Name:

Student ID:
Section:

1	2	3	4	5	6	7	8	Total

FINAL EXAM

Math 167
Temple-F08

Problem 1. (30pts) Let A be a real $n \times n$ symmetric matrix.

(a) Prove that A has only real eigenvalues.

(b) Assuming that A has n distinct eigenvalues, prove that A has an orthonormal basis of eigenvectors.

Problem 1.(Continued)

Problem 2. (25pts) Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Find an orthonormal basis of eigenvectors of A.

(c) Find an 2×2 orthogonal matrix S such that $A = SDS^T$, where D is a diagonal matrix.

(d) Find $lim_{n \to \infty} A^n$.

Problem 2.(Continued)

Problem 3. (20pts) Let A and B be $m \times n$ matrices with entries a_{ij} and b_{ij}, respectively. Prove that if $Av_i = Bv_i$ for some basis $\{v_1, \cdots, v_n\}$ of $\mathcal{R}^n$, then $a_{ij} = b_{ij}$ for every $i, j = 1 \cdots n$.

Problem 3.(Continued)

Problem 4. **(25pts)** Assume that the populations y_1 and y_2 of two interacting species of animals evolves according to the equation

$$y'(t) \equiv \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) - a \\ y_2(t) - b \end{bmatrix}. \qquad (1)$$

(a) Show that the constant populations $y_1 = a,\ y_2 = b$ solves (1).

(b) Show that $y = \begin{bmatrix} a \\ b \end{bmatrix}$ is a *stable rest point* by finding a formula for the general solution $y(t)$ of (1) and showing that

$$\lim_{t\to\infty} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

(Hint: Let $x_1 = y_1 - a,\ x_2 = y_2 - b$, and find the eigen-solutions of $x' = Ax$.)

Problem 4.(Continued)

Problem 5. (25pts) Assume that A is a square $n \times n$ matrix of real numbers, and assume that the (symmetric) product $A^T A$ is *positive definite* in the sense that $x^T A^T Ax > 0$, (*strictly* postive!), for every vector $x \in \mathcal{R}^n$.

(a) Prove that there exists on orthonormal basis $\{v_1, ..., v_n\}$ of $\mathcal{R}^n$ and n non-negaive numbers $\lambda_i > 0$ such that $A^T Av_i = \lambda_i v_i$.

(b) Prove that $u_i = Av_i$ is an *orthogonal* basis of eigenvectors of AA^T in the sense that $< u_i, u_j >= u_i \cdot u_j = 0$, (i.e. $u_i \perp u_j$), when $i \neq j$.

Problem 5.(Continued)

Problem 6. (25pts) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Compute A^n, and use this to *derive* the matrix e^A directly from the definition.

Problem 6.(Continued)

Problem 7. (25pts)

(a) If $x = c_1u_1 + \cdots + c_nu_n$ gives vector x in terms of a given orthonormal basis $\{u_1, \cdots, u_n\}$, derive a formula for c_i.

(b) Use Gramm-Schmidt to construct an orthonormal basis u_1, u_2, u_3 for $\mathcal{R}^3$ from the basis

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Problem 7.(Continued)

Problem 8. (25pts) Let $A_{m\times n}$ satisfy $m > n$, and assume the columns of A form a basis for R^n. For $\mathbf{b} \in \mathcal{R}^m$, let $A\hat{\mathbf{x}}$ denote the element of $Col(A)$ closest to $\mathbf{b}$.

(a) Show that $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$.

(b) Find the line $b = C + Dt$ that best fits the data points

$$(t_1, b_1) = (1, -1); \quad (t_2, b_2) = (-1, 1); \quad (t_3, b_3) = (1, 2)$$

in the sense that $\sum_{i=1}^{3} \{b_i - (C + Dt_i)\}^2$ is minimized.

Problem 8.(Continued)