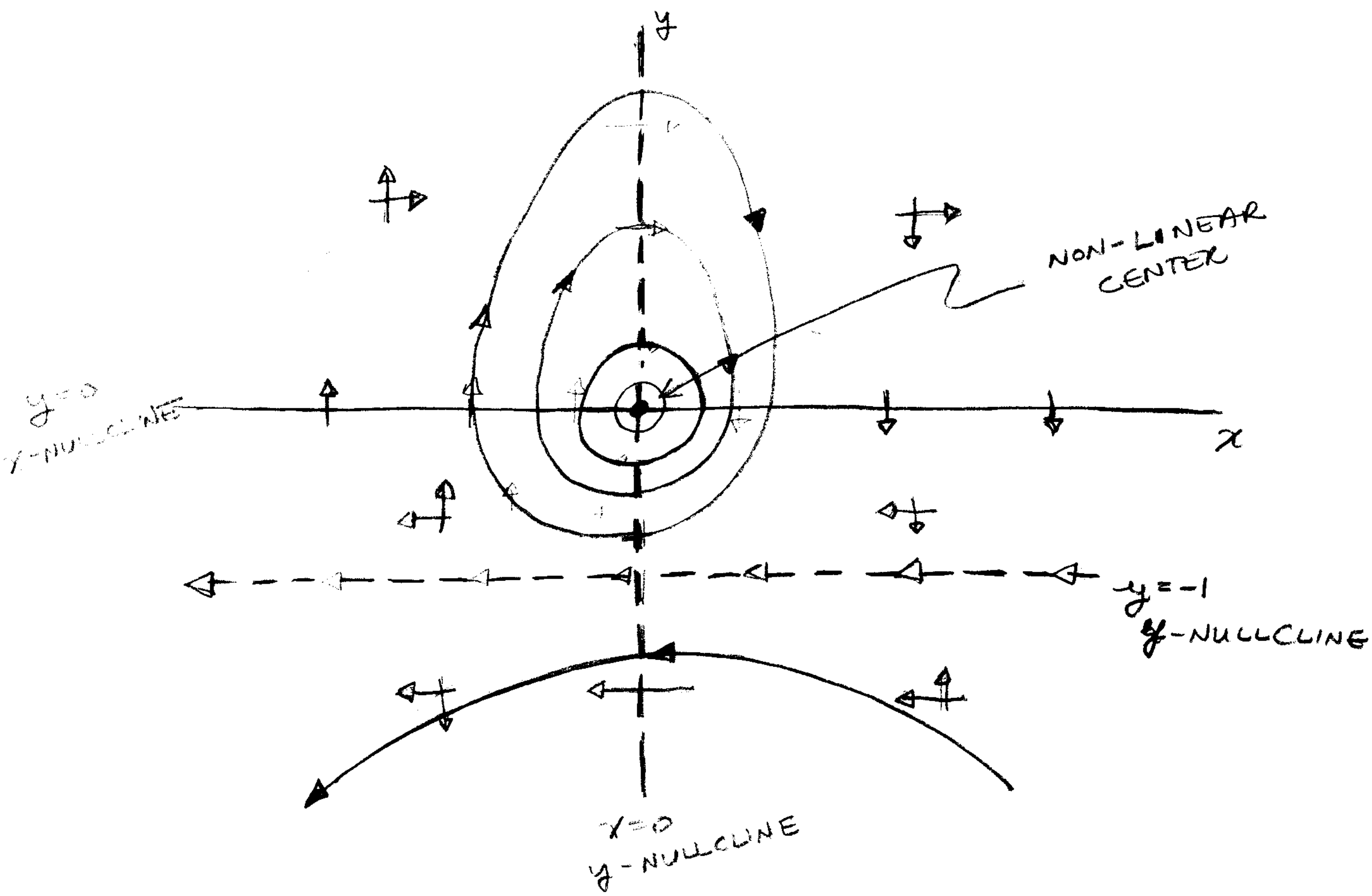
* STRICTLY SPEAKING THE SYSTEM IS NOT TIME - REVERSIBLE ...

HOWEVER SYSTEM IS INVARIANT UNDER $t \to -t$ AND $\underline{\underline{x \to -x}}$

$$\begin{aligned} +\dot{x} &= y \\ -\dot{y} &= +xy + x \end{aligned} \quad \to \quad \begin{cases} \dot{x} = y \\ \dot{y} = -xy - x \end{cases}$$

THEREFORE SYSTEM IS SYMMETRIC AROUND THE y-axis ; IF $(x(t), y(t))$ IS A SOLUTION, THEN SO IS $(-x(-t), y(-t))$ ... SO SYSTEM IS EFFECTIVELY REVERSIBLE.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x(y+1) \end{cases}$$



* REMEMBER THAT FLOW IS SYMMETRIC AROUND THE y-axis i.e. $(x(t), y(t))$ & $(-x(-t), y(-t))$. ARE TWIN TRAJECTORIES.

$\boxed{6.6.10}$ IS $(0,0)$ A NONLINEAR CENTER FOR

$$\begin{cases} \dot{x} = -y - x^2 = f(x,y) \\ \dot{y} = x \qquad\quad = g(x,y) \end{cases} \quad ?$$

NOTE THAT $f(x,y)$ is NOT ODD in y. ~~therefore~~ SYSTEM IS NOT REVERSIBLE IN THE USUAL SENSE, HOWEVER IT IS INVARIANT UNDER THE CHANGE OF VARIABLES $t \to -t$ AND $x \to -x$, i.e. IF $(x(t), y(t))$ is a SOLUTION, THEN SO is $(-x(-t), y(-t))$. SYSTEM IS SYMMETRIC AROUND THE y-axis.

- LINEAR STABILITY OF $(0,0)$

$$J = \begin{bmatrix} -2x^* & -1 \\ 1 & 0 \end{bmatrix}_{\substack{x^*=0 \\ y^*=0}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det(J - \lambda I) = \lambda^2 + 1 = 0$$

$\Rightarrow \lambda = \pm i$ and $(0,0)$ is A CENTER.

- BECAUSE THE SYSTEM IS EFFECTIVELY REVERSIBLE AND $(0,0)$ IS A "LINEAR CENTER", $(0,0)$ IS A NON-LINEAR CENTER (STROGATZ THM 6.6.1).

NOTES ON 6.6.7, 6.6.10

IN 6.6.10,

** NOTE THAT INTERCHANGING x AND y IN THE SYSTEM, WE GET

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - y^2 \end{cases},$$

WHICH IS REVERSIBLE IN THE USUAL SENSE (INVARIANT WITH RESPECT TO $t \to -t$, $y \to -y$).

SIMILARLY IN PROBLEM 6.6.7

$$\begin{cases} \dot{x} = -xy - y = -(x+1)y \\ \dot{y} = x \end{cases}$$

IS REVERSIBLE IN THE USUAL SENSE.

NOTE THAT THE ONLY TIME WE SHOULD BE CAREFUL INTERCHANGING x AND y IN THIS CONTEXT IS WHEN THEY HAVE STRICT PHYSICAL INTERPRETATIONS.